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Numerical solution of fractional variable order linear control system in state-space form

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Abstract. The aim of this paper is to introduce a matrix approach for approximate solving of non-commensurate fractional variable order linear control systems in state-space form. The approach is based on switching schemes that realize variable order derivatives. The obtained numerical solution is compared with simulation and analog model results.

Key words: variable order fractional calculus, differential equations, analog modeling.

1. Introduction

Fractional calculus generalizes traditional integer order integration and differentiation onto non-integer order operators. The idea was first mentioned in 1695 by Leibniz and de l'Hôpital. In the end of 19th century, Liouville and Riemann introduced the first definition of fractional derivative. However, only in late 1960s the idea drew attention of engineers. Theoretical background of fractional calculus can be found in, e.g., [1–3]. Fractional calculus has been found a convenient tool to model behavior of many materials and systems, particularly those involving diffusion processes. For example, ultracapacitors can be modeled more efficiently using fractional calculus, as was demonstrated in [4, 5].

Recently, the case of time-varying order begun to be studied extensively. The fractional variable order behavior can be encountered for example in chemistry, when system's properties are changing due to chemical reactions. Experimental studies of an electrochemical example of physical fractional variable order system have been presented in [6]. The variable order equations have been used to describe time evolution of drag expression in [7]. Numerical implementations of fractional variable order integrators and differentiators can be found in, e.g., [8, 9]. The fractional variable order calculus can also be used to describe variable order fractional noise [10]. In [11], the variable order interpretation of the analog realization of fractional order integrators, realized as domino ladders, has been considered. Applications of variable order derivatives and integrals arise also in control [12, 14].

In [15, 16], three general types of variable order derivative definitions have been given. Alternative definitions of variable order derivatives were introduced in [17, 18]. Numerical and analytical solutions of linear fractional variable order differential equations were presented, respectively in [19, 20] and [21].

In our paper, a method of finding a numerical solution of fractional variable order control system in a state-space form is introduced, both for time-invariant and time-variant case. Moreover, the obtained results are also valid for system of differential equations with different types of variable order derivatives. To validate our approach the fractional variable order state-space system was physically built and the experimental results were compared with numerical implementations.

The paper is organized as follows. At the beginning, in Section 2, the few types of fractional variable order derivatives are recalled, together with their discrete approximations and matrix forms. In Section 3 the solution of linear control system in state-space form for time-variant and time-invariant non-commensurate fractional variable order system is presented. An analog model of particular type of fractional variable order state-space system is introduced in Section 4. The experimental and numerical results are presented in Section 5. Finally, Section 6 summarizes the main results.

2. Fractional variable order operators

Below, we recall the already known different types of fractional constant and variable order derivatives and differences.

2.1. Definitions of variable order operators. The following fractional constant order difference of Grünwald-Letnikov type will be used as a base of generalization onto variable order

$$\Delta^{\alpha} x_l = \frac{1}{h^{\alpha}} \sum_{j=0}^{l} (-1)^j \binom{\alpha}{j} x_{l-j}, \tag{1}$$

where $\alpha \in \mathbb{R}$, $l = 0, \ldots, k$, and $h > 0$ is sample time.

We will consider the following four types of fractional variable order derivatives and their discrete approximations (differences). We admit the order changes in time, i.e., $\alpha(t) \in \mathbb{R}$ for $t > 0$; and in discrete-time domain $\alpha_l \in \mathbb{R}$ for $l = 0, \ldots, k$, where $k \in \mathbb{N}$.

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The $\mathscr{A}$-type variable-order derivative and its discrete approximation is given, respectively, by

$$ {}_0^{\mathscr{A}}\mathrm{D}_t^{\alpha(t)}x(t) = \lim_{h\to 0}\frac{1}{h^{\alpha(t)}}\sum_{j=0}^{\eta}(-1)^j\binom{\alpha(t)}{j}x(t-jh), $$

where $\eta = \lfloor t/h \rfloor$, and

$$ {}^{\mathscr{A}}\Delta^{\alpha_l}x_l = \frac{1}{h^{\alpha_l}}\sum_{j=0}^{l}(-1)^j\binom{\alpha_l}{j}x_{l-j}. $$

The $\mathscr{B}$-type variable-order derivative and its discrete approximation is given, respectively, by

$$ {}_0^{\mathscr{B}}\mathrm{D}_t^{\alpha(t)}x(t) = \lim_{h\to 0}\sum_{j=0}^{\eta}\frac{(-1)^j}{h^{\alpha(t-jh)}}\binom{\alpha(t-jh)}{j}x(t-jh) $$

and

$$ {}^{\mathscr{B}}\Delta^{\alpha_l}x_l = \sum_{j=0}^{l}\frac{(-1)^j}{h^{\alpha_{l-j}}}\binom{\alpha_{l-j}}{j}x_{l-j}. $$

The $\mathscr{D}$-type variable-order derivative and its discrete approximation is given, respectively, by

$$ {}_0^{\mathscr{D}}\mathrm{D}_t^{\alpha(t)}x(t) = \lim_{h\to 0}\left(\frac{x(t)}{h^{\alpha(t)}} - \sum_{j=1}^{\eta}(-1)^j\binom{-\alpha(t)}{j}{}_0^{\mathscr{D}}\mathrm{D}_{t-jh}^{\alpha(t)}x(t)\right) $$

and

$$ {}^{\mathscr{D}}\Delta^{\alpha_l}x_l = \frac{x_l}{h^{\alpha_l}} - \sum_{j=1}^{l}(-1)^j\binom{-\alpha_l}{j}{}^{\mathscr{D}}\Delta^{\alpha_{l-j}}x_{l-j}. $$

The $\mathscr{E}$-type variable-order derivative and its discrete approximation is given, respectively, by

$$ {}_0^{\mathscr{E}}\mathrm{D}_t^{\alpha(t)}x(t) = \lim_{h\to 0}\Big(\tfrac{x(t)}{h^{\alpha(t)}} - \textstyle\sum_{j=1}^{\eta}(-1)^j\binom{-\alpha(t-jh)}{j} \\ \tfrac{h^{\alpha(t-jh)}}{h^{\alpha(t)}}{}_0^{\mathscr{E}}\mathrm{D}_{t-jh}^{\alpha(t)}x(t)\Big) $$

and

$$ {}^{\mathscr{E}}\Delta^{\alpha_l}x_l = \frac{x_l}{h^{\alpha_l}} - \sum_{j=1}^{l}(-1)^j\binom{-\alpha_{l-j}}{j}\frac{h^{\alpha_{l-j}}}{h^{\alpha_l}}\Delta^{\alpha_{l-j}}x_{l-j}. $$

The main motivation of considering the above definitions of fractional variable order derivatives is the fact that they are widely presented in literature and can be applied in physical systems. In [22], the $\mathscr{A}$-type of fractional variable order derivative was successfully used to design the variable order PD controller in robot arm control. In [23], the heat transfer process in specific grid-holes media whose geometry is changed in time was modeled by a new $\mathscr{D}$-type definition. Moreover, these definitions have mutual duality properties described in [24], which can be adapted to solve the fractional variable order differential equations (see [21]).

2.2. Matrix forms of fractional variable order differences. The matrix form of the fractional constant order difference (1) is given as follows ([25, 26]):

$$ \begin{pmatrix} \Delta^{\alpha}x_0 \\ \Delta^{\alpha}x_1 \\ \Delta^{\alpha}x_2 \\ \vdots \\ \Delta^{\alpha}x_k \end{pmatrix} = W(\alpha,k)\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}, $$

where

$$ \begin{pmatrix} h^{-\alpha} & 0 & 0 & \dots & 0 \\ w_{\alpha,1} & h^{-\alpha} & 0 & \dots & 0 \\ w_{\alpha,2} & w_{\alpha,1} & h^{-\alpha} & \dots & 0 \\ w_{\alpha,3} & w_{\alpha,2} & w_{\alpha,1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{\alpha,k} & w_{\alpha,k-1} & w_{\alpha,k-2} & \dots & h^{-\alpha} \end{pmatrix}, $$

$W(\alpha, k) \in \mathbb{R}^{(k+1)\times(k+1)}$, and

$$ w_{\alpha,l} = \frac{(-1)^l\binom{\alpha}{l}}{h^{\alpha}}, \quad l = 0,\dots,k. \tag{2} $$

Let us define the 4-tuple $\mathscr{T} = (\mathscr{A}, \mathscr{B}, \mathscr{D}, \mathscr{E})$, where $\mathscr{T}_\ell$ is the ℓ-th element of $\mathscr{T}$ and denotes a type of variable order derivative (difference). The matrix numerical forms of the already mentioned variable order differences $\mathscr{A}$, $\mathscr{B}$, $\mathscr{D}$, $\mathscr{E}$ are the following

$$ \begin{pmatrix} {}^{\mathscr{T}_\ell}\Delta^{\alpha_0}x_0 \\ {}^{\mathscr{T}_\ell}\Delta^{\alpha_1}x_1 \\ {}^{\mathscr{T}_\ell}\Delta^{\alpha_2}x_2 \\ \vdots \\ {}^{\mathscr{T}_\ell}\Delta^{\alpha_k}x_k \end{pmatrix} = {}^{\mathscr{T}_\ell}W(\bar{\alpha},k)\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}, \quad \ell = 1,\dots 4, $$

where the matrices ${}^{\mathscr{T}_\ell}W(\bar{\alpha}, k) \in \mathbb{R}^{(k+1)\times(k+1)}$, with $\bar{\alpha} = (\alpha_0, \dots, \alpha_k)$, and $l = 1, \dots 4$, are already defined in [18, 27].

3. Solution of linear control system in state-space form

Recall the 4-tuple $\mathscr{T} = (\mathscr{A}, \mathscr{B}, \mathscr{D}, \mathscr{E})$ and define other quadruple $\tilde{\mathscr{T}} = (\mathscr{D}, \mathscr{E}, \mathscr{A}, \mathscr{B})$, where $\mathscr{T}_\ell$ and $\tilde{\mathscr{T}}_\ell$ denote the ℓ-th elements of $\mathscr{T}$ and $\tilde{\mathscr{T}}$, respectively. We also define two n-tuples $\mathbb{T} = (\mathbb{T}^1, \dots, \mathbb{T}^n)$, where $\mathbb{T}^i \in \mathscr{T}$, and $\tilde{\mathbb{T}} = (\tilde{\mathbb{T}}^1, \dots, \tilde{\mathbb{T}}^n)$, where $\tilde{\mathbb{T}}^i \in \tilde{\mathscr{T}}$, in both cases $i = 1, \dots, n$, and such that if $\mathbb{T}^i \in \mathscr{T}_\ell$ then $\tilde{\mathbb{T}}^i \in \tilde{\mathscr{T}}_\ell$ for some $\ell = \{1, \dots, 4\}$.

3.1. Time-variant control system. Now, consider a time-variant non-commensurate fractional variable order system

$$ {}^{\mathbb{T}}_{0}\mathrm{D}_t^{\alpha(t)}x = A(t)x + B(t)u, \quad x(0) = 0 \tag{3a}$$

$$ y = C(t)x + D(t)u, \tag{3b}$$

where $x = x(t) \in \mathbb{R}^n$, $u = u(t) \in \mathbb{R}^m$, $y = y(t) \in \mathbb{R}^p$, ${}^{\mathbb{T}}\mathrm{D}_t^{\alpha(t)}x = \left({}^{\mathbb{T}^1}_{0}\mathrm{D}_t^{\alpha_1(t)}x_1(t), \ldots, {}^{\mathbb{T}^n}_{0}\mathrm{D}_t^{\alpha_n(t)}x_n(t)\right)^T \in \mathbb{R}^n$, $A(t) = [a_{ij}(t)] \in \mathbb{R}^{n\times n}$, $B(t) = [b_{ir}(t)] \in \mathbb{R}^{n\times m}$, $C(t) = [c_{si}(t)] \in \mathbb{R}^{p\times n}$, $D(t) = [d_{sr}(t)] \in \mathbb{R}^{p\times m}$, for $t \in \mathbb{R}$, $i, j = 1, \ldots, n$, $r = 1, \ldots, m$, $s = 1, \ldots, p$; and $\mathbb{T}^i \in \mathscr{T}$ is a type of variable order derivative definition. We assume variable orders to be piece-wise constant functions, i.e., for $i = 1, \ldots, n$

$$\alpha_i(t) = \alpha_i^{\nu+1} \in \mathbb{R} \quad \text{for } t_\nu \le t < t_{\nu+1},\ \nu = 0, \ldots, N-1$$

where $N \in \mathbb{N}$ denotes the number of time-intervals.

System (3) can be approximated, with the discretization step time $h > 0$, by the following numerical form

$$ {}^{\mathbb{T}}_{0}\Delta_t^{\alpha(l)}x = A(l)x + B(l)u \tag{4a}$$

$$ y(l) = C(l)x + D(l)u, \tag{4b}$$

where ${}^{\mathbb{T}}_{0}\Delta_t^{\alpha(l)}x = \left({}^{\mathbb{T}^1}_{0}\Delta_t^{\alpha_1(l)}x_1(l), \ldots, {}^{\mathbb{T}^n}_{0}\Delta_t^{\alpha_n(l)}x_n(l)\right)^T \in \mathbb{R}^n$, and ${}^{\mathbb{T}^i}_{0}\Delta_t^{\alpha_i(l)}x_i(l)$ is a $\mathbb{T}^i$-type difference, and $l = 0, \ldots, k$. The ij-th entry of $A(l)$ is $a_{ij}^l = a_{ij}(lh) \in \mathbb{R}$, the ir-th entry of $B(l)$ is $b_{ir}^l = b_{ir}(lh) \in \mathbb{R}$, the si-th entry of $C(l)$ is $c_{si}^l = c_{si}(lh) \in \mathbb{R}$, and the sr-th entry of $D(l)$ is $d_{sr}^l = d_{sr}(lh) \in \mathbb{R}$.

In turn, system (4) can be rewritten in the equivalent numerical matrix form

$$ {}^{\mathbb{T}}\mathbb{W}(\alpha)\hat{x} = \mathbb{A}\hat{x} + \mathbb{B}\hat{u} \tag{5a}$$

$$ \hat{y} = \mathbb{C}\hat{x} + \mathbb{D}\hat{u}, \tag{5b}$$

where ${}^{\mathbb{T}}\mathbb{W}(\alpha) \in \mathbb{R}^{n(k+1)\times n(k+1)}$ is

$$ {}^{\mathbb{T}}\mathbb{W}(\alpha) = \operatorname{block\,diag}\left({}^{\mathbb{T}^1}W(\bar{\alpha}_1, k), \ldots, {}^{\mathbb{T}^n}W(\bar{\alpha}_n, k)\right),$$

and

$$ \bar{\alpha}_i = (\alpha_i(0), \ldots, \alpha_i(kh)), \quad i = 1, \ldots, n, \tag{6}$$

with $k \ge N$; and $\hat{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in \mathbb{R}^{n(k+1)\times 1}$, $\bar{x}_i = (x_i(0), \ldots, x_i(kh)) \in \mathbb{R}^{1\times(k+1)}$, $\hat{u} = (\bar{u}_1, \ldots, \bar{u}_m)^T \in \mathbb{R}^{m(k+1)\times 1}$, $\bar{u}_r = (u_r(0), \ldots, u_r(kh)) \in \mathbb{R}^{1\times(k+1)}$, and

$$\mathbb{A} = \begin{pmatrix} \hat{a}_{11} & \cdots & \hat{a}_{1n} \\ \vdots & \ddots & \vdots \\ \hat{a}_{n1} & \cdots & \hat{a}_{nn} \end{pmatrix} \in \mathbb{R}^{n(k+1)\times n(k+1)},$$

$$\hat{a}_{ij} = \mathbf{diag}\left(a_{ij}^0, \ldots, a_{ij}^k\right) \in \mathbb{R}^{(k+1)\times(k+1)};$$

$$\mathbb{B} = \begin{pmatrix} \hat{b}_{11} & \cdots & \hat{b}_{1m} \\ \vdots & \ddots & \vdots \\ \hat{b}_{n1} & \cdots & \hat{b}_{nm} \end{pmatrix} \in \mathbb{R}^{n(k+1)\times m(k+1)},$$

$$\hat{b}_{ir} = \mathbf{diag}(b_{ir}^0, \ldots, b_{ir}^k) \in \mathbb{R}^{(k+1)\times(k+1)};$$

$$\mathbb{C} = \begin{pmatrix} \hat{c}_{11} & \cdots & \hat{c}_{1n} \\ \vdots & \ddots & \vdots \\ \hat{c}_{p1} & \cdots & \hat{c}_{pn} \end{pmatrix} \in \mathbb{R}^{p(k+1)\times n(k+1)},$$

$$\hat{c}_{si} = \mathbf{diag}(c_{si}^0, \ldots, c_{si}^k) \in \mathbb{R}^{(k+1)\times(k+1)};$$

$$\mathbb{D} = \begin{pmatrix} \hat{d}_{11} & \cdots & \hat{d}_{1m} \\ \vdots & \ddots & \vdots \\ \hat{d}_{p1} & \cdots & \hat{d}_{pm} \end{pmatrix} \in \mathbb{R}^{p(k+1)\times m(k+1)},$$

$$\hat{d}_{sr} = \mathbf{diag}(d_{sr}^0, \ldots, d_{sr}^k) \in \mathbb{R}^{(k+1)\times(k+1)}.$$

Theorem 1. Solution of (5a), and thereby approximated solution of (3a), is

$$\hat{x} = \left(I_{n(k+1)} - {}^{\tilde{\mathbb{T}}}\mathbb{W}(-\alpha)\mathbb{A}\right)^{-1} {}^{\tilde{\mathbb{T}}}\mathbb{W}(-\alpha)\mathbb{B}\hat{u}, \tag{7}$$

where ${}^{\tilde{\mathbb{T}}}\mathbb{W}(-\alpha) \in \mathbb{R}^{n(k+1)\times n(k+1)}$ is

$$\begin{aligned} {}^{\tilde{\mathbb{T}}}\mathbb{W}(-\alpha) = \operatorname{block\,diag}\Big({}^{\tilde{\mathbb{T}}^1}W(-\bar{\alpha}_1, k), \ldots, \\ {}^{\tilde{\mathbb{T}}^n}W(-\bar{\alpha}_n, k)\Big), \end{aligned} \tag{8}$$

and $I_{n(k+1)} \in \mathbb{R}^{n(k+1)\times n(k+1)}$ stands for identity matrix. The vector of approximated values of ith state variable $x_i(t)$ is

$$\bar{x}_i = \left(\hat{x}_{(i-1)k+i}, \ldots, \hat{x}_{ik+i}\right)^T \in \mathbb{R}^{(k+1)\times 1}.$$

Proof. Multiplying from left both sides of (5a) by ${}^{\tilde{\mathbb{T}}}\mathbb{W}(-\alpha)$, and using the duality property, i.e., ${}^{\tilde{\mathbb{T}}}\mathbb{W}(-\alpha){}^{\mathbb{T}}\mathbb{W}(\alpha) = I_{n(k+1)}$, because ${}^{\tilde{\mathscr{T}}_\ell}\mathbb{W}(-\bar{\alpha}_i, k){}^{\mathscr{T}_\ell}\mathbb{W}(\bar{\alpha}_i, k) = I_{k+1}$ for $i = 1, \ldots, n$, after simple manipulations we obtain (7). □

Corollary 1. Having already calculated vector $\hat{x}$ given by (7), the output $\hat{y}$ of system (5) is given by (5b). The vector of approximated values of sth output $y_s(t)$ is

$$\bar{y}_s = \left(\hat{y}_{s(k+1)-k}, \ldots, \hat{y}_{s(k+1)}\right)^T \in \mathbb{R}^{(k+1)\times 1}.$$

Remark 1. Solution (7) admits singularities if matrix $I_{n(k+1)} - {}^{\tilde{\mathbb{T}}}\mathbb{W}(-\alpha)\mathbb{A}$ is singular. In a case of scalar system, i.e., $n = 1$, a condition under which a solution exists is given in [19].

Example 1. Let us consider system (3) with two state variables, i.e, $n = 2$, single input $u(t) = H(t)$, where $H(t)$ denotes a Heaviside step function, i.e., $m = 1$, two outputs, i.e., $p = 2$, and the matrices

$$A(t) = \begin{pmatrix} 0 & \lambda_1(t) \\ -\lambda_2(t) & -2\lambda_2(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} \lambda_1(t) \\ 10\lambda_2(t) \end{pmatrix},$$

$$C(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad D(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where

$$\lambda_1(t) = \begin{cases} \lambda_1^1 & \text{for } t \in [0,1) \\ \lambda_1^2 & \text{for } t \in [1,3] \end{cases}, \quad \lambda_2(t) = \begin{cases} \lambda_2^1 & \text{for } t \in [0,1) \\ \lambda_2^2 & \text{for } t \in [1,3]. \end{cases}$$

The types of variable order derivatives are given by the 2-tuple $\mathbb{T} = (\mathscr{D}, \mathscr{B})$, and then ${}_0^{\mathbb{T}}\mathrm{D}_t^{\alpha(t)}x = \left({}_0^{\mathscr{D}}\mathrm{D}_t^{\alpha_1(t)}x_1(t), {}_0^{\mathscr{B}}\mathrm{D}_t^{\alpha_2(t)}x_2(t)\right)^T$. The piece-wise constant variable orders $\alpha_i(t)$, $i = 1, 2$, are defined on two time-intervals ($N = 2$), that is

$$\alpha_1(t) = \begin{cases} \alpha_1^1 & \text{for } t \in [0,1) \\ \alpha_1^2 & \text{for } t \in [1,3] \end{cases}, \quad \alpha_2(t) = \begin{cases} \alpha_2^1 & \text{for } t \in [0,1) \\ \alpha_2^2 & \text{for } t \in [1,3] \end{cases}.$$

The solution of system (3) will be calculated from (7) and (5b), on time horizon $T = 3$ with sample time $h = 1$, and then $k = T/h = 3$. By (6), this yields

$$\bar{\alpha}_1 = (\alpha_1^1, \alpha_1^2, \alpha_1^2, \alpha_1^2) \quad \text{and} \quad \bar{\alpha}_2 = (\alpha_2^1, \alpha_2^2, \alpha_2^2, \alpha_2^2).$$

The matrices of equivalent matrix numerical form according to (5) are the following

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \lambda_1^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \\ -\lambda_2^1 & 0 & 0 & 0 & -2\lambda_2^1 & 0 & 0 & 0 \\ 0 & -\lambda_2^2 & 0 & 0 & 0 & -2\lambda_2^2 & 0 & 0 \\ 0 & 0 & -\lambda_2^2 & 0 & 0 & 0 & -2\lambda_2^2 & 0 \\ 0 & 0 & 0 & -\lambda_2^2 & 0 & 0 & 0 & -2\lambda_2^2 \end{pmatrix} \in \mathbb{R}^{8\times 8},$$

$$\mathbb{B} = \begin{pmatrix} \lambda_1^1 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & \lambda_1^2 \\ 10\lambda_2^1 & 0 & 0 & 0 \\ 0 & 10\lambda_2^2 & 0 & 0 \\ 0 & 0 & 10\lambda_2^2 & 0 \\ 0 & 0 & 0 & 10\lambda_2^2 \end{pmatrix} \in \mathbb{R}^{8\times 4},$$

$$\mathbb{C} = I_8 \in \mathbb{R}^{8\times 8}, \quad \mathbb{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{8\times 4}.$$

The difference matrices of (5) and (7), for $\alpha_1^1 = 1$, $\alpha_1^2 = 0.25$ and $\alpha_2^1 = 0.5$, $\alpha_2^2 = 1$, are the following

$${}^{\mathbb{T}}\mathbb{W}(\alpha) = \operatorname{block\,diag}\left({}^{\mathscr{D}}W(\bar{\alpha}_1, 3), {}^{\mathscr{B}}W(\bar{\alpha}_2, 3)\right) \in \mathbb{R}^{8\times 8},$$

where

$${}^{\mathscr{D}}W(\bar{\alpha}_1, 3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0.0938 & -0.25 & 1 & 0 \\ 0.0156 & -0.0938 & -0.25 & 1 \end{pmatrix},$$

$${}^{\mathscr{B}}W(\bar{\alpha}_2, 3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ -0.125 & -0.5 & 1 & 0 \\ -0.0625 & -0.125 & -1 & 1 \end{pmatrix}$$

and

$${}^{\tilde{\mathbb{T}}}\mathbb{W}(-\alpha) = \operatorname{block\,diag}\left({}^{\mathscr{A}}W(-\bar{\alpha}_1, 3), {}^{\mathscr{E}}W(-\bar{\alpha}_2, 3)\right) \in \mathbb{R}^{8\times 8},$$

where

$${}^{\mathscr{A}}W(-\bar{\alpha}_1, 3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0.1563 & 0.25 & 1. & 0 \\ 0.1172 & 0.1563 & 0.25 & 1 \end{pmatrix},$$

$${}^{\mathscr{E}}W(-\bar{\alpha}_2, 3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0.375 & 0.5 & 1 & 0 \\ 0.5 & 0.625 & 1 & 1 \end{pmatrix}.$$

Obviously,

$${}^{\mathscr{A}}W(-\bar{\alpha}_2, 3)\,{}^{\mathscr{D}}W(\bar{\alpha}_2, 3) = I_4,$$
$${}^{\mathscr{E}}W(-\bar{\alpha}_2, 3)\,{}^{\mathscr{B}}W(\bar{\alpha}_2, 3) = I_4$$

and then

$${}^{\tilde{\mathbb{T}}}\mathbb{W}(-\alpha)\,{}^{\mathbb{T}}\mathbb{W}(\alpha) = I_8.$$

For example, for $\lambda_1^1 = -1$, $\lambda_1^2 = 2$ and $\lambda_2^1 = 3$, $\lambda_2^2 = 0.5$, we get

$$\hat{x} = \begin{pmatrix} -9.2500 \\ -26.4688 \\ 6.9271 \\ 9.5176 \\ 8.2500 \\ 16.2188 \\ 5.3385 \\ 4.0614 \end{pmatrix}, \quad y_1 = \begin{pmatrix} -8.2500 \\ -25.4688 \\ 7.9271 \\ 10.5176 \end{pmatrix}, \quad y_2 = \begin{pmatrix} -8.2500 \\ 16.2188 \\ 5.3385 \\ 4.0614 \end{pmatrix}.$$

3.2. Time-invariant control system. In the case of time-invariant non-commensurate fractional variable order system

$$ {}_0^{\mathbb{T}}\mathrm{D}_t^{\alpha(t)} x = Ax + Bu, \quad x(0) = 0 \tag{9a}$$

$$ y = Cx + Du, \tag{9b}$$

which is a particular case of (3), i.e., for constant system matrices, we have the following result, which is a consequence of (7).

Corollary 2. Approximated numerical solution of (9) is

$$\hat{x} = \left(I_{n(k+1)} - {}^{\tilde{\mathbb{T}}}\mathbb{W}(-\alpha)(A \otimes I_{k+1}) \right)^{-1} \cdot {}^{\tilde{\mathbb{T}}}\mathbb{W}(-\alpha)(B \otimes I_{k+1})\hat{u}, \tag{10a}$$

$$\hat{y} = (C \otimes I_{k+1})\hat{x} + (D \otimes I_{k+1})\hat{u}, \tag{10b}$$

where $\otimes$ denotes Kronecker product of matrices, and ${}^{\tilde{\mathbb{T}}}\mathbb{W}(-\alpha)$ is given by (8). Obviously, solution of (9) can be calculated alternativelly from (7), but technically it seems simpler to use (10).

4. Analog model of fractional variable order state-space system

An analog model of fractional variable order state-space system has been realized directly based on $\mathscr{D}$-type fractional variable order integrator presented in Fig. 1. This multi-switching model of $\mathscr{D}$-type definition was meticulously investigated in [28]. So in this section only the major features related to this structure will be recalled and then the main experimental setup concerning fractional variable order state-space system will be investigated.

4.1. Analog model of fractional variable order integrator. The experimental setup corresponding to multi-switching form of $\mathscr{D}$-type fractional variable order integrator, shown in Fig. 1, contains:

- operational amplifiers TL071 denoted as: A_1, A_2 and A_3;
- electronic switches DG303 marked as: S_1, S_2 and S_3;

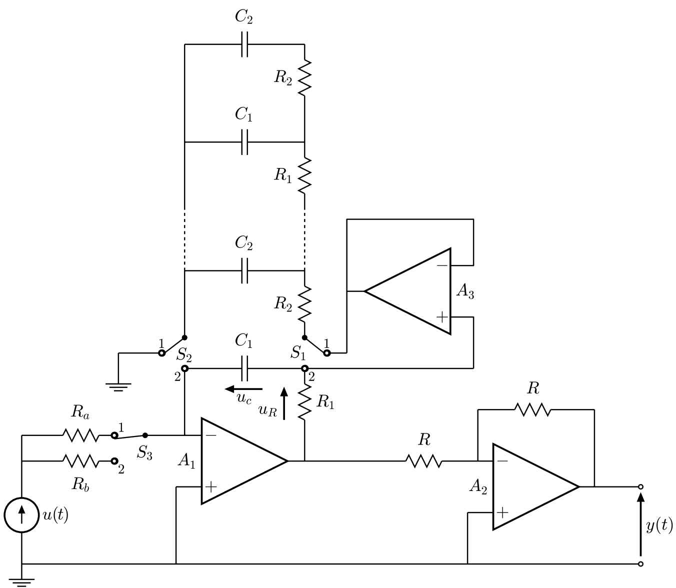

Fig. 1. Multi-switching analog realization of the $\mathscr{D}$-type fractional variable-order integral

- domino ladder structure containing branches with passive elements such as: resistors $R_1 = 2.4$ kΩ, $R_2 = 8.2$ kΩ and capacitors $C_1 = 330$ nF, $C_2 = 220$ nF;
- inverting amplifier A_3 in configuration with resistor $R = 100$ kΩ.

Depending on position of S_1 and S_2 switches it is possible to distinguish a half-order ($\alpha = 0.5$) or first-order ($\alpha = 1$) integrator. When S_1 and S_2 switches are connected to terminal 2, then the analog model represents an approximation of half-order integrator and its transfer function is expressed by

$$G(s) = \frac{\lambda_1}{s^{0.5}},$$

where λ_1 is a time-constant determined by branches with resistances R_1, R_2, R_b and capacities C_1, C_2 used in half-order impedance.

The half-order impedance containing branches with resistors R_1, R_2 and capacitors C_1, C_2 was build according to algorithm described in [29].

Otherwise, when switches S_1 and S_2 are connected to terminal 1, then the analog model based on R_1 and C_1 elements, represents the first-order integrator and its transfer function is the following

$$G(s) = \frac{\lambda_2}{s},$$

where λ_2 is a time-constant determined by resistors R_1, R_a and capacitor C_1.

Due to the different time-constants of half- and first-order integrators, the R_a and R_b resistors were used to keep the same values of system parameters. At the end, the inversion of output signal is done by operational amplifier A_2 in configuration with resistors denote as R. So, the output signal of operational amplifier A_1 is multiplied by gain equals to –1 and finally gives the output signal.

The output-signal of fractional variable order integrator shown in Fig. 1 is described by

$$y(t) = \lambda \cdot {}_0^{\mathscr{D}}\mathrm{D}_t^{-\alpha(t)} u(t), \tag{11}$$

where $\alpha(t)$ changes its value between 0.5 and 1, at time switch t_{switch}, λ is a time-constant.

4.2. Analog model of fractional variable order state-space system. Having introduced the multi-switching model of fractional order $\mathscr{D}$-type integrator, the realization of fractional variable order state-space can be done. The block diagram of fractional variable order state-space system for types $\mathbb{T} = (\mathscr{A}, \mathscr{A})$, corresponding to this one used in experiment, is shown in Fig. 2. The equivalent experimental model of such state-space system is depicted in Fig. 3.

The main part of experimental setup contains:

- data acquisition card DS1104;
- two connected in series, fractional variable order integrators shown in Fig. 1 with different time-constants λ_1 and λ_2;
- summing amplifier.

The fractional variable order integrators, which appear in state-space system (see Fig. 2 and 3), are expressed by $\mathscr{D}$-type operator. Orders $\alpha_1(t)$ and $\alpha_2(t)$ take one of two values: 0.5 or 1.

The half- and first-order impedances have they own time-constants values. So, the resistors R_a and R_b presented in Fig. 1 were adjusted to get an appropriate constant value of system parameter for each fractional variable order integrator. To set the system parameter λ_1 to 3.5, the resistors are equal to $R_a = 820$ kΩ and $R_b = 33$ kΩ, for first fractional variable order integrator. To achieve the $\lambda_2 = 2.9$ for second fractional variable order integrator, these resistors are equal to $R_a = 910$ kΩ and $R_b = 43$ kΩ. Additionally, summing amplifier in experiment inverts the input signal. There is no inverted amplifier in this configuration due to the offset voltage of operational amplifier which reduces the accuracy of whole fractional variable order state-space system.

The experimental data from such state-space system were collected with sampling time equal 0.005 sec. and input signal was $u(t) = 0.5$ V. The sampling-time is an important issue related to numerical solutions of fractional variable order systems. An uncarefully chosen value of sampling-time can cause either inaccuracy of solution or increase the complexity of computations. For example, too small value of sampling-time can improve the accuracy of solution, but it generates larger size of matrices in equations to be solved. On the other hand, not enough value of sampling-time can decrease the accuracy of solution, but can be easily solved numerically. However, from practical point of view, since analog realization is a continuous-time dynamical system, the value of sampling-time used in experimental setup concerns only data acquisition.

5. Experimental and numerical results

In this section, the experimental and numerical results for fractional variable order state-space systems are presented. Moreover, the experimental data are compared to the numerical implementations of fractional variable order state-space systems.

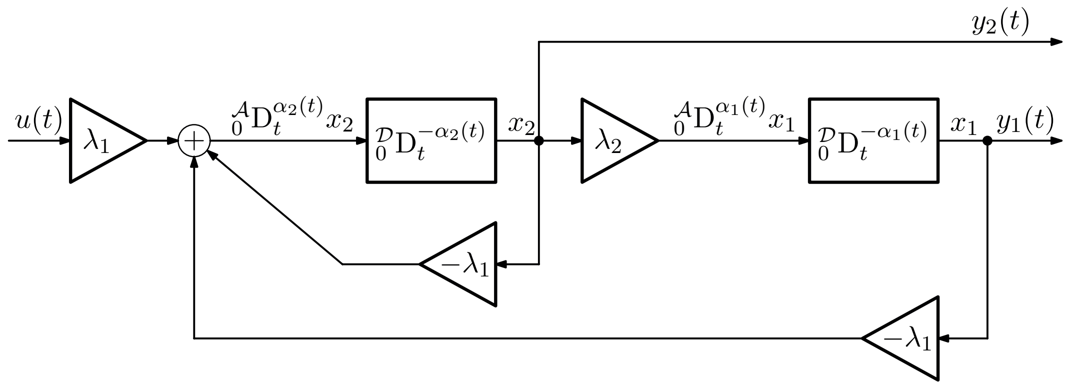

Fig. 2. Block diagram of fractional variable order state-space system for types $\mathbb{T} = (\mathscr{A}, \mathscr{A})$ equivalent to scheme in Fig. 3. Time-constants: $\lambda_1 = 3.5$ and $\lambda_2 = 2.9$

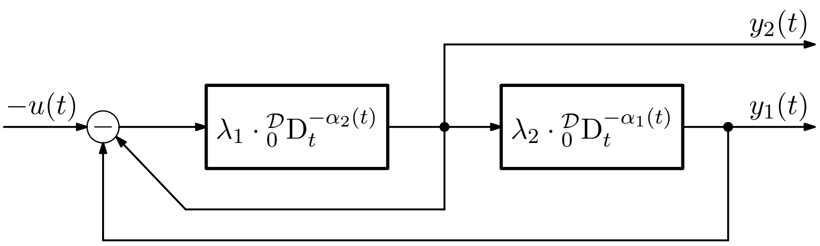

Fig. 3. Analog model of fractional variable order state-space system based on $\mathscr{D}$-type fractional variable order integrator (see Fig. 1)

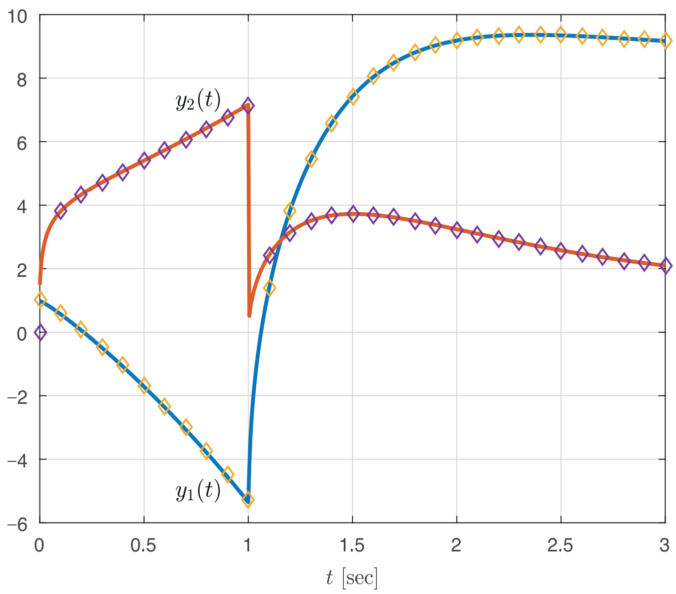

Fig. 4. Numerical solution (solid lines) of system from Ex. 2, compared with simulation results (diamond lines), for $\mathbb{T}=(\mathscr{D},\mathscr{B})$, $\alpha_1(t)\neq\alpha_2(t)$, i.e., $\alpha_1^1=1$, $\alpha_1^2=0.25$, and $\alpha_2^1=0.5$, $\alpha_2^2=1$; and $\lambda_1^1=-1$, $\lambda_1^2=2$, and $\lambda_2^1=3$, $\lambda_2^2=0.5$; for $h=0.005$

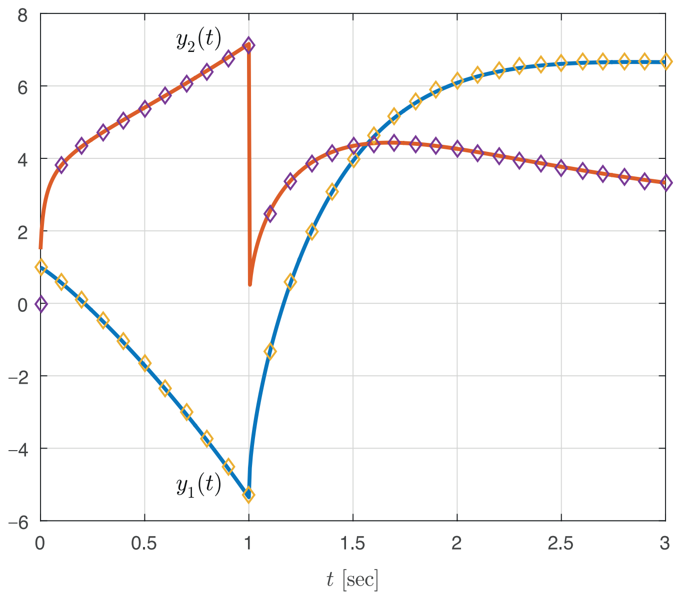

Fig. 5. Numerical solution (solid lines) of system from Ex. 2, compared with simulation results (diamond lines), for $\mathbb{T}=(\mathscr{E},\mathscr{B})$, $\alpha_1(t)\neq\alpha_2(t)$, i.e., $\alpha_1^1=1$, $\alpha_1^2=0.25$, and $\alpha_2^1=0.5$, $\alpha_2^2=1$; and $\lambda_1^1=-1$, $\lambda_1^2=2$, and $\lambda_2^1=3$, $\lambda_2^2=0.5$; for $h=0.005$

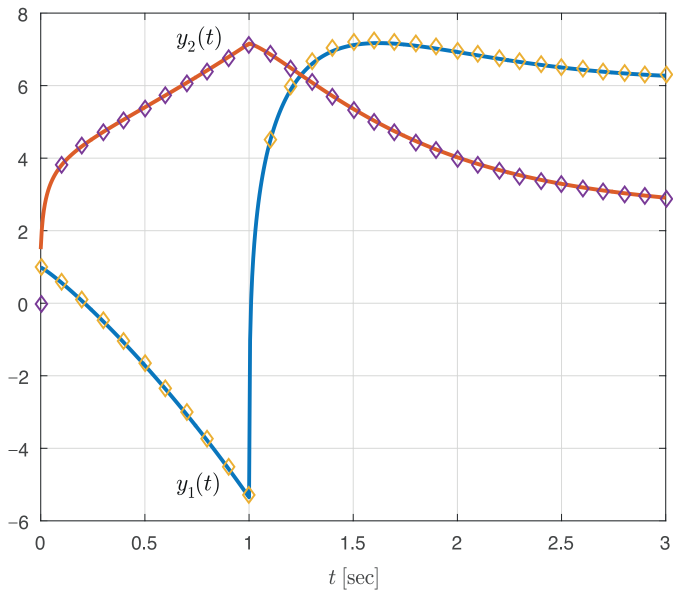

Fig. 6. Numerical solution (solid lines) of system from Ex. 2, compared with simulation results (diamond lines), for $\mathbb{T}=(\mathscr{E},\mathscr{A})$, $\alpha_1(t)\neq\alpha_2(t)$, i.e., $\alpha_1^1=1$, $\alpha_1^2=0.25$, and $\alpha_2^1=0.5$, $\alpha_2^2=1$; and $\lambda_1^1=-1$, $\lambda_1^2=2$, and $\lambda_2^1=3$, $\lambda_2^2=0.5$; for $h=0.005$

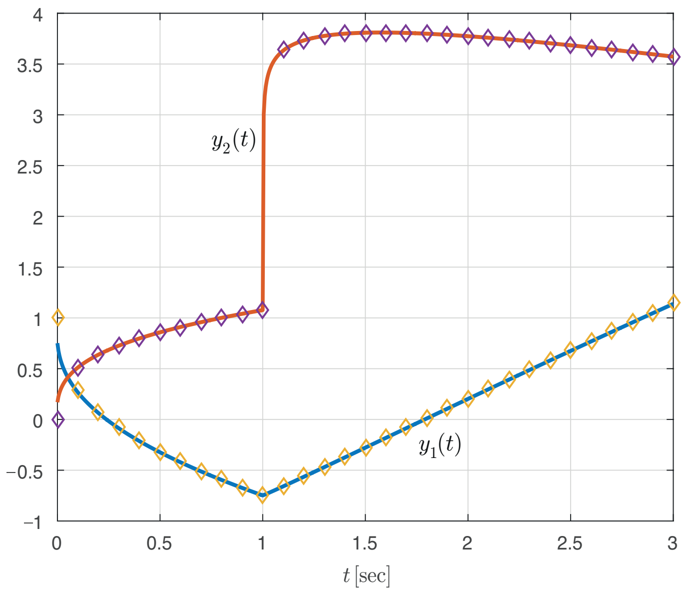

Fig. 7. Numerical solution (solid lines) of system from Ex. 2, compared with simulation results (diamond lines), for $\mathbb{T}=(\mathscr{A},\mathscr{D})$, $\alpha_1(t)\neq\alpha_2(t)$, i.e., $\alpha_1^1=0.25$, $\alpha_1^2=1$, and $\alpha_2^1=1/3$, $\alpha_2^2=1/6$; and $\lambda_1^1=-0.8$, $\lambda_1^2=0.2$, and $\lambda_2^1=0.1$, $\lambda_2^2=1$; for $h=0.005$

5.1. Numerical and simulation results. Numerical solution compared to simulation results of time-variant system described by (3) for various types $\mathbb{T}$ of variable order derivatives were presented in Ex. 2. Figs. 4–7 show the plots when both orders $\big(\alpha_1(t),\alpha_2(t)\big)$ and system parameters $\big(\lambda_1(t),\lambda_2(t)\big)$ changed their values in time.

Example 2. Consider control system from Ex. 1. Below, we present several solution plots for different variable orders and parameters scenarios, obtained both from calculations and simulations performed in Simulink.

Obviously, the plots on time-interval from 0 to 1 in Figs. 4–6 are identical. It is caused by the behavior of fractional variable order derivative, which for constant value of order is the same as the behavior of fractional constant order derivative. So, despite different types of variable order derivatives used in simulations, the results are the same until 1 sec.

5.2. Experimental results. Experimentally obtained data of fractional variable order state-space system for types $\mathbb{T} = (\mathscr{A}, \mathscr{A})$ compared to the numerical results were introduced in Ex. 3. In Fig. 8 the case is investigated, when $\alpha_1(t)$-order integrator depicted in Fig. 3 starts with first-order impedance and $\alpha_2(t)$-order integrator begins with half-order (0.5) impedance. Then, at time switch $t_{\text{switch}} = 0.5$ sec. both integrators change their orders. Thus, $\alpha_1(t)$ changes to half-order impedance and $\alpha_2(t)$ changes to first-order impedance. The orders are switched each 0.5 sec.

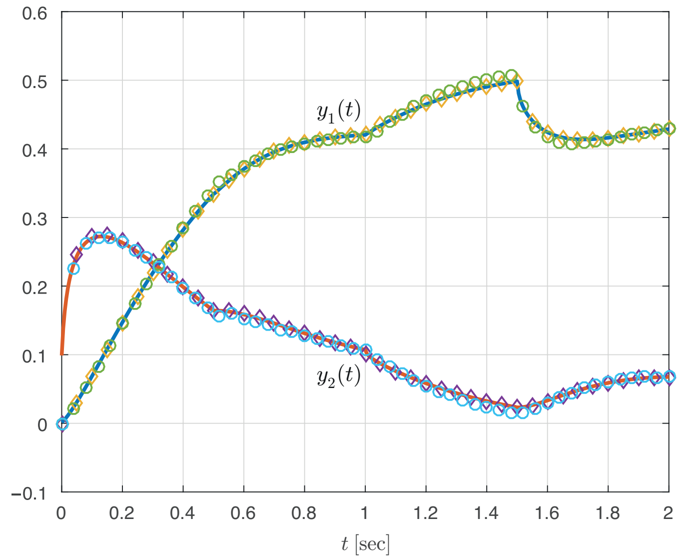

Fig. 8. Comparison of numerical solution (continuous lines), simulation (diamond lines) and experimental results (circle lines) of system (9) for $t_{\text{switch}} = 0.5$, $\alpha_1^1 = 1$, $\alpha_1^2 = 0.5$ and $\alpha_2^1 = 0.5$, $\alpha_2^2 = 1$

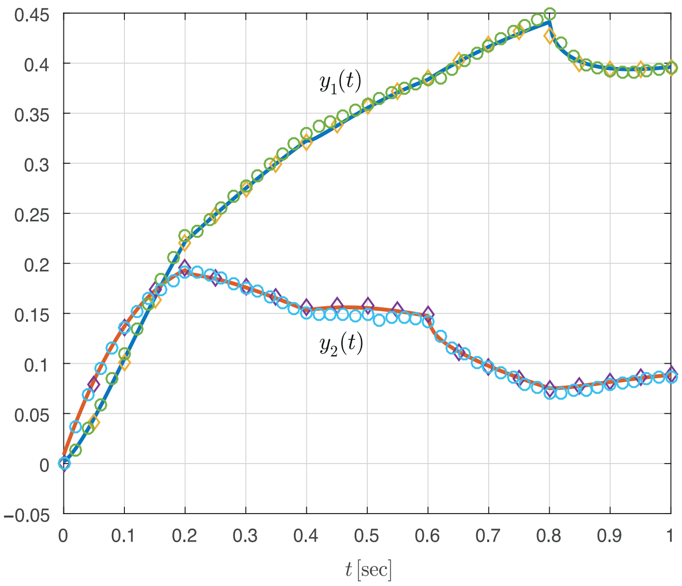

Fig. 9. Comparison of numerical solution (continuous lines), simulation (diamond lines) and experimental results (circle lines) of system (9) for $t_{\text{switch}} = 0.2$, $\alpha_1^1 = 0.5$, $\alpha_1^2 = 1$ and $\alpha_2^1 = 1$, $\alpha_2^2 = 0.5$

Another case, when $\alpha_1(t)$- and $\alpha_2(t)$-integrators start (respectively) with half- and first-order impedance were shown in Fig. 9. In this situation orders are changed each 0.2 sec. The experimental results, when both fractional variable order integrators begin with the same value of order which equals to 0.5 is considered in Fig. 10. After first switching time $t_{\text{switch}} = 0.2$ sec, the orders were changed to 1 for both integrators.

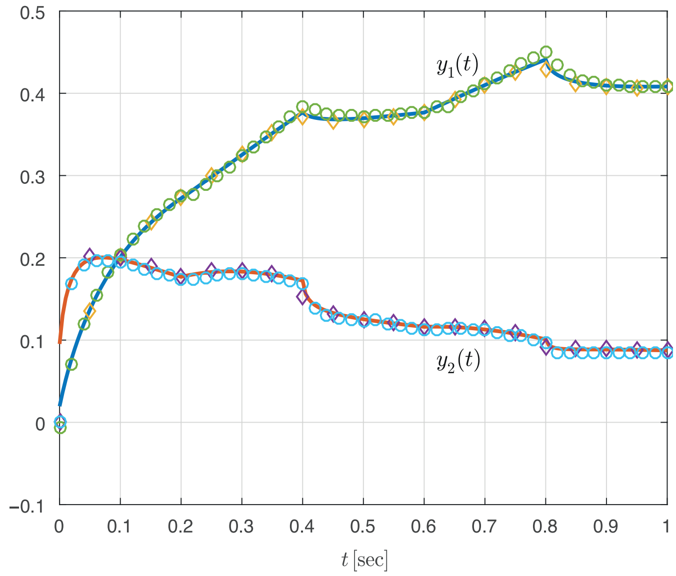

Fig. 10. Comparison of numerical solution (continuous lines), simulation (diamond lines) and experimental results (circle lines) of system (9) for $t_{\text{switch}} = 0.2$, $\alpha_1^1 = \alpha_2^1 = 0.5$, $\alpha_1^2 = \alpha_2^2 = 1$

Example 3. Consider time-invariant system (9) with variable order derivatives of types $\mathbb{T} = (\mathscr{A}, \mathscr{A})$, input signal $u = 0.5\,H(t)$, and matrices

$$A = \begin{pmatrix} 0 & 2.9 \\ -3.5 & -3.5 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 3.5 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with variable orders values defined on time intervals in the following way

$$\alpha_i(t) = \begin{cases} {}^1\alpha_i & \text{for } t \in [jt_{\text{switch}}, (j+1)t_{\text{switch}}) \\ {}^2\alpha_i & \text{for } t \in [(j+1)t_{\text{switch}}, (j+2)t_{\text{switch}}) \end{cases},$$

Calculations, simulations and experimental data were performed for step time $h = 0.005$.

The solutions of the system calculated for different scenarios of variable order values and time switches t_{switch}, are compared to simulation (in Simulink) and experimental results, and are depicted in Figs. 8–10.

6. Conclusions

In the paper, the matrix approach for numerical solution of fractional variable order linear control system in state-space form is presented. The considerations are devoted to time-variant and time-invariant control system in state-space form. Moreover, the analog model for $\mathbb{T} = (\mathscr{A}, \mathscr{A})$-types of fractional variable order state-space system based on multi-switching $\mathscr{D}$-type integral model is proposed and realized. It is possible by applying duality property between particular types of fractional variable order operators. At the end, the experimental data is compared to numerical results show high accuracy of presented approximation approach.

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References

[1] I. Podlubny, *Fractional Differential Equations*. Academic Press, 1999.

[2] S. Samko, A. Kilbas, and O. Maritchev, *Fractional Integrals and Derivative. Theory and Applications*. Gordon & Breach Sci. Publishers, 1987.

[3] T. Kaczorek, *Selected Problems of Fractional Systems Theory*. Heidelberg, Springer, 2011.

[4] H. El Brouji, J.-M. Vinassa, O. Briat, N. Bertrand, and E.Woirgard, "Ultracapacitors self discharge modelling using a physical description of porous electrode impedance", *Vehicle Power and Propulsion Conference, IEEE*, 1–6 (2008).

[5] R. Martin, J.J. Quintana, A. Ramos, and I. de la Nuez, "Modeling electrochemical double layer capacitor, from classical to fractional impedance", *The 14th IEEE Mediterranean Electrotechnical Conference*, 61–66 (2008).

[6] H. Sheng, H. Sun, C. Coopmans, Y. Chen, and G.W. Bohannan, "Physical experimental study of variable-order fractional integrator and differentiator", in *Proceedings of The 4th IFACWorkshop Fractional Differentiation and its Applications* (2010).

[7] L. Ramirez and C. Coimbra, "On the variable order dynamics of the nonlinear wake caused by a sedimenting particle", *Physica D-Nonlinear Phenomena* 240 (13), 1111–1118 (2011).

[8] C.-C. Tseng, "Design and application of variable fractional order differentiator", *Proceedings of The 2004 IEEE Asia- Pacific Conference on Circuits and Systems* 1, 405–408 (2004).

[9] C.-C. Tseng and S.-L. Lee, "Design of variable fractional order differentiator using infinite product expansion", *Proceedings of 20th European Conference on Circuit Theory and Design,* 17–20 (2011).

[10] H. Sheng, H. Sun, Y. Chen, and T. Qiu, "Synthesis of multifractional gaussian noises based on variable-order fractional operators", *Signal Processing* 91 (7), 1645–1650 (2011).

[11] D. Sierociuk, I. Podlubny, and I. Petras, "Experimental evidence of variable-order behavior of ladders and nested ladders", *IEEE Transactions on Control Systems Technology* 21 (2), 459–466 (2013).

[12] P. Ostalczyk and T. Rybicki, "Variable-fractional-order deadbeat control of an electromagnetic servo", *Journal of Vibration and Control* 14 (9–10), 1457–1471 (2008).

[13] P. Ostalczyk, "Variable-, fractional-order discrete PID controllers", *Proceedings of the IEEE/IFAC 17th International Conference on Methods and Models in Automation and Robotics* Miedzyzdroje, 534–539 (2012).

[14] P. Ostalczyk and P. Duch, "Closed-loop system synthesis with the variable-, fractional- order PID controller", *Proceedings of the IEEE/IFAC 17th International Conference on Methods and Models in Automation and Robotics,* Miedzyzdroje, 589–594 (2012).

[15] C. Lorenzo and T. Hartley, "Variable order and distributed order fractional operators", *Nonlinear Dynamics* 29 (1–4), 57–98 (2002).

[16] D. Valerio and J.S. da Costa, "Variable-order fractional derivatives and their numerical approximations", *Signal Processing* 91 (3), 470–483 (2011).

[17] D. Sierociuk, W. Malesza, and M. Macias, "On a new definition of fractional variable-order derivative", *Proceedings of the 14th International Carpathian Control Conference*, 340–345 (2013).

[18] M. Macias and D. Sierociuk, "An alternative recursive fractional variable-order derivative definition and its analog validation", *Proceedings of International Conference on Fractional Differentiation and its Applications*, Catania (2014).

[19] W. Malesza, M. Macias, and D. Sierociuk, "Matrix approach and analog modeling for solving fractional variable order differential equations", *Advances in Modelling and Control of Non-integer-Order Systems*, Lecture Notes in Electrical Engineering, eds. K.J. Latawiec, M. Lukaniszyn, and R. Stanislawski, Springer International Publishing, 71–80, 2015.

[20] D. Sierociuk, W. Malesza, and M. Macias, "Practical analog realization of multiple order switching for recursive fractional variable order derivative", in *20th International Conference on Methods and Models in Automation and Robotics*, 573–578 (2015).

[21] W. Malesza, D. Sierociuk, and M. Macias, "Solution of fractional variable order differential equation", *Proceedings of the American Control Conference IEEE*, 1537–1542 (2015).

[22] P. Ostalczyk, D. Brzezinski, P. Duch, M. Łaski, and D. Sankowski, "The variable, fractional-order discrete-time pd controller in the iisv1.3 robot arm control", *Central European Journal of Physics* 11 (6), 750–759, (2013).

[23] P. Sakrajda and D. Sierociuk, *Modeling Heat Transfer Process in Grid-Holes Structure Changed in Time Using Fractional Variable Order Calculus*. Springer International Publishing, 297–306 (2017).

[24] D. Sierociuk and M. Twardy, "Duality of variable fractional order difference operators and its application to identification", *Bull. Pol. Ac.: Tech.* 62 (4), 809–815 (2014).

[25] I. Podlubny, "Matrix approach to discrete fractional calculus", *Fractional Calculus and Applied Analysis* 3, 359–386 (2000).
[26] I. Podlubny, A. Chechkin, T. Skovranek, Y. Chen, and B.M. Vinagre Jara, "Matrix approach to discrete fractional calculus. II: Partial fractional differential equations", *Journal of Computational Physics* 228 (8), 3137–3153 (2009).
[27] D. Sierociuk, W. Malesza, and M. Macias, "On the recursive fractional variable-order derivative: Equivalent switching strategy, duality, and analog modeling", *Circuits, Systems, and Signal Processing* 34 (4), 1077–1113 (2015).
[28] D. Sierociuk, W. Malesza, and M. Macias, "Practical analog realization of multiple order switching for recursive fractional variable order derivative", *20th International Conference on Methods and Models in Automation and Robotics*, 573–578 (2015).
[29] D. Sierociuk and A. Dzielinski, "New method of fractional order integrator analog modeling for orders 0.5 and 0.25", *Proc. of the 16th International Conference on Methods and Models in Automation and Robotics,* Miedzyzdroje, 137–141 (2011).