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Problem of partitioned bases in monitoring vertical displacements for elongated structures

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Abstract: In monitoring vertical displacements in elongated structures (e.g. bridges, dams) by means of precise geometric levelling a reference base usually consists of two subgroups located on both ends of a monitored structure. The bigger the separation of the subgroups, the greater is the magnitude of undetectable displacement of one subgroup with respect to the other. With a focus on a method of observation differences the question arises which of the two basic types of computation datum, i.e. the elastic and the fixed, both applicable in this method, is more suitable in such a specific base configuration. To support the analysis of this problem, general relationships between displacements computed in elastic datum and in fixed datum are provided. They are followed by auxiliary relationships derived on the basis of transformation formulae for different computational bases in elastic datum. Furthermore, indices of base separation are proposed which can be helpful in the design of monitoring networks.

A test network with simulated mutual displacements of the base subgroups, is used to investigate behaviour of the network with the fixed and the elastic datum being applied. Also, practical guidelines are given concerning data processing procedures for such specific monitoring networks. For big separation of base subgroups a non-routine procedure is recommended, aimed at facilitating specialist interpretation of monitoring results.

Keywords: vertical displacements, control networks, partitioned bases, elastic datum, fixed datum, base separation indices

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1. Introduction

In elementary guidelines for geodetic measurements of displacements (Lazzarini, 1977), it was emphasized that reference benchmarks should be linked among themselves by direct levelling lines, preferably with minimum number of level stations. In practice, however, especially in monitoring vertical displacements of elongated engineering structures, e.g. bridges, dams, etc., one meets the situations when the reference base has to be split into two subgroups each located on one side of a structure. Although the base points within each subgroup can have direct observation links as required, the subgroups can be linked with each other only through the observations carried out between the object points. Such a base configuration decreases the accuracy of stable datum identification. And so, the bigger the separation between the subgroups, expressed in terms of a number of instrument stations along the connecting levelling lines, the greater the magnitude of undetectable mutual displacement of the subgroups. Hence, the need for answering the question, which of the basic types of datum, i.e. the elastic and the fixed, is more suitable in this specific base configuration. Although each type of datum has a property of reducing the effect of previously computed mutual displacement of the base subgroups, the one offering greater accuracy of object points displacements should be recommended for practical use. By the term "the computed mutual displacement of the base subgroups" we understand mutual displacement of the base subgroups computed with the use of elastic datum after completing the identification procedure but prior to final computation of the object points displacements. Such a displacement is a superposition of the true displacement and the apparent displacement, i.e. induced by accumulation of observation random errors.

Although the elastic datum does not deform observations when reducing the computed mutual displacement of the subgroups, it assigns residual displacements to the reference points with their accuracy characteristics being of comparable magnitude to those of the displacements of the object points. This does not correspond to the role of the reference frame where proper accuracy gradation between the reference points and the object points is expected. The fixed datum deforms observations in order to cancel the computed mutual displacement of the subgroups, but the magnitudes of these deformations are within the specified limits. With this datum the reference points, all with zero displacements, form a stable frame for the displacements of the object points and a zero-error reference for their accuracy characteristics.

It should be added that both types of computation datum enable one to verify the correctness of the reference base identified at an earlier stage of data processing.

The choice of a more suitable type of datum is difficult, specifically due to the fact that the mutual displacement between the base subgroups computed after completing stable datum identification, lies within the uncertainty interval based on measurement accuracies, network geometry and datum conditions. Thus, the true component of the mutual displacement is undetectable.

The problem of the choice between the fixed and elastic computation datum does not occur in a method of coordinate differences, since the elastic datum is the only option there.

The analysis of basic types of datum in deformation monitoring in the case of single or multiple reference bases can be found in a number of items of professional literature (e.g. Caspary, 1988; Even-Tzur, 2006; Prószyński and Kwaśniak, 2006). The present paper deals with a specific case of single base which, although being physically partitioned by the monitored object into two subgroups of points, satisfies the criteria set in the procedure of stable base identification. So for final computation of the object points displacements the base should be considered as one group of points. The problem of the choice of suitable type of datum for such situations is investigated.

However, the publications that might substantially support the research have not been encountered as yet by the author of this paper.

The paper aims at formulating the principles for handling the problem of partitioned bases in monitoring vertical displacements, with data processing based on a method of observation differences.

2. Relationships between displacements computed in elastic and in fixed datum

We shall concentrate on a method of observation differences as applied to data processing in a monitoring network with partitioned reference base (Fig. 1).

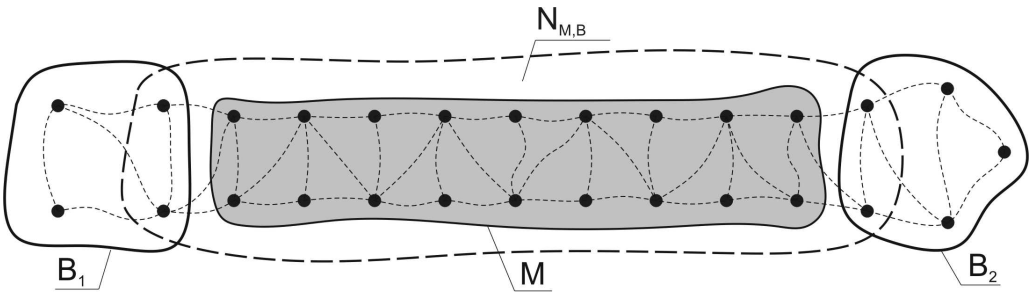

Fig. 1. Scheme of monitoring network with partitioned reference base

In the figure the following notation is used: M – the points of the monitored object (in a number of m); B_1, B_2 – the base subgroups (with b_1 and b_2 points respectively); $N_{M,B}$ – the subnetwork covering the object points and the base points which are directly connected with the object points.

Let us consider a system of standardised observation equations together with reference conditions for this network

$$\begin{bmatrix} \mathbf{A}_B & \mathbf{0} \\ \mathbf{A}_{B,M} & \mathbf{A}_{M,B} \end{bmatrix} \begin{bmatrix} \mathbf{d}_B \\ \mathbf{d}_M \end{bmatrix} = \begin{bmatrix} \Delta\mathbf{l}_B \\ \Delta\mathbf{l}_{M,B} \end{bmatrix} + \begin{bmatrix} \mathbf{v}_B \\ \mathbf{v}_{M,B} \end{bmatrix} \tag{1a}$$

$$\begin{bmatrix} \mathbf{H}_B & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{d}_B \\ \mathbf{d}_M \end{bmatrix} = \mathbf{0} \tag{1b}$$

where $\mathbf{d}_B(b\times1)$, $\mathbf{d}_M(m\times1)$ – the displacement vectors for the points B and M; $b = b_1+b_2$; $\Delta\mathbf{l}_B$ – differences of observations connecting B points themselves; $\Delta\mathbf{l}_{M,B}$ – differences of observations connecting the points in a subnetwork $N_{M,B}$; $\mathbf{A}_{M,B}$ – a coefficient matrix referring to a subnetwork $N_{M,B}$ (full rank); $\mathbf{A}_{B,M}$ – a coefficient matrix referring to two separated sets of observations connecting the points B_1 and the points B_2 directly with the points M; $\mathbf{H}_B$ – a coefficient matrix, the row dimension and the form of which depend upon the type of computation datum applied (full rank).

All the quantities in the system (1a, 1b) except for $\mathbf{d}_M$ will have a block structure corresponding to reference base partition (B_1, B_2), e.g.

$$\mathbf{d}_B = \begin{bmatrix} \mathbf{d}_{B_1} \\ \mathbf{d}_{B_2} \end{bmatrix}; \ \mathbf{A}_B = \begin{bmatrix} \mathbf{A}_{B_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{B_2} \end{bmatrix}; \ \mathbf{A}_{B,M} = \begin{bmatrix} \mathbf{A}_{B_1,M} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{B_2,M} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}; \ \mathbf{A}_{M,B} = \begin{bmatrix} \mathbf{A}_{M,B_1} \\ \mathbf{A}_{M,B_2} \\ \mathbf{A}_M \end{bmatrix} \tag{2}$$

The third row in matrices $\mathbf{A}_{B,M}$ and $\mathbf{A}_{M,B}$ refers to observations connecting M points between themselves.

The results of least squares (LS) adjustment of the system (1a, 1b) will be denoted by: $\hat{\mathbf{d}}_{B\{e\}}, \hat{\mathbf{d}}_{M\{e\}}$ – the displacement vectors for elastic datum (i.e. the datum based on a free network principle, in (Caspary, 1988) termed the minimum trace datum). The free net condition is applied here to a reference base B. The matrix $\mathbf{H}_B$ is such, that $\mathbf{A}_B\mathbf{H}_B^T = \mathbf{0}$; $\hat{\mathbf{d}}_{B\{f\}}, \hat{\mathbf{d}}_{M\{f\}}$ – the displacement vectors for fixed datum (i.e. $\hat{\mathbf{d}}_{B\{f\}} = \mathbf{0}$); $\mathbf{H}_B = \mathbf{I}$; $\hat{\mathbf{v}}_{M,B\{e\}}, \hat{\mathbf{v}}_{M,B\{f\}}$ – residuals for differences of observations $\Delta\mathbf{l}_{M,B}$, correspondingly in the elastic and in fixed datum.

The following relationships can be proved (see Appendix 1)

$$\hat{\mathbf{d}}_{M\{f\}} - \hat{\mathbf{d}}_{M\{e\}} = \mathbf{K}\hat{\mathbf{d}}_{B\{e\}}; \quad \mathbf{C}[\hat{\mathbf{d}}_{M\{f\}}] - \mathbf{C}[\hat{\mathbf{d}}_{M\{e\}}] = -\mathbf{K} \cdot \mathbf{C}[\hat{\mathbf{d}}_{B\{e\}}] \cdot \mathbf{K}^T \tag{3}$$

$$\hat{\mathbf{v}}_{M,B\{f\}} - \hat{\mathbf{v}}_{M,B\{e\}} = (\mathbf{A}_{M,B}\mathbf{K} - \mathbf{A}_{B,M})\hat{\mathbf{d}}_{B\{e\}} = -\mathbf{R}_{M,B}\mathbf{A}_{B,M}\hat{\mathbf{d}}_{B\{e\}} \tag{4}$$

where $\mathbf{K} = (\mathbf{A}_{M,B}^T\mathbf{A}_{M,B})^{-1}\mathbf{A}_{M,B}^T\mathbf{A}_{B,M}$; $\mathbf{R}_{M,B} = \mathbf{I} - \mathbf{A}_{M,B}(\mathbf{A}_{M,B}^T\mathbf{A}_{M,B})^{-1}\mathbf{A}_{M,B}^T$ is the reliability matrix for a subnetwork $N_{M,B}$; $\mathbf{C}[\hat{\mathbf{d}}_*]$ is the covariance matrix for a vector $\hat{\mathbf{d}}_*$, with a variance factor assumed to be equal to 1.

For a levelling network the matrix $\mathbf{K}$ has a specific structure, since

$$\mathbf{K} \cdot \underset{(1\times b)}{\mathbf{1}^T} = -\underset{(1\times m)}{\mathbf{1}^T}, \quad \text{or equivalently} \quad \underset{(1\times m)}{\mathbf{1}} \cdot \mathbf{K} \cdot \underset{(1\times b)}{\mathbf{1}^T} \equiv \Sigma\{\mathbf{K}\} = -m \tag{5}$$

where $\mathbf{1} = [11...1]$, $\Sigma\{\mathbf{K}\}$ is the sum of elements of the matrix $\mathbf{K}$.

One can see from formula (3) and (4) that differences between the adjustment results in elastic and in fixed datum depend upon the structure of the subnetwork $N_{M,B}$ as well as upon the magnitude of residual displacements of the base points, determined in elastic datum. It should be noted that the residual displacements result from the mutual displacement of the subgroups which can be determined in elastic datum with a reference base located in one of the subgroups. The bigger the base separation, the greater can be the apparent component in the above-mentioned mutual displacement.

From the second formula in (3) it follows that, assuming the variance factor equal to 1, the variances of displacements of the object points in a fixed datum are always smaller than those in the elastic datum.

Comparing the number of degrees of freedom for this system, when using the fixed and the elastic datum (denoted $f_{\{f\}}$ and $f_{\{e\}}$ respectively), we get:

$$f_{\{f\}} = n - s \cdot u_{(M)}; \quad f_{\{e\}} = n - [s \cdot u_{(B)} + s \cdot u_{(M)} - c] \ \rightarrow \ f_{\{f\}} - f_{\{e\}} = s \cdot u_{(B)} - c \tag{6}$$

where n – number of observation differences; $u_{(B)}$ – number of points of the reference base; $u_{(M)}$ – number of points of the monitored object; s – network dimension; c – network defect.

From the analysis of the relationship (6) for different types of networks with partitioned bases it follows that we shall always have $f_{\{f\}} > f_{\{e\}}$. Since the critical value in the global test on variance-factor estimator, i.e. $(\sigma_o)_{crit} = \chi^2_{f,\alpha}/f$, decreases with the increase in f, we get the following relationship between the critical values in such a test when using fixed or elastic computation datum

$$(\sigma_{o\{f\}})_{crit} < (\sigma_{o\{e\}})_{crit} \tag{7}$$

with the significance level α being kept the same.

It is reasonable that the global test in a system with fixed datum is more rigorous than that in a system with elastic datum, since it covers a check on displacement model.

3. Formulas for analyses of vertical displacements computed in elastic datum with different bases

For elastic datum defined on a full base B, the coefficient matrix $\begin{bmatrix} \mathbf{H_B} & \mathbf{0} \end{bmatrix}$ as in (1b) denoted here by $\mathbf{G}_B$, will take the form

$$\underset{(1\times u)}{\mathbf{G}_B} = \begin{bmatrix} \underset{(1\times b)}{\mathbf{1}} & \underset{(1\times m)}{\mathbf{0}} \end{bmatrix} \tag{8}$$

where $\mathbf{1}(1 \times b)$ as in (5).

Similarly, for elastic datum defined on a subgroup B_1 or B_2, the corresponding matrices $\mathbf{G}$ (see Eq.(8)), denoted as $\mathbf{G}_{B_1}$ and $\mathbf{G}_{B_2}$, will be as follows:

$$\mathbf{G}_{B_1} = \begin{bmatrix} \underset{(1\times b_1)}{\mathbf{1}} & \underset{(1\times b_2)}{\mathbf{0}} & \underset{(1\times m)}{\mathbf{0}} \end{bmatrix} \quad \mathbf{G}_{B_2} = \begin{bmatrix} \underset{(1\times b_1)}{\mathbf{0}} & \underset{(1\times b_2)}{\mathbf{1}} & \underset{(1\times m)}{\mathbf{0}} \end{bmatrix} \tag{9}$$

where b_1 and b_2 are the numbers of benchmarks in the subgroup B_1 and B_2, respectively.

Using the transformation formulae for displacement vectors when changing from the datum with a base B_i to that with a base B_j, and reversely, from the datum with a base B_j to that with a base B_i (e.g. Caspary, 1988; Prószyński and Kwaśniak, 2006), we obtain finally the following simple relationships:

$$\mathbf{d}_{(B_2)} = \mathbf{d}_{(B_1)} - \bar{d}_{B_2(B_1)} \cdot \underset{(u\times 1)}{\mathbf{1}}; \qquad \mathbf{d}_{(B_1)} = \mathbf{d}_{(B_2)} - \bar{d}_{B_1(B_2)} \cdot \underset{(u\times 1)}{\mathbf{1}} \tag{10}$$

$$\mathbf{d}_{(B_2)} = \mathbf{d}_{(B)} - \bar{d}_{B_2(B)} \cdot \underset{(u\times 1)}{\mathbf{1}}; \qquad \mathbf{d}_{(B)} = \mathbf{d}_{(B_2)} - \bar{d}_{B(B_2)} \cdot \underset{(u\times 1)}{\mathbf{1}} \tag{11}$$

where, for instance, $\bar{d}_{B_2(B_1)}$ denotes displacement of the gravity centre of the subgroup B_2 in a datum with a base B_1.

Combining the two formulae in (10) and those in (11) we get the expected relations

$$\bar{d}_{B_1(B_2)} = -\bar{d}_{B_2(B_1)}; \qquad \bar{d}_{B_2(B)} = -\bar{d}_{B(B_2)} \tag{12}$$

We can also prove the following relationships

$$\bar{d}_{B_2(B)} = -\frac{b_1}{b_2} \cdot \bar{d}_{B_1(B)}; \quad \bar{d}_{B_2(B_1)} = \frac{b_1 + b_2}{b_2} \bar{d}_{B(B_1)} \tag{13}$$

$$\bar{d}_{B_2(B_1)} = (1 + \frac{b_2}{b_1})\bar{d}_{B_2(B)} \tag{14}$$

We should note that the relationships (13), (14) do not explicitly depend on a number of monitored object points and are valid for any configuration of control network as well as measurement accuracy.

Concentrating on Eq. (14), we get the relationship between the standard deviations of $\bar{d}_{B_2(B_1)}$ and $\bar{d}_{B_2(B)}$, i.e.

$$\sigma[\bar{d}_{B_2(B_1)}] = (1 + \frac{b_2}{b_1}) \cdot \sigma[\bar{d}_{B_2(B)}] \tag{15}$$

which, as taken together with Eq. (14), means that the statistical significance of $\bar{d}_{B_2(B_1)}$ is identical with that of $\bar{d}_{B_2(B)}$, i.e.

$$\frac{\bar{d}_{B_2(B_1)}}{\sigma[\bar{d}_{B_2(B_1)}]} = \frac{\bar{d}_{B_2(B)}}{\sigma[\bar{d}_{B_2(B)}]} \tag{16}$$

Now, let us consider the relationship

$$\begin{bmatrix} \bar{d}_{B_1(B_i)} \\ \bar{d}_{B_2(B_i)} \end{bmatrix} = \begin{bmatrix} \frac{1}{b_1} \cdot \underset{(1\times b_1)}{\mathbf{1}} & \mathbf{0} \\ \mathbf{0} & \frac{1}{b_2} \cdot \underset{(1\times b_2)}{\mathbf{1}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{d}_{B_1(B_i)} \\ \mathbf{d}_{B_2(B_i)} \end{bmatrix} \tag{17}$$

where $\bar{d}_{B_1(B_i)}$, $\bar{d}_{B_2(B_i)}$ are displacements of the gravity centres of B_1 and B_2 in a datum with a base B_i.

Introducing the covariance matrix

$$\mathbf{C}\begin{bmatrix} \mathbf{d}_{B_1(B_i)} \\ \mathbf{d}_{B_2(B_i)} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \tag{18}$$

and applying the covariance propagation law for the relationship (17) we obtain finally

$$\mathbf{C}\begin{bmatrix} \bar{d}_{B_1(B_i)} \\ \bar{d}_{B_2(B_i)} \end{bmatrix} = \begin{bmatrix} \frac{1}{b_1^2}\sum\{\mathbf{C}_{11}\} & \frac{1}{b_1 b_2}\sum\{\mathbf{C}_{12}\} \\ \frac{1}{b_1 b_2}\sum\{\mathbf{C}_{21}\} & \frac{1}{b_2^2}\sum\{\mathbf{C}_{22}\} \end{bmatrix} \tag{19}$$

where $\sum\{\cdot\}$ denotes the sum of all the elements of the matrix in parentheses.

Since $\mathbf{H}_{B_i} \cdot \mathbf{d}_{B_i} = \mathbf{0}$ (see (1b)), we have $\mathbf{H}_{B_i} \cdot \mathbf{C}[\mathbf{d}_{B_i}] = \mathbf{0}$. Therefore, with $\mathbf{H}_{B_i}$ being $\mathbf{H}_{B_1}$, $\mathbf{H}_{B_2}$ or $\mathbf{H}_{B}$, respectively, we shall get

$$\mathbf{H}_{\mathrm{B}_1} : \sum \{\mathbf{C}_{11}\} = 0; \sum \{\mathbf{C}_{12}\} = \sum \{\mathbf{C}_{21}\} = 0 \tag{20a}$$

$$\mathbf{H}_{\mathrm{B}_2} : \sum \{\mathbf{C}_{22}\} = 0; \sum \{\mathbf{C}_{12}\} = \sum \{\mathbf{C}_{21}\} = 0 \tag{20b}$$

$$\mathbf{H}_{\mathrm{B}} : \sum \{\mathbf{C}_{11}\} = \sum \{\mathbf{C}_{22}\} = -\sum \{\mathbf{C}_{12}\} = -\sum \{\mathbf{C}_{21}\} \tag{20c}$$

From Eqs (19) and (20c) we get

$$\frac{\sigma[\bar{d}_{\mathrm{B}_1(\mathrm{B})}]}{\sigma[\bar{d}_{\mathrm{B}_2(\mathrm{B})}]} = \frac{b_2}{b_1} \tag{21}$$

which could be obtained also from Eq. (13).

4. Indices of base separation defined in elastic datum

To evaluate inconsistency of the reference base B due to its partitioning $\{\mathrm{B}_1, \mathrm{B}_2\}$ by the monitored object, the following indices, of absolute and relative character, are proposed

$$\mu_{\mathrm{a}} = \sigma[\bar{d}_{\mathrm{B}_2(\mathrm{B}_1)}] \qquad \mu_{\mathrm{r}} = \frac{\sigma[\bar{d}_{\mathrm{B}_2(\mathrm{B}_1)}]}{\bar{\sigma}[d_{\mathrm{B}(ij)}]} \tag{22}$$

where: $\sigma[\bar{d}_{\mathrm{B}_2(\mathrm{B}_1)}]$ – the standard deviation of the displacement of gravity centre of the base B_2 in elastic datum with a base B_1, $\bar{\sigma}[d_{\mathrm{B}(ij)}]$ – the average value of standard deviations of mutual displacements for pairs of benchmarks within base B_1 and within base B_2.

The more elongated is the monitored object, and hence, the greater the inconsistency of the reference base, the greater are the values of both indices. The index μ_{r} enables one to compare base inconsistencies for different control networks.

For largely elongated objects, the value of μ_{a} may provide basis for checking at the design stage the possibility of strengthening the network by introducing intermediate measurements of height differences between the points of both the base subgroups (e.g. by GPS levelling). By strengthening the network we might decrease the interval of undetectable mutual displacement of these subgroups.

5. Responses to displacement between the base subgroups in elastic and in fixed datum

As yet the author has not encountered the theoretically supported and empirically verified rules for the choice between the elastic and the fixed datum, the more so for the case of monitoring networks with partitioned reference bases. However, despite the lack of suitable knowledge, the attempt will be made here to compare the functioning of both types of datum with respect to this specific case. We shall use the formulae derived earlier in this paper (Section 2), as well as the conclusions drawn from numerical tests carried out on simulated networks (Twardziak, 2010).

The more elongated is the monitored object, the greater is the chance for mutual displacement between the base subgroups, due to different ground conditions on both sides of the monitored object. Obviously, the greater is also the magnitude of undetectable displacement between the subgroups. The quantity $\bar{d}_{B_2(B_1)}$ which we compute in elastic datum is a resultant displacement, being a superposition of the true and the apparent displacement, as was mentioned in the Introduction. With $\bar{d}_{B_2(B_1)}$ of the magnitude being within the corresponding uncertainty interval, it is not possible to isolate the components.

Let us now consider the formulae (3) and (4), which enable one to compare the results of using the elastic and the fixed datum. Introducing the notation

$$\Delta \mathbf{l}_{B,M} = \mathbf{A}_{B,M}\hat{\mathbf{d}}_{B\{e\}} = \begin{bmatrix} \mathbf{A}_{B_1,M} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{B_2,M} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{d}}_{B_1\{e\}} \\ \hat{\mathbf{d}}_{B_2\{e\}} \end{bmatrix} = \begin{bmatrix} \Delta \mathbf{l}_{B_1,M} \\ \Delta \mathbf{l}_{B_2,M} \\ \mathbf{0} \end{bmatrix} \tag{23}$$

where $\Delta \mathbf{l}_{B,M}$ is a vector of changes in connecting observations due to displacements of the reference benchmarks in elastic datum, we get

$$\hat{\mathbf{d}}_{M\{f\}} - \hat{\mathbf{d}}_{M\{e\}} = (\mathbf{A}_{M,B}^{T}\mathbf{A}_{M,B})^{-1}\mathbf{A}_{M,B}^{T}\mathbf{A}_{B,M}\hat{\mathbf{d}}_{B\{e\}} = \mathbf{N}_{M,B}\mathbf{A}_{B,M}\hat{\mathbf{d}}_{B\{e\}} = \mathbf{N}_{M,B} \cdot \Delta \mathbf{l}_{B,M}$$

$$\hat{\mathbf{v}}_{M,B\{f\}} - \hat{\mathbf{v}}_{M,B\{e\}} = -\mathbf{R}_{M,B}\mathbf{A}_{B,M}\hat{\mathbf{d}}_{B\{e\}} = -\mathbf{R}_{M,B} \cdot \Delta \mathbf{l}_{B,M}$$

The changes (23) can be interpreted as initial deformations imposed on a subnetwork $N_{M,B}$ at its both ends, due to zeroing the displacements of the reference benchmarks in a fixed datum. We notice immediately that with $\hat{\mathbf{d}}_{B\{e\}} \to \mathbf{0}$, and hence $\Delta \mathbf{l}_{B,M} \to \mathbf{0}$, the adjustment results in fixed and in elastic datum tend to be identical (obviously within the effect of random errors). It should be noted that small values of $\hat{\mathbf{d}}_{B\{e\}}$ which can be obtained with $\bar{d}_{B_2(B_1)}$ of small magnitude, may occur when the true and the apparent component in $\bar{d}_{B_2(B_1)}$ cancel each other. In such cases the true displacement of the base subgroups is hidden by the accumulation of observation random errors.

Having the above in mind, we shall take for testing the following three characteristic cases of the structure of $\bar{d}_{B_2(B_1)}$:

a) the true displacement is zero; the apparent displacement is of high value,
b) the true displacement and the apparent displacement being of high value each, almost cancel out,
c) the true displacement is of high value; the apparent displacement is almost zero.

The 10-span levelling control network (Fig. 2) with simulated observations has been used for testing the above cases. In all the three cases the displacements of the object points M relative to base B_1 were kept equal to zero. The base separation indices for the network were as follows:

$$\mu_a = 0.43; \qquad \mu_r = 2.42$$

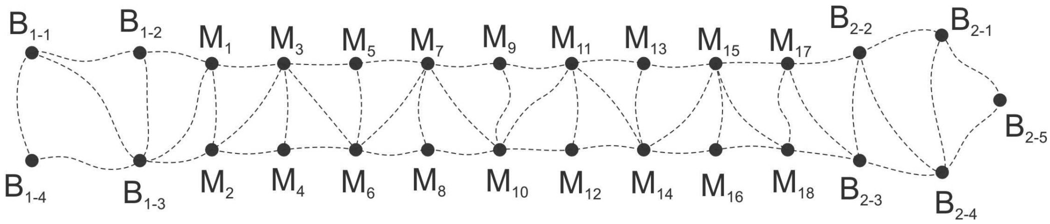

Fig. 2. The test network with simulated observations

It should be noted that in each of the cases all the necessary checks (i.e. for outlier detection and for stable base identification), preceding the main displacement computations, have been carried out successfully.

To compare how closely the computed object displacements adhere to their true values (i.e. zeroes) the following measures were determined for each analysed case:

$$\varepsilon_{\{e\}} = \sqrt{\frac{\sum_{i=1}^{n} (d_{M,i\{e\}})^2}{n}}; \qquad \varepsilon_{\{f\}} = \sqrt{\frac{\sum_{i=1}^{n} (d_{M,i\{f\}})^2}{n}}$$

where $d_{M,i\{e\}}$, $d_{M,i\{f\}}$ – the displacements of the i-th object point computed in elastic and in fixed datum, respectively, each datum with a reference base B; n – number of object points (here $n = 18$).

The diagrams below show the displacements computed in both the datums.

The tests enable one to formulate the following conclusions:

- with the true component of $\bar{d}_{B_2(B_1)}$ being zero or of negligible magnitude, the fixed datum yields object points displacements being closer to their true values ($\varepsilon_{\{f\}}$= 0.17 mm, $\varepsilon_{\{e\}}$= 0.29 mm). It is quite natural, since the on-site situation corresponds to the model assumption of absolute stability of the reference points. The elastic datum, although reduces the effect of $\bar{d}_{B_2(B_1)}$, but operates on its apparent component being a randomly induced quantity;
- with the two components canceling each other the object displacements computed in both the datums are almost identical. However, they do not adhere so closely to their true values as in the case a) ($\varepsilon_{\{f\}} = \varepsilon_{\{e\}} = 0.39$ mm);
- with the true component being of high value and the apparent one being negligibly small, the more suitable (in terms of adherence of the computed object points displacements to their true values) is the elastic datum ($\varepsilon_{\{e\}} = 0.17$ mm, $\varepsilon_{\{f\}} = 0.24$ mm). Here, it reduces the effect of the true component, whereas the fixed datum brings it to zero, ignoring the actual on-site situation.

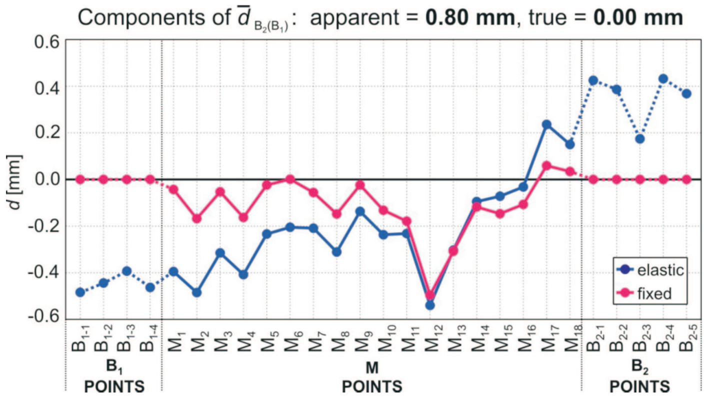

Fig. 3. Displacements computed for the case a)

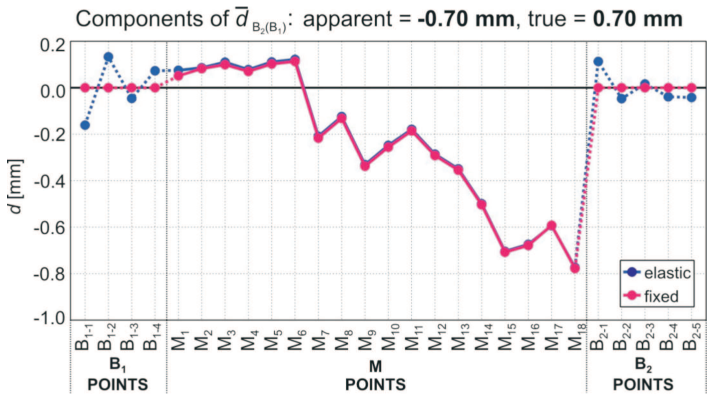

Fig. 4. Displacements computed for the case b)

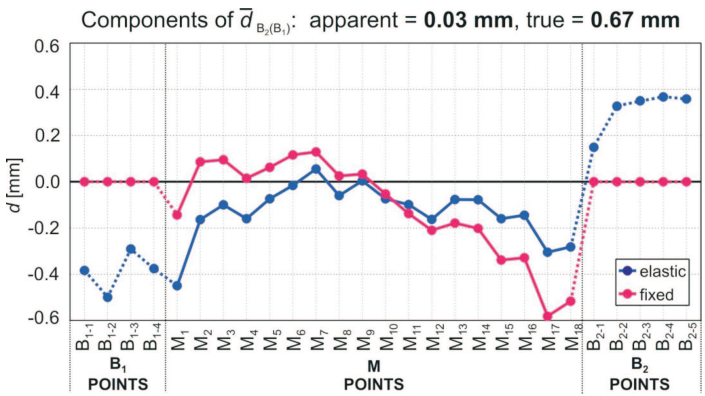

Fig. 5. Displacements computed for the case c)

6. Practical recommendations for operating with partitioned bases

Maintaining all the diagnostic procedures commonly used in data processing in monitoring networks, the following specific precautions for the case of partitioned bases can be recommended:

- the stable base identification should be carried out by analysis of mutual displacements for all pairs of the base points. The procedure known of its high reliability seems to be specially suited to partitioned bases. Its results will be verified in a final computing of displacements in a chosen datum (i.e. either fixed or elastic);
- in case that there are well-grounded indications of mutual stability or instability of the base subgroups, respectively the fixed datum and the elastic datum should be given priority;
- the numbers of benchmarks in base subgroups should not differ much and best be identical. Then, as follows from the previous analysis, the reference level for displacements which is established in elastic datum reduces by half the computed mutual displacement between the subgroups;
- with partitioned bases, the use of elastic datum results in a lack of distinct gradation in displacement accuracy characteristics between the reference benchmarks and the object benchmarks (see Introduction). Although it is an unavoidable effect of this type of base configuration, it may impair the confidence of a non-surveyor in the reference frame used for the determination of the object displacements. Therefore, this effect should not be displayed to specialists, who analyse the results of monitoring surveys in order to learn about the performance of the object and assess the degree of its safety. So, instead of showing the computed displacements and their standard deviations for the reference benchmarks, we should confine ourselves to a statement, that magnitude of each displacement is insignificant with respect to accuracy level of the measurement method applied, or more precisely, that it falls within its confidence interval.

It is obvious that all the detailed information concerning the completed monitoring surveys (the indices of reference base separation being included), should be kept as a professional documentation in the archives of engineering surveyor.

- for large separations of base subgroups, expressed by high value of base separation index μ_a, it may be reasonable to check whether it might be advantageous to strengthen the network by spanning the subgroups directly with the use of other levelling technique;
- for extremely large separations of the base subgroups, an extended strategy can be recommended. It might contain routine computation of displacements but both in elastic and in fixed datum, analysis of the discrepancies, and additionally, the determination of some other quantities, useful in professional interpretation of the monitoring results but less influenced by the separation of the subgroups. The best choice can be mutual displacements for selected pairs of the object points and their standard deviations (the use of elastic datum would be preferred for this purpose).

7. Concluding remarks

The recommendations formulated in the present paper are not conclusive and there is a need for further research on the problem of monitoring networks with partitioned bases. To formulate rigorous rules that could facilitate the choice of the most appropriate data processing strategy, it is necessary to apply more thorough and advanced statistical reasoning. That should lead to developing practical instructions, operating with specified values of base separation indices.

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Problem rozdzielonych baz w monitorowaniu przemieszczeń pionowych wydłużonych budowli

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Streszczenie

W monitorowaniu przemieszczeń pionowych wydłużonych budowli (np. mostów, zapór wodnych) przy zastosowaniu precyzyjnej niwelacji geometrycznej, baza odniesienia zazwyczaj składa się z dwóch podgrup ulokowanych na obu krańcach monitorowanej budowli. Im większe jest oddalenie tych podgrup, tym większa jest wielkość niewykrywalnego przemieszczenia jednej podgrupy względem drugiej. Przy koncentracji uwagi na metodzie różnic obserwacji powstaje problem, który z dwóch podstawowych rodzajów obliczeniowego układu odniesienia, tj. elastyczny i sztywny, stosowalnych w tej metodzie, jest układem bardziej odpowiednim przy tej specyficznej konfiguracji bazy. Celem stworzenia podstaw do analizy tego problemu, sformułowano związki natury ogólnej między przemieszczeniami obliczonymi w elastycznym i w sztywnym układzie odniesienia. Podane są następnie zależności wyprowadzone na podstawie wzorów transformacyjnych dla różnych baz obliczeniowych w elastycznym układzie odniesienia. Ponadto, przedstawiono propozycje wskaźników separacji bazy, które mogą być pomocne w projektowaniu sieci. Do badania zachowania się sieci przy zastosowaniu elastycznego i sztywnego układu odniesienia użyto sieć testową z symulowanymi przemieszczeniami podgrup bazy.

Podane są także praktyczne wskazówki dotyczące procedur opracowania danych dla takich specyficznych sieci do monitorowania przemieszczeń. Dla dużych separacji podgrup zalecona jest procedura odbiegająca od rutynowej, nakierowana na ułatwienie specjalistycznej interpretacji wyników monitorowania.

Appendix 1

Proof for properties (3) and (4)

For a fixed datum we get from (1a) the following system of observation equations

$$\mathbf{A}_{M,B}\mathbf{d}_{M\{f\}} = \Delta\mathbf{l}_{M,B} + \mathbf{v}_{M,B\{f\}} \tag{24}$$

and finally the LS estimators

$$\hat{\mathbf{d}}_{M\{f\}} = (\mathbf{A}^T_{M,B}\mathbf{A}_{M,B})^{-1}\mathbf{A}^T_{M,B}\Delta\mathbf{l}_{M,B} \tag{25}$$

$$\hat{v}_{M,B\{f\}} = -[\mathbf{I} - \mathbf{A}_{M,B}(\mathbf{A}^T_{M,B}\mathbf{A}_{M,B})^{-1}\mathbf{A}^T_{M,B}] \cdot \Delta\mathbf{l}_{M,B} = -\mathbf{R}_{M,B} \cdot \Delta\mathbf{l}_{M,B} \tag{26}$$

Assuming the elastic datum we shall form for (1a, 1b) the system of extended normal equations with non-singular matrix of coefficients, i.e.

$$\begin{bmatrix} \mathbf{A}^T_B\mathbf{A}_B + \mathbf{A}^T_{B,M}\mathbf{A}_{B,M} & \mathbf{A}^T_{B,M}\mathbf{A}_{M,B} & \mathbf{H}^T_B \\ \mathbf{A}^T_{M,B}\mathbf{A}_{B,M} & \mathbf{A}^T_{M,B}\mathbf{A}_{M,B} & \mathbf{0} \\ \mathbf{H}_B & \mathbf{0} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{d}_{B\{e\}} \\ \mathbf{d}_{M\{e\}} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^T_B\Delta\mathbf{l}_B + \mathbf{A}^T_{B,M}\Delta\mathbf{l}_{M,B} \\ \mathbf{A}^T_{M,B}\Delta\mathbf{l}_{M,B} \\ \mathbf{0} \end{bmatrix} \tag{27}$$

where $\mathbf{k}$ is a vector of correlates; $\mathbf{H}_B$ is such that $\mathbf{A}_B\mathbf{H}^T_B = \mathbf{0}$.

The second equation in (27) will have a form

$$\mathbf{A}^T_{M,B}\mathbf{A}_{B,M}\mathbf{d}_{B\{e\}} + \mathbf{A}^T_{M,B}\mathbf{A}_{M,B}\mathbf{d}_{M\{e\}} = \mathbf{A}^T_{M,B}\Delta\mathbf{l}_{M,B} \tag{28}$$

This equation should be satisfied by the LS estimators $\hat{\mathbf{d}}_{B\{e\}}, \hat{\mathbf{d}}_{M\{e\}}$. So, with $\mathbf{A}_{M,B}$ being of full rank, we get from (28)

$$\hat{\mathbf{d}}_{M\{e\}} = (\mathbf{A}^T_{M,B}\mathbf{A}_{M,B})^{-1}[\mathbf{A}^T_{M,B}\Delta\mathbf{l}_{M,B} - \mathbf{A}^T_{M,B}\mathbf{A}_{B,M}\hat{\mathbf{d}}_{B\{e\}}] = (\mathbf{A}^T_{M,B}\mathbf{A}_{M,B})^{-1}\mathbf{A}^T_{M,B}\Delta\mathbf{l}_{M,B} - \mathbf{K}\cdot\hat{\mathbf{d}}_{B\{e\}} \tag{29}$$

and substituting the relationship (25) we obtain finally

$$\hat{\mathbf{d}}_{M\{f\}} - \hat{\mathbf{d}}_{M\{e\}} = (\mathbf{A}^T_{M,B}\mathbf{A}_{M,B})^{-1}\mathbf{A}^T_{M,B}\mathbf{A}_{B,M}\hat{\mathbf{d}}_{B\{e\}} = \mathbf{K} \cdot \hat{\mathbf{d}}_{B\{e\}} \quad \blacksquare \tag{30}$$

From the second equation in (1a) we obtain

$$\mathbf{A}_{B,M}\hat{\mathbf{d}}_{B\{e\}} + \mathbf{A}_{M,B}\hat{\mathbf{d}}_{M\{e\}} = \Delta\mathbf{l}_{M,B} + \hat{\mathbf{v}}_{M,B\{e\}}$$

and after substituting (29)

$$\mathbf{A}_{B,M}\hat{\mathbf{d}}_{B\{e\}} + \mathbf{A}_{M,B}(\mathbf{A}^T_{M,B}\mathbf{A}_{M,B})^{-1}\mathbf{A}^T_{M,B}\Delta\mathbf{l}_{M,B} - \mathbf{A}_{M,B}\mathbf{K} \cdot \hat{\mathbf{d}}_{B\{e\}} = \Delta\mathbf{l}_{M,B} + \hat{\mathbf{v}}_{M,B\{e\}}$$

which can be written as

$$\mathbf{A}_{B,M}\hat{\mathbf{d}}_{B\{e\}} - \mathbf{R}_{M,B}\Delta\mathbf{l}_{M,B} - \mathbf{A}_{M,B}\mathbf{K} \cdot \hat{\mathbf{d}}_{B(e)} = \hat{\mathbf{v}}_{M,B\{e\}}$$

or taking into account (26) and carrying out some simple operations we got

$$\hat{\mathbf{v}}_{M,B\{f\}} - \hat{\mathbf{v}}_{M,B\{e\}} = (\mathbf{A}_{M,B}\mathbf{K} - \mathbf{A}_{B,M})\hat{\mathbf{d}}_{B\{e\}} = -\mathbf{R}_{M,B}\mathbf{A}_{B,M}\hat{\mathbf{d}}_{B\{e\}} \quad \blacksquare$$

Now we shall write (30) in the form

$$\hat{\mathbf{d}}_{M\{f\}} = \mathbf{K} \cdot \hat{\mathbf{d}}_{B\{e\}} + \hat{\mathbf{d}}_{M\{e\}}$$

or, equivalently

$$\hat{\mathbf{d}}_{M\{f\}} = \begin{bmatrix} \mathbf{K} & \mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{d}}_{B\{e\}} \\ \hat{\mathbf{d}}_{M\{e\}} \end{bmatrix} \tag{31}$$

Applying the covariance propagation law to equation (31), we obtain

$$\mathbf{C}[\hat{\mathbf{d}}_{M\{f\}}] = \mathbf{K}\cdot\mathbf{C}[\hat{\mathbf{d}}_{B\{e\}}]\cdot\mathbf{K}^T + \mathbf{Cov}[\hat{\mathbf{d}}_{M\{e\}}, \hat{\mathbf{d}}_{B\{e\}}]\cdot\mathbf{K}^T + \mathbf{K}\cdot\mathbf{Cov}[\hat{\mathbf{d}}_{B\{e\}}, \hat{\mathbf{d}}_{M\{e\}}] + \mathbf{C}[\hat{\mathbf{d}}_{M\{e\}}] \tag{32}$$

To find the relationship between $\mathbf{C}[\hat{\mathbf{d}}_{B\{e\}}]$ and $\mathbf{Cov}[\hat{\mathbf{d}}_{B\{e\}}, \hat{\mathbf{d}}_{M\{e\}}]$, we shall write the inverse of the coefficient matrix as in (27) showing only the elements useful for the proof, i.e.

$$\begin{bmatrix} \mathbf{C}[\mathbf{d}_{B\{e\}}] & \mathbf{Cov}[\mathbf{d}_{B\{e\}}, \mathbf{d}_{M\{e\}}] & * \\ \mathbf{Cov}[\mathbf{d}_{M\{e\}}, \mathbf{d}_{B\{e\}}] & \mathbf{C}[\mathbf{d}_{M\{e\}}] & * \\ * & * & * \end{bmatrix} \tag{33}$$

Multiplying the second row of the coefficient matrix in (27) by the first column of the inverse (33), we obtain

$$\mathbf{A}_{M,B}^T\mathbf{A}_{B,M}\mathbf{C}[\mathbf{d}_{B\{e\}}] + \mathbf{A}_{M,B}^T\mathbf{A}_{M,B}\mathbf{Cov}[\mathbf{d}_{M\{e\}}, \mathbf{d}_{B\{e\}}] = \mathbf{0}$$

and after simple operations

$$\mathbf{Cov}[\mathbf{d}_{M\{e\}}, \mathbf{d}_{B\{e\}}] = -(\mathbf{A}_{M,B}^T\mathbf{A}_{M,B})^{-1}\mathbf{A}_{M,B}^T\mathbf{A}_{B,M}\mathbf{C}[\mathbf{d}_{B\{e\}}] = -\mathbf{K}\cdot\mathbf{C}[\mathbf{d}_{B\{e\}}] \tag{34}$$

Substituting (34) into equation (32) and reordering terms, we get

$$\mathbf{C}[\hat{\mathbf{d}}_{M\{f\}}] - \mathbf{C}[\hat{\mathbf{d}}_{M\{e\}}] = \mathbf{K}\cdot\mathbf{C}[\hat{\mathbf{d}}_{B\{e\}}]\cdot\mathbf{K}^T - \mathbf{K}\cdot\mathbf{C}[\hat{\mathbf{d}}_{B\{e\}}]\cdot\mathbf{K}^T - \mathbf{K}\cdot\mathbf{C}[\hat{\mathbf{d}}_{B\{e\}}]\cdot\mathbf{K}^T = -\mathbf{K}\cdot\mathbf{C}[\hat{\mathbf{d}}_{B\{e\}}]\cdot\mathbf{K}^T \quad \blacksquare$$

We proved the properties (3), (4) of the system (1a, 1b) as structured primarily for levelling networks. However, the formulas used throughout the proof do not contain matrices corresponding to base partition, and no restrictive assumptions as to network dimension have been made within the proof. Therefore, we can conclude that the properties (3) and (4) can be applied to 1D, 2D and 3D networks, with reference bases either partitioned or not. Several numerical tests carried out on 1D and 2D networks confirm the above conclusion.