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A Multiple State Model for Premium Calculation when Several Premium-Paid States are Involved

Joanna Dębicka*, Beata Zmyślona†

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Abstract

The aim of this contribution is to derive a general matrix formula for the net period premium paid in more than one state. In order to avoid "overpayment" which implies higher premiums we give a formula for replacement of lump sum benefit into annuity benefits paid in more than one state. The obtained result is useful for example to more advanced models of dread disease insurances allowing period premiums paid by both healthy and ill person (e.g. not terminally yet). As an application, we supply analysis of dread disease insurances against the risk of lung cancer based on the actual data for the Lower Silesian Voivodship in Poland.

Keywords: modified multiple state model, net premium, life annuity, critical illness insurance, accelerated death benefits

JEL Classification: G22, I13

*Wrocław University of Economics; e-mail: joanna.debicka@ue.wroc.pl
†Wrocław University of Economics

1 Introduction

The insurance market is constantly expanding. Insurers offer more flexible contracts taking into account various situations that may arise in life. An example would be a serious illness, in case of which the priorities of the insured person may change considerably. In particular, it may be that the death benefit becomes less important while the life benefit becomes the most important. On the insurance market exist different kind of solutions to protect the insured against financial problems in this difficult situation. According to one of them, an insurer in such unexpected situations during the insurance period may offer purchase of an additional option called Accelerated Death Benefits (ADBs) to life insurance policyholder, which provides an acceleration of all or of a part of the basic death benefit to the insured before his death. By another alternative, the insured may buy dread disease insurance (or critical illness insurance) which provides the policyholder with a lump sum in case of dread disease which is included in a set of diseases specified by the policy conditions, such as heart attack, cancer or stroke (see Dash and Grimshaw (1993), Haberman and Pitacco (2012), Pittaco (1994), Pitacco (2014)). It implies that the dread disease policy does not meet any specific needs and does not protect the policyholder against such financial losses as loss of earnings or reimbursement of medical expenses. In both cases conditions of this insurance products state that the benefit is paid on diagnosis of a specified condition, rather than on disablement. This is understandable, because this type of insurance is sensitive to the development of medicine, not all dread diseases are as mortal as a few years ago. Thus insurers introduce strict conditions for the right to receive benefits associated with a severe disease. One of popular conditions is that benefits are paid not only on the diagnosis but also the expected future lifetime depends on the stage of the disease. Then the insurer has to take into account that probability of death of a dread disease sufferer depends on the duration of the disease. Depending on the conditions insurance premium may be paid in various forms by: healthy or sick (but not terminally) person, living person or healthy person. This article focuses on accurate valuation of such insurance products.

Multiple state modelling is a stochastic tool for designing and implementing insurance products. The multistate methodology is commonly used in calculation of actuarial values of different types of life and health insurances. A general approach to calculation of moments of the cash value of the future payment streams (including benefits, annuities and premiums) arising from a multistate insurance contract can be found in e.g. Dębicka (2013). This methodology, developed for the discrete-time model (where insurance payments are excercised at the end of time intervals), is based on an *modified multiple state model* (or *extended multiple state model*), for which matrix formulas for actuarial values can be derived. This approach to costing contracts not only makes calculations easier, but also enables us to factorize the stochastic nature of the evolution of the insured risk and the interest rate, which can be observed in the derived formulas.

The aim of this contribution is to derive a general matrix formula for the net period premium paid in more than one state, which can be applied to any type of insurance being modeled by the multiple state model. In a special case, when the insured pays a single premium in advance or period premiums under the condition that he is healthy (or active), the valuation of the contract may be done by the use of results derived in Dębicka (2013). In this paper we extend results of Dębicka (2013) in order to cover more advanced models of dread disease insurances (as e.g. ADB's form) that allow period premiums paid by both healthy and ill person (e.g. not terminally yet). A related problem of the premium paid in several states for multiple life insurance contract is considered in Gala (2013). In Section 5, we conduct a discussion describing the differences between these two approaches. We also indicate the benefits resulting from the use of methodology of insurance contracts valuation presented in this paper. Since dread disease insurance policies provide the policyholder with an additional lump sum in case of a severe illness, it is important to avoid a situation of "overpayment" which implies higher expected cost and hence higher premiums. Such a situation could take place when death occurs within a very short period after pay off of the additional benefit. The solution for this problem is achieved by replacing the lump sum payment with a series of payments (for example several annual payments), each payment being conditional on survival of the insured. Importantly, in case of advanced models of dread disease insurances, at the period of realising annuity payments, the insured risk can be present in different states of set space. Thus, we derive a general matrix formula for the a rate of such annual payments paid in more than one state.
The paper is organized as follows. In Section 2 we describe the modified multiple state model and its probabilistic structure. This modification allows us to use matrix-form approach to costing insurance contract. In Section 3 we derive general matrix expressions for the net period premium paid in more than one state and the replacement of lump sum benefit into annuity benefits realized in several states. Section 4 deals with the study of dread disease insurances against the risk of lung cancer. The modified multiple state model for dread disease insurances is presented in Section 4.1. The probability structure of the analyzed model is built under conditions that the probability of death for a dread disease sufferer depends on the duration of the disease and the payment of benefits associated with a severe disease depends both on the diagnosis and on the disease stage presented in Dębicka and Zmyślona (2015) (Section 4.2). In Section 4.3, the results obtained in Section 3 are applied to costing of different types of critical illness policies based on the actual data for the Lower Silesian Voivodship in Poland. Discussion and suggestions for further possible applications of obtained results are presented in Section 5.

2 Multiple state model

Following Haberman and Pitacco (2012), with a given insurance contract we assign a *multiple state model*. That is, at any time the insured risk is in one of a finite number of states labelled by $1, 2, \ldots, N$ or simply by letters. Let $\mathcal{S}$ be the *state space*. Each state corresponds to an event which determines the cash flows (premiums and benefits). Additionally, by $\mathcal{T}$ we denote a *set of direct transitions* between states of the state space. Thus $\mathcal{T}$ is a subset of the set of pairs (i, j), i.e., $\mathcal{T} \subseteq \{(i, j) \mid i \neq j; i, j \in \mathcal{S}\}$. The pair $(\mathcal{S}, \mathcal{T})$ is called a *multiple state model*, and describes all possible insured risk events as far as its evolution is concerned (usually up to the end of insurance). This model is structured so that it is a possibility to assign any cash flow arising from the insurance contract to one of the states (annuity, premiums), or the transition between them (lump sums). That it was possible to use matrix formulas for actuarial values, the multiple state model must be constructed so that each cash flow must be related to one of the states. Observe that for the lump sum the information that the insured risks is in a particular state at moment k is not enough to determine the benefit at time k, because we need additional information about where the insured risk was at previous moment $k-1$. Matrix is a two-dimensional structure, thus it is not possible to determine the exact moment of realization of lump sum benefit by using above three pieces of information. It appears that each $(\mathcal{S}, \mathcal{T})$ model can be easily (by the recursive procedure proposed in Dębicka (2013)) extended to *modified multiple state model* $(\mathcal{S}^*, \mathcal{T}^*)$ in which the lump sum benefit is affiliated with particular state and not a direct transition between states.

In this paper we consider an insurance contract issued at time 0 (defined as the time of issue of the insurance contract) and terminating according to the plan at a later time n (n is the term of policy). Moreover, x is the age of the insured person at a policy issue.

We focus on discrete-time model. Let $X^*(k)$ denote the state of an individual (the policy) at time k ($k \in \mathrm{T} = \{0, 1, 2, \ldots, n\}$). Hence the evolution of the insured risk is given by a discrete-time stochastic process $\{X^*(k); k \in \mathrm{T}\}$, with values in the finite set $\mathcal{S}^* = \{1, 2, \ldots, N^*\}$. In order to describe the probabilistic structure of $\{X^*(k)\}$, for any moment $k \in \{0, 1, 2, \ldots, n\}$, we introduce $\mathbb{P}^*_j(k) = \mathbb{P}(X^*(k) = j)$ and vector

$$\boldsymbol{P}(k) = (\mathbb{P}^*_1(k), \mathbb{P}^*_2(k), \mathbb{P}^*_3(k), \ldots, \mathbb{P}^*_{N^*}(k))^T \in \mathbb{R}^{N^*}.$$

Note that $\boldsymbol{P}(0) \in \mathbb{R}^{N^*}$ is a vector of the initial distribution (usually it is assumed that state 1 is an initial state, that is $\boldsymbol{P}(0) = (1, 0, 0, \ldots, 0)^T \in \mathbb{R}^{N^*}$).

Under the assumption that $\{X^*(k)\}$ is a nonhomogeneous Markov chain (see, e.g. Dębicka (2013), Hoem (1969), Hoem (1988), Waters (1984), Wolthuis (1994)) we have $\boldsymbol{P}^T(t) = \boldsymbol{P}^T(0) \prod_{k=0}^{t-1} \boldsymbol{Q}^*(k)$, where $\boldsymbol{Q}^*(k) = (q^*_{ij}(k))_{i,j=1}^{N^*}$ with $q^*_{ij}(k) = \mathbb{P}(X^*(k+1) = j | X^*(k) = i)$ being the transition probability. The above transition probabilities can be determined using a *multiple increment-decrement table* (or *multiple state life table*).

3 Matrix formula for cash flows paid in several states

Before presenting the matrix formula for cash flows paid in several states (like premiums and annuities) we need to introduce some notation (cf. Dębicka (2013)).
In order to describe the probabilistic structure of $\{X^*(k)\}$ we introduce matrix

$$\boldsymbol{D} = \begin{pmatrix} \boldsymbol{P}(0)^T \\ \boldsymbol{P}(1)^T \\ \ldots\ldots \\ \boldsymbol{P}(n)^T \end{pmatrix} \in \mathbb{R}^{(n+1)\times(N^*)}. \tag{1}$$

The individual's presence in a given state may have some financial effect. For k-th unit of time (it means for period $[k-1,k)$), we distinguish between the following types of cash flows: a cash flow paid in advance at time $k-1$ if $X^*(k-1) = i$ (premiums and life annuity due) and a cash flow paid from below at time k if $X^*(k) = j$ (lump sum and immediate life annuity). Note that the insurance policy gives rise to two payment streams. Firstly, a stream of premium payments, which flows from the insured to the insurer. Secondly, in the opposite direction, a stream of actuarial payment functions, where fixed amounts under the annuity product and lump sum benefits are considered as a series of future cash flows.
One of important quantities is the *total loss* $\mathcal{L}$ of the insurance contract, defined as the difference between the present value of future benefits and the present value of future premiums. In particular, the stream of actuarial payment functions is an *inflow* representing an income to $\mathcal{L}$ and it takes positive values, while the stream of premium payments is an *outflow* representing an outgo from $\mathcal{L}$ and it takes negative values.
Let $cf_j^*(k)$ be the future cash flow payable at time k if $X^*(k) = j$ $(k = 0, 1, \ldots, n)$ and

$$\boldsymbol{C} = \begin{pmatrix} cf_1^*(0) & cf_2^*(0) & \cdots & cf_{N^*}^*(0) \\ cf_1^*(1) & cf_2^*(1) & \cdots & cf_{N^*}^*(1) \\ cf_1^*(2) & cf_2^*(2) & \cdots & cf_{N^*}^*(2) \\ \ldots & \ldots & \ldots & \ldots \\ cf_1^*(n) & cf_2^*(n) & \cdots & cf_{N^*}^*(n) \end{pmatrix}$$

denote $(n+1) \times N^*$ cash flows matrix.
From the financial point of view, the cash flow $cf_i^*(k)$ is a sum of *inflows* representing an income to a particular fund and *outflows* representing an outgo from a particular fund. Hence

$$\boldsymbol{C} = \boldsymbol{C}_{in} + \boldsymbol{C}_{out}, \tag{2}$$

where $\boldsymbol{C}_{in}$ consists only of an income to a particular fund and $\boldsymbol{C}_{out}$ consists only of an outgo from a particular fund. We note that for $\mathcal{L}$, $\boldsymbol{C}_{in}$ includes the benefits and

$\boldsymbol{C}_{out}$ includes the premiums.
Let $Y(t)$ denote the rate of interest in time interval $[0,t]$. Then the discount function $v(t)$ is of the form $v(t) = \mathrm{e}^{-Y(t)}$. It is useful to introduce the following notation

$$\boldsymbol{Y} = (e^{-Y(0)}, e^{-Y(1)}, \ldots, e^{-Y(n)})^T \in \mathbb{R}^{n+1}$$

and

$$\mathbb{E}(Y) = \boldsymbol{M} = (m_0, m_1, \ldots, m_n)^T \in \mathbb{R}^{n+1},$$

with $m_k = \mathbb{E}(e^{-Y(k)})$.
We refer to Dębicka (2003) for the exact forms of the matrix $\boldsymbol{M}$ when $Y(t)$ is a Gaussian stochastic process with stationary increments and positive drift function. Two special cases of process $Y(t)$ are used to model the stochastic interest rate. In the first model it is assumed that $Y(t) = \sigma W(t) + \mu t$, where $W(t)$ is a standard Wiener process, μ is the mean rate of interest and σ is the volatility. In the second model it is assumed that $Y(t) = \int_0^t V(s)ds$, where the force of interest $V(s)$ is given by $V(s) = \sigma U(s) + \mu$ with $U(s)$ an Ornstein-Uhlenbeck process.
Let us note that for constant interest rate, we have $v(0,k) = v^k$ and $\boldsymbol{M} = (1, v, v^2, \ldots, v^n)^T$.
Additionally, let

$$\begin{aligned}
\boldsymbol{S} &= (1,1,1,\ldots,1)^T \in \mathbb{R}^{N^*}, \\
\boldsymbol{I}_{k+1} &= (0,0,\ldots,\underbrace{1}_{k+1},\ldots,0)^T \in \mathbb{R}^{n+1}, \\
\boldsymbol{J}_j &= (0,0,\ldots,\underbrace{1}_{j},\ldots,0)^T \in \mathbb{R}^{N^*},
\end{aligned}$$

for each $j = 1,2,\ldots,N^*$ and $k = 0,1,2,\ldots,n$.
Furthermore, for any matrix $\boldsymbol{A} = (a_{ij})_{i,j=1}^{n+1}$ let $Diag(\boldsymbol{A})$ be a diagonal matrix

$$Diag(\boldsymbol{A}) = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \ldots & a_{n+1n+1} \end{pmatrix}.$$

Insurance premiums are called *net premiums* if the *equivalence principle* is satisfied, i.e. $\mathbb{E}(\mathcal{L}) = 0$. In order to study the first moment of $\mathcal{L}$ we make the following standard assumptions (see also Dębicka (2013), Frees (1990) or Parker (1994)):

Assumption A1 Random variable $X^*(t)$ is independent of $Y(t)$.

Assumption A2 First moment of the random discounting function $e^{-Y(t)}$ is finite.

The *net single premium* paid in advance (at time 0, when $X^*(0) = 1$) for the insurance modelled by $(\mathcal{S}^*, \mathcal{T}^*)$ equals (cf. Dębicka (2013))

$$\pi = \pi_1(0) = \boldsymbol{M}^T Diag\left(\boldsymbol{C}_{in}\boldsymbol{D}^T\right)\boldsymbol{S}. \tag{3}$$

Additionally, by Dębicka (2013), the *net period premium* payable in advance at the beginning of the time unit during the first m units ($m \leq n$) if $X^*(t) = 1$ equals

$$p = \frac{\boldsymbol{M}^T Diag\left(\boldsymbol{C}_{in}\boldsymbol{D}^T\right)\boldsymbol{S}}{\boldsymbol{M}^T\left[\boldsymbol{I} - \sum_{k=m+1}^{n+1}\boldsymbol{I}_k\boldsymbol{I}_k^T\right]\boldsymbol{D}\boldsymbol{J}_1}, \tag{4}$$

where the denominator in (4) is equal to the actuarial value of a temporary (m-year) life annuity-due contract $\ddot{a}_{11}(0, m-1)$.
In case of each type of insurance, the net single premium can be calculated using formula (3). Importantly, formula for net period premium has to be modified, because premiums may be paid not only if $X^*(k) = 1$, but also when $X^*(k)$ is in other states (of course those in which an insured person is alive). We derive formula for period premium in Section 4.3 based on Theorem 1.
Let $\ddot{a}_{1(i)}(k_1, k_2)$ denote the actuarial value of the stream of unit benefits arising from life annuity-due contract payable in period $[k_1, k_2)$ if $X^*(k) = i$ for $k \in [k_1, k_2)$. Actuarial value is calculated at the beginning of the insurance period ($k = 0$). We tacitly assume that $X^*(0) = 1$.

Lemma 1. *Suppose that A1-A2 hold and $X^*(0) = 1$. Then for $(\mathcal{S}^*, \mathcal{T}^*)$ we have*

a) a temporary life annuity due

$$\ddot{a}_{1(i)}(k_1, k_2) = \boldsymbol{M}^T\left(\sum_{t=k_1}^{k_2-1}\boldsymbol{I}_{t+1}\boldsymbol{I}_{t+1}^T\right)\boldsymbol{D}\boldsymbol{J}_i, \tag{5}$$

b) an immediate life annuity

$$a_{1(i)}(k_1, k_2) = \boldsymbol{M}^T\left(\sum_{t=k_1+1}^{k_2}\boldsymbol{I}_{t+1}\boldsymbol{I}_{t+1}^T\right)\boldsymbol{D}\boldsymbol{J}_i. \tag{6}$$

Proof. Let us observe that under assumption A1-A2 we have

$$\ddot{a}_{1(i)}(k_1, k_2) = \sum_{t=k_1}^{k_2-1}\mathbb{E}\left(\mathrm{e}^{-Y(t)}\right)\cdot\mathbb{P}(X^*(t) = i \mid X^*(0) = 1). \tag{7}$$

Since $X^*(0) = 1$, then $\mathbb{P}(X^*(t) = i \mid X^*(0) = 1) = \mathbb{P}_i^*(t)$. Moreover

$$\mathbb{E}\left(\mathrm{e}^{-Y(t)}\right) = \boldsymbol{M}^T \boldsymbol{I}_{t+1}, \tag{8}$$

$$\mathbb{P}_i^*(t) = \boldsymbol{I}_{t+1}^T \boldsymbol{D} \boldsymbol{J}_i. \tag{9}$$

Applying (8) and (9) to (7) we have

$$\ddot{a}_{1(i)}(k_1, k_2) = \sum_{t=k_1}^{k_2-1} \boldsymbol{M}^T \boldsymbol{I}_{t+1} \boldsymbol{I}_{t+1}^T \boldsymbol{D} \boldsymbol{J}_i = \boldsymbol{M}^T \left(\sum_{t=k_1}^{k_2-1} \boldsymbol{I}_{t+1} \boldsymbol{I}_{t+1}^T \right) \boldsymbol{D} \boldsymbol{J}_i,$$

which completes the proof of (5). Proof (6) is analogous to proof of (5). □

Let $\mathcal{S}^p \subset \mathcal{S}^*$ be such that $X^*(k) \in \mathcal{S}^p$ for $k = 0, 1, \ldots, m-1$ implies that the period premium $p_{\mathcal{S}^p}$ is paid. The formula for such a premium is presented in Theorem 1.

Theorem 1. *Suppose that equivalence principle holds and assumptions A1-A2 are satisfied. Moreover, for the modified multiple state model $(\mathcal{S}^*, \mathcal{T}^*)$ the cash flows matrix is defined for the insurer's total loss fund, and insurance premiums are paid for $k = 0, 1, 2, \ldots, m-1$ if $X^*(k) = i$ and $i \in \mathcal{S}^p$. Then the formula for net period premium $p_{\mathcal{S}^p}$ paid during the first m units of the insurance period has the following form*

$$p_{\mathcal{S}^p} = \frac{\boldsymbol{M}^T Diag\left(\boldsymbol{C}_{in} \boldsymbol{D}^T\right) \boldsymbol{S}}{\sum_{i \in \mathcal{S}^p} \ddot{a}_{1(i)}(0, m)}, \tag{10}$$

where $\ddot{a}_{1(i)}(0, m) = \boldsymbol{M}^T \left(\sum_{k=0}^{m-1} \boldsymbol{I}_{k+1} \boldsymbol{I}_{k+1}^T \right) \boldsymbol{D} \boldsymbol{J}_i$.

Proof. For the multistate insurance, the equivalence principle $\mathbb{E}(\mathcal{L}) = 0$ may be written in the following form (Theorem 2 in Dębicka (2013))

$$\boldsymbol{M}^T Diag\left(\boldsymbol{C} \boldsymbol{D}^T\right) \boldsymbol{S} = 0,$$

which combined with (2) gives

$$\boldsymbol{M}^T Diag\left(-\boldsymbol{C}_{out} \boldsymbol{D}^T\right) \boldsymbol{S} = \boldsymbol{M}^T Diag\left(\boldsymbol{C}_{in} \boldsymbol{D}^T\right) \boldsymbol{S}. \tag{11}$$

Let $p = p_{\mathcal{S}^p}$ be the net period premium payable in advance at the beginning of a unit time during the first m units, when $X^*(k) = i$ and $i \in \mathcal{S}^p$. Since $X^*(0) = 1$, then the premium associated with state i for the first time may be paid at moment $k = \delta(1, i)$, which equals the length of the shortest possible sequence of transitions from state 1 to state i . Clearly $\delta(1, i)$ has to be such that $\delta(1, i) \leq m$ where $\delta(1, i) = \inf\{l \geq 0 : \mathbb{P}(X^*(l) = i) > 0\}$.

Let $\boldsymbol{C}_{out(p,i,m)} = -p \cdot \boldsymbol{C}_{(\ddot{1},i,m)}$, where cash flow matrix $\boldsymbol{C}_{(\ddot{1},i,m)}$ containing unit annuities paid in advance when the insured risk is at state i during the first m units of the insurance period. Thus is, for $j = 1, 2, \ldots, N^*$,

$$\boldsymbol{C}_{(\ddot{1},i,m)}\boldsymbol{J}_j = \begin{cases} (0,\ldots,0,\underbrace{1}_{\delta(1,i)},1,\ldots,1,\underbrace{1}_{m-1},0,\ldots,0)^T & \text{for} \quad j = i \\ (0,\ldots,0)^T & \text{for} \quad j \neq i \end{cases}.$$

Assuming that the premiums may be paid during the first m units of the insurance period if $X^*(k) = i$ and $i \in \mathcal{S}^p$, we have

$$\boldsymbol{C}_{out} = \sum_{i\in\mathcal{S}^p} \boldsymbol{C}_{out(p,i,m)} = -p \sum_{i\in\mathcal{S}^p} \boldsymbol{C}_{(\ddot{1},i,m)}. \tag{12}$$

Applying (12) to left side of equation (11) we obtain

$$\begin{aligned} &\boldsymbol{M}^T Diag\left(\left(p\sum_{i\in\mathcal{S}^p}\boldsymbol{C}_{(\ddot{1},i,m)}\right)\boldsymbol{D}^T\right)\boldsymbol{S} = \\ &= p\sum_{i\in\mathcal{S}^p}\boldsymbol{M}^T Diag\left(\boldsymbol{C}_{(\ddot{1},i,m)}\boldsymbol{D}^T\right)\boldsymbol{S} = \\ &= p\sum_{i\in\mathcal{S}^p}\boldsymbol{M}^T\sum_{k=0}^{n}\boldsymbol{I}_{k+1}\boldsymbol{I}_{k+1}^T\boldsymbol{C}_{(\ddot{1},i,m)}\boldsymbol{D}^T\boldsymbol{I}_{k+1}. \end{aligned} \tag{13}$$

Moreover, we have

$$\boldsymbol{I}_{k+1}^T\boldsymbol{C}_{(\ddot{1},i,m)}\boldsymbol{D}^T\boldsymbol{I}_{k+1} = \begin{cases} \mathbb{P}_i^*(k) & \text{for} \quad k = \delta(1,i), \delta(1,i)+1, \ldots, m-1 \\ 0 & \text{for} \quad k = 0, 1, \ldots, \delta(1,i)-1 \text{ and} \\ & \qquad\;\; k = m, m+1, \ldots, n \end{cases}. \tag{14}$$

Combination of (9) and (14) to (13) leads to

$$\begin{aligned} p\sum_{i\in\mathcal{S}^p}\boldsymbol{M}^T\sum_{k=0}^{n}\boldsymbol{I}_{k+1}\boldsymbol{I}_{k+1}^T\boldsymbol{C}_{(\ddot{1},i,m)}\boldsymbol{D}^T\boldsymbol{I}_{k+1} &= p\sum_{i\in\mathcal{S}^p}\boldsymbol{M}^T\sum_{k=\delta(1,i)}^{m-1}\boldsymbol{I}_{k+1}\boldsymbol{I}_{k+1}^T\boldsymbol{D}\boldsymbol{J}_i = \\ &= p\sum_{i\in\mathcal{S}^p}\boldsymbol{M}^T\left(\sum_{k=\delta(1,i)}^{m-1}\boldsymbol{I}_{k+1}\boldsymbol{I}_{k+1}^T\right)\boldsymbol{D}\boldsymbol{J}_i. = \\ &= p\sum_{i\in\mathcal{S}^p}\ddot{a}_{1(i)}(\delta(1,i),m). \end{aligned} \tag{15}$$

Note that i-th column of matrix $\boldsymbol{D}$ equals to

$$\begin{aligned} \boldsymbol{D}\boldsymbol{J}_i = \; & (\mathbb{P}(X^*(0)=i,\ldots,\mathbb{P}(X^*(\delta(1,i)-1)=i),\mathbb{P}(X^*(\delta(1,i))=i),\ldots, \\ & \mathbb{P}(X^*(n)=i))^T = (0,0,\ldots,0,\mathbb{P}(X^*(\delta(1,i))=i),\ldots,\mathbb{P}(X^*(n)=i))^T. \end{aligned}$$

Thus from Lemma 1 for $l \in \{0, 1, \ldots \delta(1,i)\}$ we obtain $\ddot{a}_{1(i)}(l,m) = \ddot{a}_{1(i)}(0,m)$. Then (15) we can rewrite as follows

$$p \sum_{i \in \mathcal{S}^p} \ddot{a}_{1(i)}(\delta(1,i), m) = p \sum_{i \in \mathcal{S}^p} \ddot{a}_{1(i)}(0,m) \tag{16}$$

and by (16) and (11) we straightforwardly obtain (10). □

Remark 1. *In applications, parameter m in Theorem 1 has to be selected so that at least one premium can be made in each of the states belonging to the subspace of states $\mathcal{S}^p$.*
Note that multiple state model reminds a directed graph, where the states correspond to the vertices (the nodes) of the graph, and the direct transitions correspond to the edges between the nodes. Therefore, in order to find the shortest way (path) between the states we use the graph optimization methodology.
Let $w = ((i_0, i_1), (i_1, i_2), \ldots, (i_{k-1}, i_k))$, where $i_0, i_1, \cdots_k \in \mathcal{S}^*$, denote a path (way) from state i_0 to state i_k in the model $(\mathcal{S}^*, \mathcal{T}^*)$. By $d(w)$ let us denote the path length of w i.e.

$$d(w) = \sum_{(i,j) \in \mathcal{T}^*} \mathbb{1}_{\{(i,j) \subset w\}}. \tag{17}$$

Additionally, let $\delta(i_0, i_k) = \min_w d(w)$ be the length of the shortest path from i_0 to i_k, where the minimum runs throughout all the paths leading from i_0 to i_k. Observe that the shortest path, if exists, must be a straight path, i.e. such that all the nodes of the path are different. This shortest path can be determined by Dijkstra's algorithm Dijkstra (1959).
Thus $m \geq \max_{\{i \in \mathcal{S}^{cf}\}} \delta(1,i)$ where $\delta(1,i)$ the length of the shortest path from state 1 to state i.

The general multiple state model for multistate insurances covers both lump sum benefits and annuity benefits. In order to avoid a situation of "overpayment", the lump sum $c_i(k)$ is replaced with a series of payments $b_i(k)$ (the annuity), which are connected with the stay of the insured risk in state i. In more complex multistate models, the lump sum can be converted into an annuity which is paid in more than one state. Let $\mathcal{S}^b \subset \mathcal{S}^*$ be such that $X^*(k) \in \mathcal{S}^b$ for $k = 1, \ldots, n$ implies that the annuity $b_{\mathcal{S}^b}$ is paid up to the end of the insurance contarct instead of a given lump sum $c_i(k)$. The formula for such a change is presented in Theorem 2.

Theorem 2. *Suppose that equivalence principle holds and assumptions A1-A2 are satisfied. For the multiple state model $(\mathcal{S}^*, \mathcal{T}^*)$, let $c_i(k)$ be the lump sum paid in advance when $X^*(k) = j$ for $k = 1, 2, \ldots, n$ and the cash flows matrix $\boldsymbol{C}_{in}$ consist only of lump sums to be converted into an annuity. Then the formula for a series of* *payments b paid when* $X^*(k) \in \mathcal{S}^b$ *(such as* $j \in \mathcal{S}^b$*) is given by*

$$b_{\mathcal{S}^b} = \frac{\boldsymbol{M}^T Diag\left(\boldsymbol{C}_{in}\boldsymbol{D}^T\right)\boldsymbol{S}}{\sum_{i\in\mathcal{S}^b} a_{1(i)}(0,n)}, \tag{18}$$

where $a_{1(i)}(0,n) = \boldsymbol{M}^T \left(\sum_{k=1}^{n} \boldsymbol{I}_{k+1}\boldsymbol{I}_{k+1}^T\right)\boldsymbol{D}\boldsymbol{J}_i$.

Proof. Actuarial value of the lump sum has to be equal to the actuarial value of the series of annuity payments. Thus

$$\boldsymbol{M}^T Diag\left(\boldsymbol{C}_{in}\boldsymbol{D}^T\right)\boldsymbol{S} = \boldsymbol{M}^T Diag\left(\sum_{i\in\mathcal{S}^b} \boldsymbol{C}_{in(b,i,n)}\boldsymbol{D}^T\right)\boldsymbol{S}, \tag{19}$$

where $\boldsymbol{C}_{in(b,i,n)} = b\cdot\boldsymbol{C}_{(1,i,n)}$ and the cash flow matrix $\boldsymbol{C}_{(1,i,n)}$ contain unit annuities paid from below when the insured risk is at state i during the insurance period n. By a similar argument as used in the proof of Theorem 1, using (6) we transform (19) into

$$\boldsymbol{M}^T Diag\left(\boldsymbol{C}_{in}\boldsymbol{D}^T\right)\boldsymbol{S} = b\sum_{i\in\mathcal{S}^b} a_{1(i)}(0,n).$$

This completes the proof of (18). □

Theorem 1 extends findings of Dębicka (2013) to the case where premiums are paid not only in the initial state, and life annuity-due rates are paid in more that one state. The matrix form derived in Theorem 1 and Theorem 2 not only provides a concise formula for premiums and annuity-due rates, but also factorizes the double stochastic nature of actuarial values of the total payment stream arising from the insurance contract. Matrix $\boldsymbol{D}$ depends only on the distribution of process $\{X^*(t)\}$, while $\boldsymbol{M}$ depends only on the interest rate. Moreover, matrices $\boldsymbol{C}$, $\boldsymbol{C}_{in}$, $\boldsymbol{C}_{out}$ depend on cash flows and describe the type (the case) of the insurance contract.

4 Applications

In this section we apply results derived in Theorem 1 and Theorem 2 to dread disease insurances on the example of the critical insurance against a lung cancer.

4.1 Actuarial model for dread disease insurance

Dread disease (or 'critical illness') policies provide the policyholder with a lump sum in case of dread disease which is included in a set of diseases specified by the policy conditions, such as heart attack, cancer or stroke (see Dash and Grimshaw (1993), Haberman and Pitacco (2012), Pittaco (1994), Pitacco (2014)). Typically conditions

of this insurance products state that the benefit is paid on diagnosis of a specified condition, rather than on disablement. It implies that dread disease policy does not meet any specific needs and does not protect the policyholder against such financial losses as loss of earnings or reimbursement of medical expenses. Individual critical illness insurance can take one of two main forms: a *stand-alone* cover or a *rider benefit* for a basic life insurance. The rider benefit (also called *living benefit*) may provide an acceleration of all or of a part of the basic life cover (Accelerated Death Benefits – ADBs), or it may be an additional benefit.
Dread disease insurances are of a long-term type, hence they are sensitive to the development of medicine, not all dread diseases are as mortal as a few years ago. Thus insurers introduce strict conditions for the right to receive benefits associated with a severe disease. One of popular conditions is that benefits are paid not only on the diagnosis but also the expected future lifetime depends on the stage of the disease. Then the insurer has to take into account that probability of death of a dread disease sufferer depends on the duration of the disease. Let us recall that in the classical notation, used for critical illness insurances, statuses are labeled by letters, where a means that the insured is active (or healthy), i indicates that the insured person is ill and suffers from a dread disease and d is related to the death of the insured; see e.g. Haberman and Pitacco (2012), Pitacco (2014). In this paper we distinguish between states

$d(O, D)$ - the death of the insured person who is ill and his expected future lifetime is at least 4 years ($e_s \geq 4$) or due to other cases, and

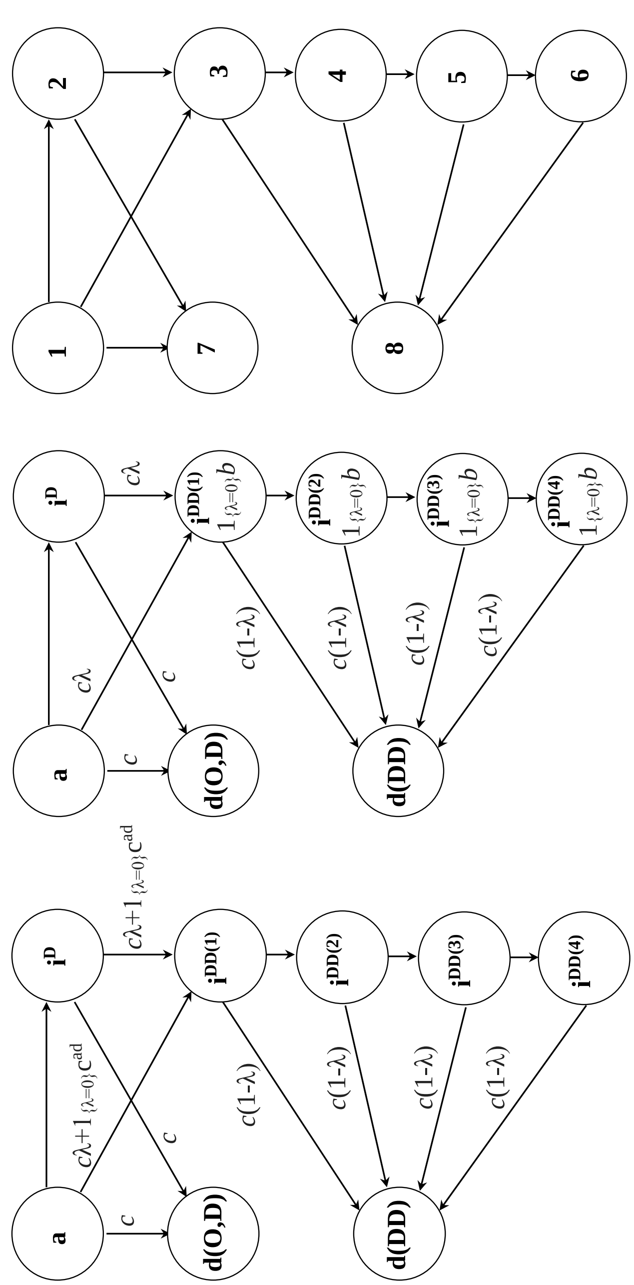
$d(DD)$ - the death of the insured person who is ill and his expected future lifetime is less than 4 years ($e_s < 4$),

where e_s is the expected future lifetime of s-years-old person. Moreover, following Dębicka and Zmyślona (2015), state i is divided into five states:

i^D - the insured person is ill and his expected future lifetime is at least 4 years ($e_s \geq 4$). In this stage the remission of the disease is still possible, although return to health state is impossible.

$i^{DD(h)}$ ($h = 1, 2, 3, 4$) - the insured is terminally sick and his expected lifetime is less than $4-(h-1)$ years . In this stages the remission of the disease is very unlikely.

This leads to a multiple state model for dread disease insurance derived in Dębicka and Zmyślona (2015); see Figure 1. Note that states $i^{DD(h)}$ are reflex (that is strictly transitional and after one unit of time, the insured risk leaves this state). Unbundling of the four states $i^{DD(h)}$ results from the fact that typically an insurer pays the benefit to a insured sick whose expected future lifetime is no longer than four years or, in some cases, two years (depending on medical circumstances). The difference results from the definition of a terminally ill person. On the one hand, for example the HIV+ patients with more

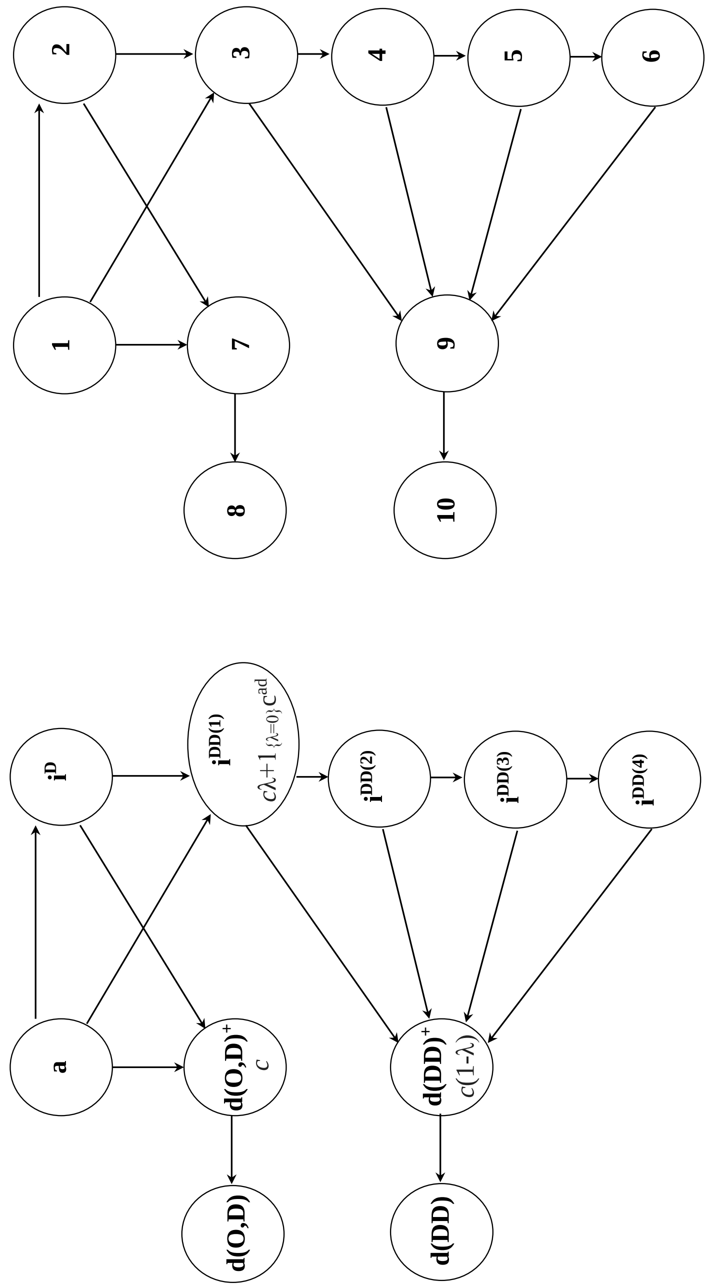


a. MSM with CI lump sum benefits b. MSM with CI annuity benefits c. MSM after numeration of state space

Figure 1: A multiple state model for dread disease insurance with a DD benefit

than 4,5 years of life expectancy, are treated as patients in a relatively good health. On the other hand, the term *terminally ill* in the context of health care refers to a person who is suffering from a serious illness and whose life is not expected to go beyond 2 years at the maximum.
The general multiple state model for critical illness insurances covers both disease lump sum benefits (Figure 1a) and disease annuity benefits (Figure 1b). Next to the arcs are marked benefits related to the transition between states, where c is a given lump sum (*death benefit*) and c^{ad} is an additional lump sum (*disease benefit*). To avoid a situation of 'overpayment' (that could take place when death occurs within a very short period after disease inception to the terminal phases of the dread disease), the single cash payment c^{ad} is replaced with a series of payments b (the annuity), which are connected with staying of the insured risk in states $i^{DD(h)}$. In particular, the model presented in Figure 1b may be applied to critical illness insurance contract with increasing ($b_3 < b_4 < b_5 < b_6$) or decreasing ($b_3 > b_4 > b_5 > b_6$) annuity benefits, where b_j is an annuity rate realized at state $j = 3, 4, 5, 6$.
By $\lambda \in [0, 1]$ we denote the so called *acceleration parameter*. The amount $c\lambda + \mathbb{1}_{\{\lambda=0\}}c^{ad}$ is payable after the dread disease diagnosis, while the remaining amount $c(1-\lambda)$ is payable after death, if both random events occur within the policy term n. Note that the multiple state model presented in Figure 1 covers all forms of DD insurances. Namely, if $\lambda = 0$, then the model describes a rider benefit as an additional benefit. If $0 < \lambda < 1$, then the model describes a rider benefit as an acceleration of part of the basic life cover. For $\lambda = 1$, the model becomes a stand-alone cover. In this case state $i^{DD(1)}$ is absorbing, because the whole insurance cover ceases immediately after the terminal stage dread disease diagnosis.
In order to simplify notation, let us label states according to Figure 1c.
Since the multiple state model presented in Figure 1c is extensive, in order to determine appropriate actuarial values, it is worth using matrix notation. For this purpose, we have to modify the model, replacing lump sum benefits by benefits associated with staying of the insured risk in particular states (according procedure presented in Dębicka (2013)). As a result, *the modified multiple state model* $(\mathcal{S}^*, \mathcal{T}^*)$ for dread disease insurance assumes the form presented in Figure 2a (with DD lump sum benefits).
Following the procedure of extending the multiple state model presented in Dębicka (2013), we introduce states $d(O, D)^+$ and $d(DD)^+$. State $d(O, D)^+$ denotes death of the insured person who is ill and his expected future lifetime is at least 4 years ($e_s \geq 4$). If the insured risk is in this state, the death benefit c is paid. State $d(O, D)$ in this model denotes that the insured has been dead for at least one year. Although states $d(O, D)^+$ and $d(D, O)$ deal with the same event, they differ by the fact that the lump sum is realized only when the insured risk is at state $d(O, D)^+$. States $d(DD)^+$ and $d(DD)$ are interpreted correspondingly. Note that $d(O, D)^+$ and $d(DD)^+$ are reflex. Because $i^{DD(1)}$ is a reflex state, there is no need to create state $i^{DD(1)^+}$.

Figure 2: An extended multiple state model for dread disease insurance



a. EMSM with a DD lump benefits

b. EMSM after numeration of state space

The extended multiple state model with dread disease annuity benefits is the same as in Figure 2a. In order to simplify the notation in what follows we enumerate the set space, as presented in Figure 2b.

4.2 Dread disease insurance against risk of lung cancer

Malignant tumours constitute the second cause (after cardiovascular diseases) of death in developed countries; see Dębicka and Zmyślona (2015). In particular, lung cancer falls into the group of tumours characterized by the highest morbidity and mortality rates. In many European countries, it is the most frequent in population of men and the second frequent in population of women after breast cancer. Moreover, lung cancer is a tumour with unfavourable prognosis. Because of the high prevalence and mortality rates, the relatively short survival time after the diagnosis, lung cancer is a perfect example of the deadly disease, which could be covered by DD insurances.
Age, sex and region of residence should be taken into account in the analysis of the etiology of lung cancer. The analysis of geographic data shows a significant diversity of incidence and mortality rates to be observed in different regions of Europe. For example, in Poland, the morbidity and mortality vary significantly among particular provinces (voivodships).
In case of DD disease insurance for lung cancer, the model (presented in Figure 2b) has six states associated with health situation of the insured person, which means that the insured:

1 - is alive and not sick with malignant lung tumour,

2 - is diagnosed of lung cancer without finding of metastasis to lymph nodes, brain, bones or so-called distant metastases,

3 - is diagnosed of lung cancer and the existence of distant metastases are observed and his/her expected lifetime is less than 4 years ($e_y < 4$),

4 - has a lung cancer with distant metastases and $e_y < 3$,

5 - has a lung cancer with distant metastases and $e_y < 2$,

6 - has a lung cancer with distant metastases and $e_y < 1$,

Other states are associated with the death of the insured person.
Following Dębicka and Zmyślona (2015), in order to estimate elements of transition matrix $\boldsymbol{Q}^*(k)$ we used databases Life Tables of Poland (2008), Wojciechowska and Didkowska (2014) and NHF (2014).
The transition probabilities in $\boldsymbol{Q}^*(k)$ can be determined using a *multiple increment-decrement table* (or *multiple state life table*). Such a table, referring to x years old

person for $(\mathcal{S},\mathcal{T})$ presented in Figure 1c, takes the following form (cf. Dębicka and Zmyślona (2015))

$$\{l^1_{x+k}, l^2_{x+k}, l^3_{x+k}, l^4_{x+k}, l^5_{x+k}, l^6_{x+k}, d^{12}_{x+k}, d^{13}_{x+k}, d^{17}_{x+k}, d^{23}_{x+k}, d^{27}_{x+k}, d^{34}_{x+k}, d^{38}_{x+k}, d^{45}_{x+k}, d^{48}_{x+k}, d^{56}_{x+k}, d^{58}_{x+k}\}_{k\geq 0} \tag{20}$$

where l^i_{x+k} denotes the number of lives in state i at age $x+k$ and d^{ij}_{x+k} denotes the number of lives at age $x+k$, who during period $[x+k, x+k+1)$ left the state i and transit to state j.

Notice that after application of the procedure of extension of $(\mathcal{S},\mathcal{T})$, state 8 became state 9 in $(\mathcal{S}^*,\mathcal{T}^*)$ (cf. Figure 2b). Thus for $(\mathcal{S}^*,\mathcal{T}^*)$ the multiple increment-decrement table is as follows

$$\{l^1_{x+k}, l^2_{x+k}, l^3_{x+k}, l^4_{x+k}, l^5_{x+k}, l^6_{x+k}, d^{12}_{x+k}, d^{13}_{x+k}, d^{17}_{x+k}, d^{23}_{x+k}, d^{27}_{x+k}, d^{34}_{x+k}, d^{39}_{x+k}, d^{45}_{x+k}, d^{49}_{x+k}, d^{56}_{x+k}, d^{59}_{x+k}\}_{k\geq 0}. \tag{21}$$

Generally, the multiple increment-decrement table, that refers to x years old person for a multiple state model $(\mathcal{S},\mathcal{T})$ consists of functions described for each transient state $i \in \mathcal{S}$. It appears that life table (20) is enough to describe probabilistic structure for the model $(\mathcal{S}^*,\mathcal{T}^*)$ presented in Figure 2b even though this table does not contain l^7_{x+k}, d^{78}_{x+k} and l^9_{x+k}, d^{910}_{x+k}. This is possible due to the fact that both states 7 and 9 are reflex (i.e. i is transient and $q^*_{ii}(k) = 0$) for which there exists only one possibility to leave. Hence l^7_{x+k} and d^{78}_{x+k} are unambiguously defined by d^{17}_{x+k} and d^{27}_{x+k} in the following way $l^7_{x+k} = d^{17}_{x+k-1} + d^{27}_{x+k-1} = d^{78}_{x+k}$. Correspondingly l^9_{x+k} and d^{910}_{x+k} are connected by the relation $l^9_{x+k} = d^{39}_{x+k-1} + d^{49}_{x+k} + d^{59}_{x+k-1} + d^{69}_{x+k} = d^{910}_{x+k}$. The transition matrix of $\{X^*(k)\}$ for DD disease insurance model presented in Figure 2b has the following form

$$\boldsymbol{Q}^*(k) = \begin{pmatrix} q^*_{11}(k) & q^*_{12}(k) & q^*_{13}(k) & 0 & 0 & 0 & q^*_{17}(k) & 0 & 0 & 0 \\ 0 & q^*_{22}(k) & q^*_{23}(k) & 0 & 0 & 0 & q^*_{27}(k) & 0 & 0 & 0 \\ 0 & 0 & 0 & q^*_{34}(k) & 0 & 0 & 0 & 0 & q^*_{39}(k) & 0 \\ 0 & 0 & 0 & 0 & q^*_{45}(k) & 0 & 0 & 0 & q^*_{49}(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & q^*_{56}(k) & 0 & 0 & q^*_{59}(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{22}$$

where

$$q^*_{ij}(k) = \begin{cases} \dfrac{l^i_{x+k+1} - \sum_{j:(i,j)\in\mathcal{T}} d^{ij}_{x+k}}{l^i_{x+k}} & \text{for} \quad j = i \\ \dfrac{d^{ij}_{x+k}}{l^i_{x+k}} & \text{for} \quad j \neq i \end{cases},$$

and l^{i}_{x+k}, d^{ij}_{x+k} come from (21).
If a multiple state life table is not available, then estimation of $\boldsymbol{Q}^*(k)$ is needed. We refer to Dębicka and Zmyślona (2015) where this problem is analysed in detail in case of lung cancer disease for multiple increment-decrement tables (20); see also multi-state life tables for lung cancer disease developed in Dębicka and Zmyślona (2016).

4.3 Net premiums

In what follows we analyse three scenarios, where

- premium is paid only if $X^*(k) = 1$ (the insured is healthy / active), i.e. $\mathcal{S}_p = \{1\}$,
- premium is paid only if $X^*(k) = 1, 2$ (the insured is healthy or has not a lung cancer with distant metastases), i.e. $\mathcal{S}_p = \{1, 2\}$,
- premium is paid if $X^*(k) = 1, 2, \ldots, 6$ (the insured is alive, independently on his health status), i.e. $\mathcal{S}_p = \{1, 2, 3, 4, 5, 6\}$.

Then, by Theorem 1, the net period premium, paid during the first m units of the insurance contract, has the following form

$$p_{\mathcal{S}_p} = \begin{cases} \dfrac{\boldsymbol{M}^T Diag\left(\boldsymbol{C}_{in}\boldsymbol{D}^T\right)\boldsymbol{S}}{\ddot{a}_{1(1)}(0,m-1)} & \text{for} \quad \mathcal{S}_p = \{1\} \quad \text{and} \quad 1 \leq m \leq n \\ \dfrac{\boldsymbol{M}^T Diag\left(\boldsymbol{C}_{in}\boldsymbol{D}^T\right)\boldsymbol{S}}{\ddot{a}_{1(1)}(0,m-1)+\ddot{a}_{1(2)}(0,m-1)} & \text{for} \quad \mathcal{S}_p = \{1,2\} \quad \text{and} \quad 2 \leq m \leq n \\ \dfrac{\boldsymbol{M}^T Diag\left(\boldsymbol{C}_{in}\boldsymbol{D}^T\right)\boldsymbol{S}}{\sum_{i=1}^{6}\ddot{a}_{1(i)}(0,m-1)} & \text{for} \quad \mathcal{S}_p = \{1,\ldots,6\} \quad \text{and} \quad 5 \leq m \leq n \end{cases}, \qquad (23)$$

where $\ddot{a}_{1(i)}(0, m-1)$ is defined by (5) and m is limited from bottoms in accordance with Remark 1.
Premiums calculated in Section 4.3.1 and Section 4.3.2 are based on formulas (3) and (23), where

$\boldsymbol{M}$ is described under the assumption, that the interest rate is constant and equal 1%,

$\boldsymbol{D}$ is calculated for a 40-year-old person ($x = 40$) and a 25-year-insurance period ($n = 25$), based on results presented in Section 4.2 (apart from Table 2, where matrix $\boldsymbol{D}$ is calculated for 20, 30, 50 and 60 age at entry additionally).

We assume that period premiums are paid up to the end of insurance period ($n = m = 25$).

4.3.1 Accelerated benefit for temporary life insurances

We assume that the living benefit provides an acceleration of part λ of the basic life cover 1 unit. If the insured person's death occurred in time interval $[k, k+1)$, $k = 0, 1, 2, \ldots, n-1$, before the end of the insurance contract, then at time $k+1$ the insurer pays benefit 1 (i.e. $c = 1$). Then cash flows matrix, which consists only of an income to the total loss found, has the form

$$\boldsymbol{C}_{in} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 1 & 0 & 1-\lambda & 0 \\ \multicolumn{10}{c}{\dotfill} \\ 0 & 0 & \lambda & 0 & 0 & 0 & 1 & 0 & 1-\lambda & 0 \end{pmatrix} \in \mathbb{R}^{26\times 10}.$$

The premiums for such an insurance, depending on acceleration parameter λ and sex of the insured person, are presented in Table 1.

Table 1: Premiums for acceleration benefit for temporary life insurances

Premium	π		$p_{\{1\}}$		$p_{\{1,2\}}$		$p_{\{1,2,3,4,5,6\}}$	
λ	Woman	Man	Woman	Man	Woman	Man	Woman	Man
0.001	0.11555	0.23896	0.00543	0.01184	0.00542	0.01182	0.00541	0.01181
0.25	0.11640	0.23951	0.00547	0.01187	0.00546	0.01185	0.00545	0.01184
0.5	0.11724	0.24005	0.00551	0.01189	0.00549	0.01188	0.00549	0.01186
0.75	0.11809	0.24060	0.00555	0.01192	0.00553	0.01190	0.00553	0.01189
0.999	0.11894	0.24115	0.00559	0.01195	0.00557	0.01193	0.00557	0.01192
1	0.11894	0.24115	0.00559	0.01195	0.00557	0.01193	–	–

A detailed analysis of mortality and morbidity of lung cancer disease is presented in Dębicka and Zmyślona (2015) and Dębicka and Zmyślona (2016). It is obvious that the mortality rate increases in the population of patients suffering from cancer. However, the probability of death is particularly high in case of diagnosis of the so-called distant metastases. A patient without diagnosed distant metastases rarely dies of cancer. The mortality rate is only slightly increased and it could be connected with the treatment process. However, a patient with diagnosed metastases dies of cancer in the terminal state, even if circulatory arrest or immunity reduced by chemotherapy were a direct cause of death (states 3-6 mark off such a period in the life of a patient, in which the risk of death is very high). All the mentioned remarks related to mortality of patients with lung cancer have a direct impact on the amount of period premiums calculated in Table 1. The little difference between $p_{\{1\}}$ and $p_{\{1,2\}}$ is related to the fact that the mortality rate among people with cancer without distant metastases is slightly increased. However, the small difference between $p_{\{1,2\}}$ and $p_{\{1,2,3,4,5,6\}}$ is due to high mortality rate among people with cancer with distant metastases (i.e. when the insured risk is at states $3-6$). There is a small chance that the insured in the

terminal state of illness will survive another year, and thus pay the premium. Note that the relative difference between $p_{\{1\}}$ and $p_{\{1,2,3,4,5,6\}}$ is about 0.37% for women and 0.25% for men.
For the same reason the acceleration parameter does not have a big impact on the premium. If the insured decides to take λ part of the death benefit upon being diagnosed distant metastases, then the remaining part of the benefit will be paid out in a short period of time to the beneficiary. For example, premiums for $\lambda = 0,001$ and $\lambda = 1$ differ relatively by less than 3% for women and 1% for men.
Lung cancer belongs to the group of tumors characterized by highest morbidity and mortality rates. It is the most frequent in population of men and the second frequent in population of women after breast cancer. Epidemiological data shows the existence of significant differences between the incidence of lung cancer in men and women populations. The morbidity rate is several times higher in male population than in female. This fact is reflected in the premiums calculated in Table 1, which for women are about 51% lower than for men.
Also, the incidence rate depends strongly on age. Lung cancer occurs very rarely among patients up to forty years of age, and then the incidence begins to increase after the age of fifty. The peak incidence occurs at the sixth and seventh decades of life. All these facts have an impact on the amount of premiums, which can be seen in Table 2. It is not surprising that the values of premiums for men and women are

Table 2: Net single premiums depending on the age at entry ($\lambda = 0.5$, $n = 25$)

x	π^{woman}	π^{man}	$\left(\frac{\pi^{man}}{\pi^{woman}} - 1\right) \cdot 100$
20	0.014088	0.045039	220
30	0.044563	0.109622	146
40	0.117245	0.240053	105
50	0.244143	0.433692	78
60	0.494406	0.672507	36

increasing with the age at entry. The calculations show strong influence of sex and the specifics of incidence rate correlated with age on the amount of net premiums. We can observe that, regardless of age, net single premiums are lower for women. The difference between premiums for male and female decreases with the rise of the age at entry from 220% to 36%.

4.3.2 Additional DD benefit for life insurances

Let $c^{ad}(k)$ denote the lump sum benefit payable at time k on condition that the insured person's irreversible phase of dread disease occurred in time interval $[k-1,k)$, $k = 1, 2, \ldots, n-1$ and $X^*(k) = 3$, enabling the use of a more expensive and more complete diagnosis. Sometimes, the single cash payment $c^{ad}(k)$ is replaced by a series

of payments $b_j(k)$ (the annuity payable for period $[k, k+1)$ if the insured is terminally ill at time k, i.e. $X^*(k) = 3, 4, 5, 6$) conditional on the survival of the insured.
Moreover, let $c(k+1)$ denote the benefit payable at time $k+1$ if the insured person's death occurred in time interval $[k, k+1)$, $k = 0, 1, 2, \ldots, n-1$ and $X^*(k) = 7, 9$, before the end of the insurance contract.
Let d be the pure endowment benefit payable at time n if the insured person is still alive at that time (i.e. $X^*(n) = 1, 2, 3, 4, 5, 6$).
We combine DD insurance for $\lambda = 0$ with life insurance and analyse the following cases:

Case 1 Additional lump sum DD benefit for n-year temporary life insurance ($c^{ad}(k) = 1$ and $c(k+1) = 1$)

$$\boldsymbol{C}_{in} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \multicolumn{10}{c}{\dotfill} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{26\times 10}.$$

Case 2 Additional annuity DD benefit for n-year temporary life insurance ($b_j(k) = b$ for $k = n, n-1, \ldots, j-2$, $j = 3, 4, 5, 6$ and $c(k+1) = 1$)

$$\boldsymbol{C}_{in} = \begin{pmatrix} 0 & 0 & b & b & b & b & 1 & 0 & 1 & 0 \\ 0 & 0 & b & b & b & b & 1 & 0 & 1 & 0 \\ \multicolumn{10}{c}{\dotfill} \\ 0 & 0 & b & b & b & b & 1 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{26\times 10}.$$

Note that from Theorem 2 we have

$$b = \frac{c^{ad} a_{1(3)}(0, 25)}{\sum_{i=3}^{6} a_{1(i)}(0, 25)}. \tag{24}$$

Then using (24) for $c^{ad} = 1$ we obtain $b = 0.80542$ for man and $b = 0.81445$ for woman.

Case 3 Additional lump sum DD benefit for n-year endowment insurance ($c^{ad}(k) = 1$, $c(k+1) = 1$, $d = 1$)

$$\boldsymbol{C}_{in} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \multicolumn{10}{c}{\dotfill} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{26\times 10}.$$

Table 3: Premiums for combination additional DD benefit and life insurances

Premium	π		$p_{\{1\}}$		$p_{\{1,2\}}$		$p_{\{1,2,3,4,5,6\}}$	
Sex	Woman	Man	Woman	Man	Woman	Man	Woman	Man
Case 1 - 2	0.14361	0.25702	0.00674	0.01273	0.00673	0.01271	0.00672	0.01270
Case 3	0.81648	0.81771	0.03835	0.04051	0.03826	0.04045	0.03821	0.04041

The premiums for the described cases, depending on sex of the insured person, are presented in Table 3. The replacement of the single cash payment c^{ad} (Case 1) with a series of payments b (Case 2) does not influence premiums because the actuary value of annuity payments b equals 1 (the additional lump sum DD benefit). So the high rate of annuities ($b \simeq 0.8$) confirms that mortality of terminally ill people (when the insured risk is at states $3-6$) is very high for both women and men.
Note that insurance contract described in Case 3 is a combination of insurance contract from Case 1 and the pure endowment insurance. In Case 1 the relative difference between net single premiums for women and men is about 79% (for period premiums it is about 89%). In Case 3, however, this difference is very small and amounts to about 0.15% for net single premium and from $5,6\%$ to $5,8\%$ for period premiums. This situation is related to the fact that the survival probability up to the end of the insurance period for women is higher than for the men (according to Life Tables of Poland (2008) the probabilities are equal: 0.891017 for women and 0.735806 for men). Thus, the chance of paying the pure endowment benefit is higher for women. In Case 3, the amount of premiums for women and men is comparable, because women pay lower premiums for life and health insurance, but higher for pure endowment contract and for men it is the opposite.
The cash flow matrices introduced in Case 1 -Case 3 can be straightforwardly applied to the calculation of actuarial values such as reserves and elements of profit testing (the process of adjusting the features of a contract) as e.g. in Dębicka (2010), Dębicka (2013) and Dębicka *et al.* (2016).

5 Discussion

Combination of life insurance contract and supplementary insurances leads to complex protecting packages. The $(\mathcal{S}^*, \mathcal{T}^*)$ model considered in this paper also allows to incorporate options related to the health status of the insured, such as disability or permanent inability to work. The extended multiple state model presented in Figure 2 can be used in the analysis of cash flows arising from various contracts on both the primary and secondary financial and insurance markets. This is possible mainly because the modelling of morbidity and mortality is related to the stages of the disease. Two potential stages in the history of the disease are distinguished, namely a mild stage without diagnosed distant metastases and a critical stage with diagnosed distant metastases. This way of modelling gives an opportunity to create insurance products that ensure payment of benefits not only after a diagnosis but also in the event of a deterioration of the disease. This model is also useful in the field of health economics. Financial and insurance products based on the model can ensure an important source of co-financing treatment by a patient. Moreover, the model can be used to estimate the costs of treating a disease depending on its course.

Another application might be to use a part of the modified multiple state model concerning the population of those suffering from lung cancer with metastasis (e.g. states $3, 4, 5, 6, 9, 10$) for modeling contracts on the secondary market of life insurances (the viatical market). For example, it can be applied to derive the value of viatical settlement payments under the condition that the insured person is at state 3, or the expected cost of premiums and benefits for investors (see e.g. Dębicka and Heilpern (2017)).

The matrix approach not only allows for valuation of discussed insurance and finance contracts, but can be used for other insurance contracts which are modelled by the multiple state model. In particular, it can be useful for a *general buy-back accelerated critical illness* model presented in Brink (2010). Theorem 2 can be applied for calculation of the value of annuity paid in any subset of states belonging to the state space $\mathcal{S}^*$ for marriage insurance or marriage reverse annuity contracts.

A related contribution that deals with the calculation of period premiums paid in more than one state is given by Gala (2013), where the total future cash flows Z that arise from insurance contract is treated as a set of separate streams of cash flows of particular types (e.g. premiums, annuities, lump sums). This way of valuation of the insurance contract represents an *actuarial* approach to the analysis of cash flows arising from insurance contract. Whereas from the *financial* point of view Z is treated as a sum of current values of the total cash flows realized at individual moments of the duration of the insurance contract, the total cash flow at any given time is the sum of premiums and different kinds of benefits realized at a given time in the insurance period. The distinction between these two approaches to the analysis of Z is important when the insurance contract guarantees lump sum benefits directly related to the occurrence of a random event covered by the insurance contract. For the lump sum the information that the insured risk is in a particular state at moment k is not enough to determine the benefit at moment k because one needs additional information about the state of the insured risk at time $k-1$. Therefore, in Gala (2013), in order to describe all benefits resulting from the contract, for each moment of the contract it is necessary to specify a vector of annuity benefits (related to the stay of the insured risk in each state) and a matrix of lump sum benefits (related to the transition of the insurance risk between states). Hence, if we analyze n year insurance contract, then we have to describe a collection of n N-dimensional vectors and n matrices of size $N \times N$ (N is the size of the state space of the multiple state model) while in the approach proposed in this paper it is enough to determine one $(n+1) \times N^*$ benefit cash flow matrix (N^* is the size of the state space of the modified multiple state model). It seems that a financial approach to the analysis of cash flows resulting from the conclusion of an insurance contract gives a simpler and more transparent record of actuarial values based on the addition and multiplication of matrices (in the actuarial approach an additional Hadamard product is necessary). Numerical analysis of insurance risk for lung cancer shows a significant impact of gender for the calculation of the premium. Due to high mortality rate of people suffering from lung cancer disease with distant metastases and the slightly increased mortality rate of patients without diagnosed distant metastases there is very little cost difference between contracts paid according to the scenario when premiums are paid only by a healthy insured person and the one when premiums are paid regardless of the health status of the insured.

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