

Positive and stable electrical circuits with state-feedbacks

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Abstract: The analysis of the positivity and stability of linear electrical circuits by the use of state-feedbacks is addressed. Generalized Frobenius matrices are proposed and their properties are investigated. It is shown that if the state matrix of an electrical circuit has generalized Frobenius form then the closed-loop system matrix is not positive and asymptotically stable. Different cases of modification of the positivity and stability of linear electrical circuits by state-feedbacks are discussed and necessary conditions for the existence of solutions to the problem are established.

Key words: positivity, stability, linear, electrical circuit, state-feedback

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state-of-the-art in positive theory is given in the monographs [3, 10]. Variety of models having positive behavior can be found in engineering, especially in electrical circuits [16], economics, social sciences, biology and medicine, etc. [3, 10].

The positive electrical circuits have been analyzed in [4–6, 8–13]. The constructability and observability of standard and positive electrical circuits has been addressed in [5], the decoupling zeros in [6] and minimal-phase positive electrical circuits in [8]. A new class of normal positive linear electrical circuits has been introduced in [9]. Positive fractional linear electrical circuits have been investigated in [12] and positive unstable electrical circuits in [13]. Infinite eigenvalue assignment by output-feedback for singular systems has been analyzed in [7]. Zeroing of state variables in descriptor electrical circuits has been addressed in [14]. Controller synthesis for positive linear systems with bounded controls has been investigated in [1]. Stability of continuous-time and discrete-time linear systems with inverse state matrices has been analyzed in [15].

In this paper the analysis of the positivity and stability of linear electrical circuits by state-feedbacks will be investigated.

The paper is organized as follows: in section 2 the basic definitions and properties of positive electrical circuits are recalled. The generalized Frobenius matrices are introduced and their properties are analyzed in section 3. The linear electrical circuits with state-feedbacks are investigated in section 4. and the general case is presented in section 5. Concluding remarks are given and some open problems are formulated in section 6.

The following notation will be used: \mathfrak{R} is the set of real numbers, $\mathfrak{R}^{n \times m}$ is the set of $n \times m$ real matrices, $\mathfrak{R}_+^{n \times m}$ is the set of $n \times m$ real matrices with nonnegative entries and $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$, M_n is the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n is the $n \times n$ identity matrix.

2. Positive electrical circuits

Consider the linear continuous-time electrical circuit described by the state equations:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (1a)$$

$$y(t) = Cx(t) + Du(t), \quad (1b)$$

where $x(t) \in \mathfrak{R}^n$, $u(t) \in \mathfrak{R}^m$, $y(t) \in \mathfrak{R}^p$ are the state, input and output vectors and $A \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times m}$, $C \in \mathfrak{R}^{p \times n}$, $D \in \mathfrak{R}^{p \times m}$.

It is well-known [16] that any linear electrical circuit composed of resistors, coils, capacitors and voltage (current) sources can be described by the state equations (1). Usually as the state variables $x_1(t), \dots, x_n(t)$ (the components of the state vector $x(t)$) the currents in the coils and voltages on the capacitors are chosen.

Definition 2.1. [16] The electrical circuit (1) is called (internally) positive if $x(t) \in \mathfrak{R}_+^n$ and $y = y(t) \in \mathfrak{R}_+^p$, $t \in [0, +\infty]$ for any $x_0 = x(0) \in \mathfrak{R}_+^n$ and every $u(t) \in \mathfrak{R}_+^m$, $t \in [0, +\infty]$.

Theorem 2.1. [16] The electrical circuit (1) is positive if and only if

$$A \in M_n, \quad B \in \mathfrak{R}_+^{n \times m}, \quad C \in \mathfrak{R}_+^{p \times n}, \quad D \in \mathfrak{R}_+^{p \times m}. \quad (2)$$

Theorem 2.2. [16] The linear electrical circuit composed of resistors, coils and voltage sources is positive for any values of the resistances, inductances and source voltages if the number of coils is less or equal to the number of its linearly independent meshes and the direction of the mesh currents are consistent with the directions of the mesh source voltages.

Theorem 2.3. [16] The linear electrical circuit composed of resistors, capacitors and voltage sources is not positive for all values of its resistances, capacitances and source voltages if each of its branches contains a resistor, capacitor and a voltage source.

Theorem 2.4. [16] The R, L, C, e electrical circuits are not positive for any values of its resistances, inductances, capacitances and source voltages if at least one of its branches contains coil and capacitor.

Definition 2.2. [16] The positive electrical circuit for $u(t) = 0$ is called asymptotically stable if

$$\lim_{t \rightarrow \infty} x(t) = 0 \quad \text{for any } x_0 \in \mathfrak{R}_+^n. \quad (3)$$

Theorem 2.5. [16] The positive electrical circuit is asymptotically stable if all coefficients of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 \quad (4)$$

are positive, i.e. $a_k > 0$ for $k = 0, 1, \dots, n - 1$.

3. Generalized Frobenius matrices

Definition 3.1. The following matrices:

$$\begin{aligned}
 A_1 &= \begin{bmatrix} 0 & b_1 & 0 & \cdots & 0 \\ 0 & 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{n-1} \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, & A_2 = A_1^T &= \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ b_1 & 0 & \cdots & 0 & -a_1 \\ 0 & b_2 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n-1} & -a_{n-1} \end{bmatrix}, \\
 A_3 &= \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ b_1 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n-1} & 0 \end{bmatrix}, & A_4 = A_3^T &= \begin{bmatrix} -a_{n-1} & b_1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & b_{n-1} \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad (5) \\
 & & & & b_k > 0, \quad k = 1, \dots, n - 1
 \end{aligned}$$

are called the matrices in generalized Frobenius form.

It is easy to verify that

$$\det[I_n s - A_j] = s^n + a_{n-1} s^{n-1} + \dots + a_1 b_2 \dots b_{n-1} s + a_0 b_1 \dots b_{n-1} \quad \text{for } j = 1, \dots, 4 \quad (6)$$

and the coefficients of the polynomial are positive if and only if $a_k > 0$ and $b_k > 0$ for $k = 1, \dots, n - 1$.

Theorem 3.1.

1. The inverse matrix of the generalized Frobenius matrix is also the generalized Frobenius matrix.
2. The inverse matrix of the generalized Frobenius matrix is asymptotically stable if and only if the generalized Frobenius matrix is also asymptotically stable.

Proof. The proof will be given only for the matrix A_1 . The proof for the remaining matrices (5) is similar.

It is easy to verify that

$$\begin{aligned}
 A_1^{-1} &= \begin{bmatrix} 0 & b_1 & 0 & \cdots & 0 \\ 0 & 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{n-1} \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}^{-1} \\
 &= \begin{bmatrix} -a_0^{-1}b_1^{-1}a_1 & -a_0^{-1}b_2^{-1}a_2 & \cdots & -a_0^{-1}b_{n-1}^{-1}a_{n-1} & -a_0^{-1} \\ b_1^{-1} & 0 & \cdots & 0 & 0 \\ 0 & b_2^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n-1}^{-1} & 0 \end{bmatrix}. \tag{7}
 \end{aligned}$$

By Theorem 2.5 the matrix A_1 is asymptotically stable if and only if $a_k > 0$ for $k = 0, \dots, n-1$. By definition $b_k > 0$ and $b_k^{-1} > 0$ for $k = 1, \dots, n-1$. Therefore, from (7) it follows that all coefficients of the first row are negative if and only if all coefficients of the n -th row of the matrix A_1 are negative.

Theorem 3.2. If $s_{ji}, j = 1, \dots, 4, i = 1, \dots, n$ are the nonzero eigenvalues of the generalized Frobenius matrix A_j then s_{ji}^{-1} are the eigenvalues of the inverse matrix $A_j^{-1}, j = 1, \dots, 4$.

Proof. Let $s_{ji}, j = 1, \dots, 4, i = 1, \dots, n$ be the zeros of the characteristic equation of the matrix A_j

$$\det[I_n s_j - A_j] = 0 \quad \text{for } j = 1, \dots, 4. \tag{8}$$

Then multiplying (8) by $\det[-s_j^{-1}A_j^{-1}]$ we obtain:

$$\det[I_n s_j - A_j] \det[-s_j^{-1}A_j^{-1}] = \det[(I_n s_j - A_j)(-s_j^{-1}A_j^{-1})] = \det[I_n s_j^{-1} - A_j^{-1}] = 0. \tag{9}$$

Therefore, if $s_{ji}, j = 1, \dots, 4, i = 1, \dots, n$ are the eigenvalues of the matrix A_j then s_{ji}^{-1} are the eigenvalues of the matrix $A_j^{-1}, j = 1, \dots, 4$.

Theorem 3.3. The characteristic polynomial of the inverse matrices in the generalized Frobenius forms (5) is given by

$$\det[I_n s - A_j^{-1}] = s^n + a_0^{-1}b_1^{-1}a_1s^{n-1} + \dots + a_0^{-1}b_1^{-1} \dots b_{n-1}^{-1}a_{n-1}s + a_0^{-1}b_1^{-1} \dots b_{n-1}^{-1} \tag{10}$$

for $j = 1, \dots, 4$.

Proof. Using (7) and developing the determinant with respect to the first row we obtain:

$$\begin{aligned}
 \det[I_n s - A_1^{-1}] &= \det \begin{bmatrix} s + a_0^{-1}b_1^{-1}a_1 & a_0^{-1}b_2^{-1}a_2 & \cdots & a_0^{-1}b_{n-1}^{-1}a_{n-1} & a_0^{-1} \\ -b_1^{-1} & s & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & s & 0 \\ 0 & 0 & \cdots & -b_{n-1}^{-1} & s \end{bmatrix} = \\
 &= s^n + a_0^{-1}b_1^{-1}a_1s^{n-1} + \dots + a_0^{-1}b_1^{-1} \dots b_{n-1}^{-1}a_{n-1}s + a_0^{-1}b_1^{-1} \dots b_{n-1}^{-1}. \tag{11}
 \end{aligned}$$

Similar results we obtain for $j = 2, 3, 4$.

Example 3.1. The characteristic polynomial of the generalized Frobenius matrix

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ -1 & -4 & -5 \end{bmatrix} \quad (12)$$

has the form:

$$\det[I_3 s - A] = \begin{vmatrix} s & -2 & 0 \\ 0 & s & -2 \\ 1 & 4 & s+5 \end{vmatrix} = s^3 + 5s^2 + 8s + 4 \quad (13)$$

and its zeros are $s_1 = -1$, $s_2 = s_3 = -2$.

The inverse matrix of (12) has the form:

$$A^{-1} = \begin{bmatrix} -2 & -2.5 & -1 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \quad (14)$$

and its characteristic polynomial

$$\det[I_3 s - A^{-1}] = \begin{vmatrix} s+2 & 2.5 & 1 \\ -0.5 & s & 0 \\ 0 & -0.5 & s \end{vmatrix} = s^3 + 2s^2 + 1.25s + 0.25, \quad (15)$$

with zeros $s_1^{-1} = -1$, $s_2^{-1} = s_3^{-1} = -0.5$.

4. Electrical circuits with state-feedbacks

The problem under consideration can be formulated as follows: given the unstable electrical circuits described by (1a) with $A \in \mathfrak{R}^{n \times n}$ and $B \in \mathfrak{R}_+^{n \times m}$ find the state-feedback

$$u(t) = Kx(t), \quad K \in \mathfrak{R}^{m \times n} \quad (16)$$

such that the closed-loop system

$$\dot{x}(t) = A_C x(t), \quad (17a)$$

where

$$A_C = A + BK \in M_n \quad (17b)$$

is asymptotically stable.

Theorem 4.1. If $A \in \mathfrak{R}^{n \times n}$ has the generalized Frobenius form then does not exist a state-feedback matrix $K \in \mathfrak{R}^{1 \times n}$ for any $B \in \mathfrak{R}_+^{n \times 1}$ such that the closed-loop system (17a) is positive and asymptotically stable.

Proof. Let A has the generalized Frobenius form A_1 , i.e. $A = A_1$. Then $A_{C1} = A_1 + BK \in M_n$ has also the same form if and only if $B = [0 \ \dots \ 0 \ b]^T$ and

$$A_{C1} = \begin{bmatrix} 0 & b_1 & 0 & \dots & 0 \\ 0 & 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{n-1} \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} [k_1 \ k_2 \ \dots \ k_n]$$

$$= \begin{bmatrix} 0 & b_1 & 0 & \dots & 0 \\ 0 & 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{n-1} \\ bk_1 - a_0 & bk_2 - a_1 & -a_2 & \dots & bk_n - a_{n-1} \end{bmatrix} \in M_n \tag{18}$$

if and only if

$$bk_i - a_{i-1} \geq 0 \quad \text{for } i = 1, \dots, n. \tag{19}$$

If the condition (19) is satisfied then by Theorem 2.5 the matrix (18) is unstable.

Example 4.1. Consider the electrical circuit shown in Figure 1.

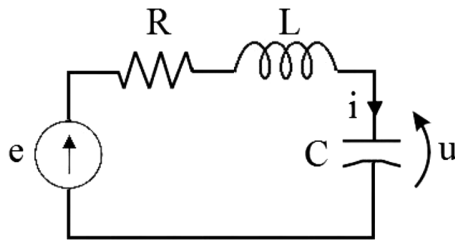


Fig. 1. Electrical circuit of Example 4.1

As the state variable we choose the voltage u on the capacitor with a given capacitance, C , and the current i in the coil with a given inductance L . Using Kirchhoff's laws we obtain the equations:

$$e = Ri + L \frac{di}{dt} + u, \tag{20a}$$

$$i = C \frac{du}{dt}, \tag{20b}$$

which can be written in the form:

$$\frac{d}{dt} \begin{bmatrix} u \\ i \end{bmatrix} = A \begin{bmatrix} u \\ i \end{bmatrix} + Be, \tag{21a}$$

where

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}. \tag{21b}$$

Note that the matrix A has the generalized Frobenius form.

For $R = 0$ the matrix has the form:

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \quad (22)$$

and it is unstable.

For state-feedback matrix $K = [k_1 \ k_2]$ we obtain:

$$A_C = A + BK = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{L} \\ \frac{1}{L} & 0 \end{bmatrix} [k_1 \ k_2] = \begin{bmatrix} 0 & \frac{1}{C} \\ \frac{1}{L}(k_1 - 1) & \frac{k_2}{L} \end{bmatrix} \in M_2 \quad (23)$$

for $k_1 \geq 1$. In this case the matrix (23) is unstable since the characteristic polynomial

$$\det[I_2s - A_C] = \det \begin{bmatrix} s & -\frac{1}{C} \\ \frac{1}{L}(1 - k_1) & s - \frac{k_2}{L} \end{bmatrix} = s^2 - \frac{k_2}{L}s + \frac{1 - k_1}{LC} \quad (24)$$

for $k_1 \geq 1$ has nonpositive coefficient $\frac{1 - k_1}{LC}$.

The problem under the consideration can be divided into the following two subproblems:

Subproblem 1. Given the matrix $A \in \mathfrak{X}^{n \times n}$ with some negative off-diagonal entries and $B \in \mathfrak{X}_+^{n \times m}$ find a gain matrix $K_1 \in \mathfrak{X}^{m \times n}$ of the state-feedback such that

$$A_{C1} = A + BK_1 \in M_n. \quad (25)$$

Subproblem 2. Given $A_{C1} \in M_n$ and $B \in \mathfrak{X}_+^{n \times m}$ find a gain matrix $K_2 \in \mathfrak{X}^{m \times n}$ of the state-feedback such that the matrix

$$A_{C2} = A_{C1} + BK_2 \in M_n \quad (26)$$

is asymptotically stable (is Hurwitz matrix).

The following theorem gives a necessary condition for the existence of the solution of Subproblem 1.

Theorem 4.2. There exists a gain matrix $K_1 \in \mathfrak{X}^{m \times n}$ such that (25) holds only if for each row with at least one off-diagonal negative entry of A the corresponding row of B is nonzero.

Proof. Let the i -th row of A have at least one off-diagonal negative entry and the corresponding i -th row of B is zero. Then from (25) it follows that by suitable choice of K we are not able to eliminate this negative entry of A .

Example 4.2. Consider the electrical circuit shown in Figure 2 with a given resistance R , inductances L_1, L_2 , capacitance C and source voltage e .

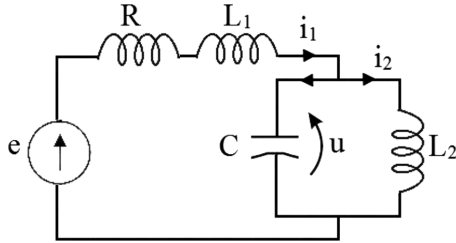


Fig. 2. Electrical circuit of Example 4.2

As the state variables we choose the currents i_1 , i_2 and voltage u . Using Kirchhoff's laws we can write the equations:

$$\begin{aligned} e &= Ri_1 + L_1 \frac{di_1}{dt} + u, \\ u &= L_2 \frac{di_2}{dt}, \\ i_1 &= i_2 + C \frac{du}{dt}, \end{aligned} \quad (27)$$

which can be written in the form:

$$\frac{d}{dt} \begin{bmatrix} u \\ i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} u \\ i_1 \\ i_2 \end{bmatrix} + Be, \quad (28a)$$

where

$$A = \begin{bmatrix} 0 & \frac{1}{C} & -\frac{1}{C} \\ -\frac{1}{L_1} & -\frac{R}{L_1} & 0 \\ \frac{1}{L_2} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L_1} \\ 0 \end{bmatrix}. \quad (28b)$$

Note that in the first row of A one entry is negative $\left(-\frac{1}{C}\right)$ and the first row of B is zero. Therefore, by Theorem 4.2 does not exist a gain matrix $K_1 \in \mathfrak{R}^{1 \times 3}$ such that (25) holds.

Example 4.3. Consider the electrical circuit shown in Figure 3 with known resistances R_1 , R_2 , R_3 , inductances L_1 , L_2 , capacitances C_1 , C_2 and source voltages e_1 , e_2 .

Using Kirchhoff's laws we may write the equations:

$$\begin{aligned} i_1 &= C_1 \frac{du_1}{dt}, & i_2 &= C_2 \frac{du_2}{dt}, \\ e_1 &= (R_1 + R_2)i_1 - R_3i_2 + L_1 \frac{di_1}{dt} + u_1, \\ e_2 &= (R_2 + R_3)i_2 - R_3i_1 + L_2 \frac{di_2}{dt} + u_2, \end{aligned} \quad (29)$$

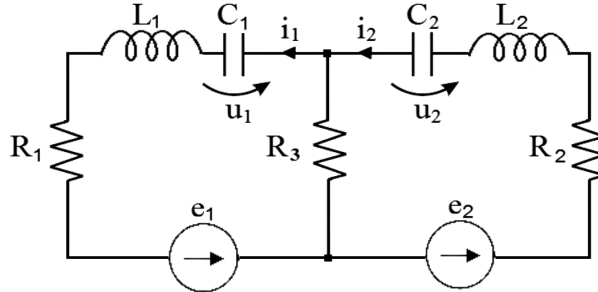


Fig. 3. Electrical circuit of Example 4.3

which can be written in the form:

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \\ i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \\ i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad (30a)$$

where

$$A = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ -\frac{1}{L_1} & 0 & -\frac{R_1 + R_3}{L_1} & \frac{R_3}{L_1} \\ 0 & -\frac{1}{L_2} & \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \quad (30b)$$

Note that the third and fourth rows of the matrix A contain negative off-diagonal entries but the corresponding rows of B contain nonzero entries. Therefore, the necessary condition of Theorem 4.2 is satisfied.

Let

$$A_{C1} = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ a_{31} & a_{32} & -a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & -a_{44} \end{bmatrix} \in M_4 \quad \text{for } a_{ij} \geq 0, \quad i = 3, 4, \quad j = 1, 2, 3, 4 \quad (31)$$

then from (25) omitting the first zero rows we obtain:

$$\begin{bmatrix} a_{31} + \frac{1}{L_1} & a_{32} & \frac{R_1 + R_3}{L_1} - a_{33} & a_{34} - \frac{R_3}{L_1} \\ a_{41} & a_{42} + \frac{1}{L_2} & a_{43} - \frac{R_3}{L_2} & \frac{R_2 + R_3}{L_2} - a_{44} \end{bmatrix} = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} K_1 \quad (32)$$

and

$$K_1 = \begin{bmatrix} 1 + L_1 a_{31} & L_1 a_{32} & R_1 + R_3 - L_1 a_{33} & L_1 a_{34} - R_3 \\ L_2 a_{41} & 1 + L_2 a_{42} & L_2 a_{43} - R_3 & R_2 + R_3 - L_2 a_{44} \end{bmatrix}. \quad (33)$$

Note that the matrix (31) is unstable for all $a_{ij} \geq 0, i = 3, 4, j = 1, 2, 3, 4$.

Theorem 4.3. By suitable choice of the matrix $K_1 \in \mathfrak{R}^{m \times n}$ it is possible to modify not more than $q = \text{rank} B$ rows of the matrix A .

Proof. By the Kronecker-Capelli theorem the equation

$$A_{C1} - A = BK_1 \quad (34)$$

has a solution K_1 if and only if

$$\text{rank}[A_{C1} - A \ B] = \text{rank} B. \quad (35)$$

Therefore, the maximal number of rows of the matrix A , which can be modified by suitable choice of K_1 is $q = \text{rank} B$.

For example in Example 4.3 the rank of the matrix B given by (30b) is two and by suitable choice of the matrix K_1 only two rows of A have been modified.

Example 4.4. Consider the electrical circuit shown in Figure 4 with given resistances $R_k, k = 1, 2, 3$, inductances L_1, L_2 and source voltage e .

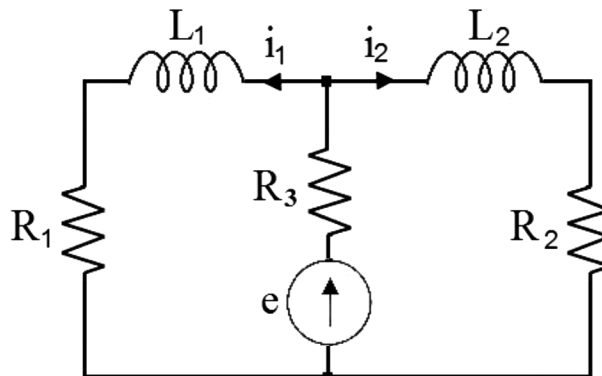


Fig. 4. Electrical circuit of Example 4.4

Using Kirchhoff's laws we may write the equations:

$$\begin{aligned} e &= R_3(i_1 + i_2) + L_1 \frac{di_1}{dt} + R_1 i_1, \\ e &= R_3(i_1 + i_2) + L_2 \frac{di_2}{dt} + R_2 i_2, \end{aligned} \quad (36)$$

which can be written in the form:

$$\frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + Be, \quad (37a)$$

where

$$A = \begin{bmatrix} -\frac{R_1 + R_3}{L_1} & -\frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} \\ \frac{1}{L_2} \end{bmatrix}. \quad (37b)$$

The matrix A is asymptotically stable but it is not the Metzler matrix. It will be shown that the gain matrix

$$K = [k_1 \ k_2] \quad (38)$$

of the state-feedback can be chosen so that the closed-loop electrical circuit will be positive and asymptotically stable.

Using (37b) and (38) we obtain:

$$\begin{aligned} A_C = A + BK &= \begin{bmatrix} -\frac{R_1 + R_3}{L_1} & -\frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ \frac{1}{L_2} \end{bmatrix} [k_1 \ k_2] \\ &= \begin{bmatrix} \frac{k_1 - (R_1 + R_3)}{L_1} & \frac{k_2 - R_3}{L_1} \\ \frac{k_1 - R_3}{L_2} & \frac{k_2 - (R_2 + R_3)}{L_2} \end{bmatrix}. \end{aligned} \quad (39)$$

Note that the matrix (39) is a Metzler matrix if and only if

$$k_1 \geq R_3 \quad \text{and} \quad k_2 \geq R_3. \quad (40)$$

If the condition (40) is satisfied then the closed-loop electrical circuit is asymptotically stable since the coefficients of the characteristic polynomial

$$\begin{aligned} \det[I_2s - A_C] &= \begin{vmatrix} s + \frac{R_1 + R_3 - k_1}{L_1} & \frac{R_3 - k_2}{L_1} \\ \frac{R_3 - k_1}{L_2} & s + \frac{R_2 + R_3 - k_2}{L_2} \end{vmatrix} \\ &= s^2 + \frac{(R_1 + R_3 - k_1)L_2 + (R_2 + R_3 - k_2)L_1}{L_1L_2} s \\ &\quad + \frac{R_1R_2 + R_1(R_3 - k_2) + R_2(R_3 - k_1)}{L_1L_2} \end{aligned} \quad (41)$$

are positive.

In the analysis of Subproblem 2 the crucial role plays the following theorem [10]:

Theorem 4.4. The positive electrical circuit (1a) with $A \in M_n$, $B \in \mathfrak{R}_+^{n \times m}$ is asymptotically stable if and only if there exists a vector $v \in \mathfrak{R}^n$ with positive components $v^T = [v_1 \ \cdots \ v_n]$, $v_k > 0$, $k = 1, \dots, n$ such that

$$Av < 0. \quad (42)$$

From Theorem 4.4 it follows that the positive electrical circuit is unstable if at least one diagonal entry of $A \in M_n$ is not negative.

The following theorem gives a necessary condition for the stabilization by the state-feedbacks of the positive electrical circuits.

Theorem 4.5. The positive electrical circuit with $A \in M_n$ and $B \in \mathfrak{R}_+^{n \times m}$ can be stabilized by a state-feedback matrix $K \in \mathfrak{R}^{m \times n}$ only if for all rows of A with nonnegative diagonal entries the corresponding rows of B are nonzero.

Proof. Note that if the k -th row of the matrix B is zero then the k -th diagonal entry of the closed-loop matrix $A_C = A + BK$ is equal to the diagonal entry of A . In this case the positive electrical circuit cannot be stabilized by the state-feedback matrix K .

Example 4.5. (continuation of Example 4.3) The matrix (31) is an unstable Metzler matrix since its two diagonal entries are zero. The matrix cannot be stabilized by state-feedback since the first two rows of the matrix B (defined by (30b)) are zero.

To find the state-feedback matrix $K \in \mathfrak{R}^{m \times n}$ for a given $A = [a_{ij}] \in M_n$ and $B = [b_1 \ \cdots \ b_n]^T \in \mathfrak{R}_+^{n \times m}$ we may use the following procedure:

Procedure 4.1.

Step 1. Knowing A and B we find

$$d = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} > 0 \quad \text{and} \quad N_k \in \mathfrak{R}^m, \quad k = 1, \dots, n, \quad N = \sum_{k=1}^n N_k, \quad (43)$$

such that

$$Ad + BN < 0 \quad (44)$$

and

$$a_{ij}d_j + b_iN_j \geq 0 \quad \text{for} \quad i \neq j, \quad i, j = 1, \dots, n. \quad (45)$$

Step 2. Knowing d and $N_k, k = 1, \dots, n$ we find

$$K = [N_1 \ \cdots \ N_n] \begin{bmatrix} d_1^{-1} \\ \vdots \\ d_n^{-1} \end{bmatrix} = [N_1d_1^{-1} \ \cdots \ N_nd_n^{-1}], \quad (46)$$

Proof. From (44) we have $BKd = BN$ and by Theorem 4.4 the electrical circuit is asymptotically stable if and only if

$$(A + BK)d < 0 \quad \text{for} \quad d > 0. \quad (47)$$

The matrix $A + BK \in M_n$ if and only if

$$a_{ij} + b_iN_jd_j^{-1} = a_{ij} + b_iN_j = (A + BK)_{ij} \geq 0 \quad \text{for} \quad i \neq j, \quad (48)$$

where $(A + BK)_{ij}$ denotes the (i, j) entry of the matrix $A + BK$.

The procedure can be solved as a standard linear programming problem [2].

Example 4.6. Consider the electrical circuit shown in Figure 5 with a given resistance R , inductance L , capacitance C and source voltage e .

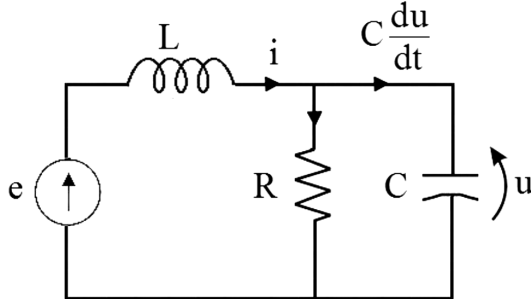


Fig. 5. Electrical circuit of Example 4.6

Using Kirchhoff's laws we may write the equations:

$$\begin{aligned} e &= L \frac{di}{dt} + u, \\ i &= \frac{u}{R} + C \frac{du}{dt}, \end{aligned} \quad (49)$$

which can be written in the form:

$$\frac{d}{dt} \begin{bmatrix} u \\ i \end{bmatrix} = A \begin{bmatrix} u \\ i \end{bmatrix} + Be, \quad (50a)$$

where

$$A = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \in \mathfrak{R}_+^2. \quad (50b)$$

Note that the matrix A has the generalized Frobenius form. Applying the state-feedback

$$e = K \begin{bmatrix} u \\ i \end{bmatrix}, \quad K = [k_1 \ k_2] \quad (51)$$

to the electrical circuit we obtain:

$$A_C = A + BK = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} [k_1 \ k_2] = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ \frac{k_1 - 1}{L} & \frac{k_2}{L} \end{bmatrix}. \quad (52)$$

From (52) it follows that for $k_1 \geq 1$ and $k_2 < 0$ the closed-loop electrical circuit is positive and asymptotically stable since the diagonal entries are negative and off-diagonal entries are nonnegative.

5. Positive electrical circuits with given stable state matrices

Consider the positive electrical circuits described by (1a) with state feedbacks

$$u(t) = Kx(t), \quad (53)$$

where $K \in \mathfrak{R}^{m \times n}$.

Find the matrix K such that the closed-loop electrical circuit with the matrix

$$A_c = A + BK \in M_n \quad (54)$$

is asymptotically stable.

Theorem 5.1. Let the closed-loop state matrix (54) be asymptotically stable. Then there exists a state-feedback matrix K such that the matrix (54) is asymptotically stable if and only if $m = n$ and

$$\det B \neq 0. \quad (55)$$

Proof. By the Kronecker-Cappelli theorem the equation

$$BK = A_c - A \quad (56)$$

has a solution K if and only if

$$\text{rank}[B \ A_c - A] = \text{rank} B \quad (57)$$

and the condition (57) is satisfied for any A_c (and $A_c - A$) if and only if the condition (55) is satisfied.

Example 5.1. Consider the electrical circuit shown in Fig. 6 with given resistances R_1 , R_2 , inductance L , capacitance C and source voltages e_1 , e_2 .

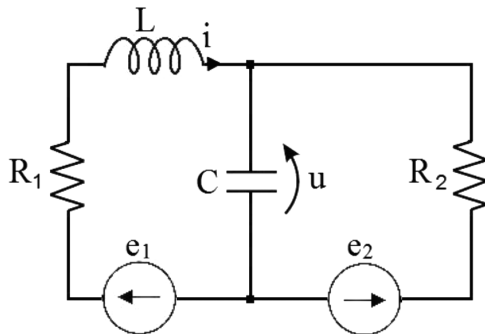


Fig. 6. Electrical circuit of Example 5.1

Using Kirchhoff's laws for the electrical circuit we may write the equations:

$$\begin{aligned} e_1 &= R_1 i + L \frac{di}{dt} + u, \\ R_2 C \frac{du}{dt} - R_2 i + u &= e_2 \end{aligned} \quad (58)$$

which can be written in the form:

$$\frac{d}{dt} \begin{bmatrix} i \\ u \end{bmatrix} = A \begin{bmatrix} i \\ u \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad (59a)$$

where

$$A = \begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_2 C} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{R_2 C} \end{bmatrix}. \quad (59b)$$

Note that the matrix A given by (59b) is not a Metzler matrix since its off-diagonal entry $\left(-\frac{1}{L}\right)$ is negative.

Find a state-feedback matrix K such that the closed-loop circuit is positive and asymptotically stable with the matrix

$$A_c = \begin{bmatrix} -a_1 & 0 \\ 0 & -a_2 \end{bmatrix} \in M_2, \quad a_k > 0 \quad \text{for } k = 1, 2. \quad (60)$$

By Theorem 5.1 the problem has a solution since the matrix B given by (59b) is nonsingular. In this case using (56) and (60) we obtain:

$$\begin{aligned} K = B^{-1}[A - A_c] &= \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{R_2 C} \end{bmatrix}^{-1} \begin{bmatrix} a_1 - \frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & a_2 - \frac{1}{R_2 C} \end{bmatrix} \\ &= \begin{bmatrix} a_1 L - R_1 & -1 \\ R_2 & a_2 R_2 C - 1 \end{bmatrix}. \end{aligned} \quad (61)$$

6. Concluding remarks and open problems

The analysis of the positivity and stability of linear electrical circuits by the use of state-feedbacks has been investigated. The notion of Frobenius matrices has been extended and the properties of the generalized Frobenius matrices have been analyzed. It has been shown that if the state matrix of a linear electrical circuit has the generalized Frobenius form then the closed-loop electrical circuit is not positive and asymptotically stable (Theorem 4.1). Different necessary conditions for positivity and stabilization by state-feedbacks of linear electrical circuits have been established (Theorems 4.2, 4.5). Necessary and sufficient conditions for the existence of state-feedback such the closed-loop electrical circuit is positive and asymptotically stable have been also given (Theorem 5.1). Procedure for computation of the state-feedback matrix stabilizing the closed-loop electrical circuit has been proposed (Procedure 4.1). The considerations have been illustrated by examples of the linear electrical circuits.

Open problems:

1. Find a class or classes of linear electrical circuits of order $n > 2$ with state matrices of the forms of the generalized Frobenius matrices. In this paper only such electrical circuits of order $n = 2$ (Examples 4.1 and 4.6) are presented.
2. Find a class or classes of positive linear electrical circuits of order $n > 2$ with Metzler state matrices.
3. It is easy to show that the state matrix (50b) of the electrical circuit shown in Figure 5 is the same, if the coil and the capacitor are interchanged. Find a class or classes of linear electrical circuits with this feature for $n > 2$.

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