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INVITED PAPER

Mathematical modeling of traveling autosolitons in fractional-order activator-inhibitor systems

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Abstract. In the article, basic properties of traveling spatially nonhomogeneous auto-wave solutions in nonlinear fractional-order reaction-diffusion systems are investigated. Such solutions, called autosolitons, arise in a stability region of the system and can coexist with the spatially homogeneous states. By a linear stability analysis and computer simulation, it is shown that the order of the fractional derivative can substantially change the properties of such auto-wave solutions and significantly enrich nonlinear system dynamics. The results of the linear stability analysis are confirmed by computer simulations of the generalized fractional van der Pol-FitzHugh-Nagumo model. A common picture of traveling auto-waves including series in time-fractional two-component activator-inhibitor systems is presented. The results obtained in the article for the distributed system have also been of interest for nonlinear dynamical systems described by fractional ordinary differential equations.

Key words: fractional derivative, nonlinear dynamics, instability, autosoliton.

1. Introduction

In recent years, the interest in studying dynamical mathematical models with fractional derivatives has been increasing. This interest is mainly determined by the attempts to understand phenomena in fractal, irregular and hereditary media [1–8]. Among the fractional differential models, much attention has been given to the fractional reaction-diffusion systems (RDS) [9–14]. On the basis of mathematical modeling of the standard RDS, a lot of noteworthy nonlinear self-organization phenomena in physical, biological and chemical systems have been explained [15–18]. Moreover, the investigations of the spatio-temporal order in such nonlinear systems and mechanisms of pattern formation are a top-priority theme of research studies in many modern technological applications [17–19]. Due to this fact, studying such systems with fractional derivatives is important both from the scientific perspective and from the point of view of its applications.

By now, for nonlinear time-fractional activator-inhibitor RDS the spatially-temporal nonhomogeneous solutions, which spontaneously arise in fractional RDS (FRDS) as a result of instability of spatially homogeneous states of the system, were investigated [20–23]. In this case, it was shown that the fractional systems can demonstrate much more complex nonlinear dynamics than the classical RDS and the fractional derivative order is an additional bifurcation parameter which can qualitatively change the properties of spatially homogeneous [20, 22] as well as nonhomogeneous [21, 24] steady solutions.

It is also well known that in monostable nonequilibrium systems, stable localized spatially nonhomogeneous states, called autosolitons, may arise [17, 18]. An autosoliton is a steady solitary inhomogeneous state of the dissipative systems which can be excited by an external localized impulse of sufficiently large amplitude in domain where homogeneous state is stable. The characteristics of autosoliton (shape, amplitude, velocity, etc) depend entirely on the parameters of the system, and do not depend on the properties of the initial perturbation gave rise to it. Similarly to periodic auto-oscillations, which correspond to stable limit cycles in phase space of dynamic variables, autosolitons correspond to attractors in the configuration space each point of it is associated with certain function which describe distributions of the parameters of the system with respect to coordinates.

The nature of autosolitons is extremely diverse [16–18]: electric impulses running in a nerve fiber, solitary strata of current density in gas discharge or semiconductor systems, complex chemical concentration autowaves in self-catalyzed chemical reactions, etc. These phenomena on the base of classical RDS have been studied for many years. For fractional RDS, such investigations have only started [25].

In the present article, we investigate the traveling autowave solutions (signals in fractional distributed systems) in a time-fractional RDS with positive and negative feedbacks. As a base model, the generalized van der Pol-FitzHugh-Nagumo (FHN) one with time-fractional derivatives is considered. The purpose of this paper is to study the properties of travelling solitary auto-wave solutions in this basic mathematical model and to establish their main characteristic features for fractional RDS of the general type.

2. Mathematical model

Let us consider the following FRDS called generalized van der Pol-FitzHugh-Nagumo model with time-fractional deriv-

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atives expressed, for convenience, in a time non-dimensional form [16–18]:

$$\varepsilon\frac{\partial^\alpha u}{\partial t^\alpha} = l^2\frac{\partial^2 u}{\partial x^2} + W(u, v, \mathrm{A}), \tag{1}$$

$$\frac{\partial^\alpha v}{\partial t^\alpha} = L^2\frac{\partial^2 v}{\partial x^2} + Q(u, v, \mathrm{A}), \tag{2}$$

where $u = u(x,t)$ is a fast variable called the activator variable and $v = v(x,t)$ is a slow variable called the inhibitor, $\varepsilon = \tau_u/\tau_v$, τ_u, τ_v and l,L are characteristic times and diffusion lengths of the system, $0 \le x \le \mathscr{L}_x$, $\mathscr{L}_x$ is the system length, A is an external parameter. When the inhibitor variable diffusion length is zero and $\alpha = 1$, the system (1), (2) corresponds to classical FitzHugh-Nagumo (FHN) model [16, 17].

In this case, the source term for the activator variable is nonlinear and it is linear for the inhibitor [16, 17]:

$$W(u,v) = u - u^3/3 - v, \qquad Q(u,v,\mathrm{A}) = -v + \beta u + \mathrm{A}. \tag{3}$$

The time derivatives $\partial^\alpha u/\partial t^\alpha$, $\partial^\alpha v/\partial t^\alpha$ on the left-hand side of equations (1) and (2) are the Caputo fractional derivatives in time of the order $0 < \alpha < 2$ and are represented as

$$\frac{\partial^\alpha}{\partial t^\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \tag{4}$$

where $m - 1 < \alpha < m$, $m = 1, 2$ [26, 27].

The system (1), (2) is subject to Neumann:

$$\partial u/\partial x|_{x=0,\mathscr{L}_x} = \partial v/\partial x|_{x=0,\mathscr{L}_x} = 0, \tag{5}$$

or periodic

$$\begin{aligned} u|_{x=0} &= u|_{x=\mathscr{L}_x}, & \partial u/\partial x|_{x=0} &= \partial u/\partial x|_{x=\mathscr{L}_x}, \\ v|_{x=0} &= v|_{x=\mathscr{L}_x}, & \partial v/\partial x|_{x=0} &= \partial v/\partial x|_{x=\mathscr{L}_x}, \end{aligned} \tag{6}$$

boundary conditions. At these boundary conditions the system is maximally autonomous and allows coexistence of spatially homogeneous and nonhomogeneous solutions [15, 17]. The initial conditions will be taken according to the purpose of each particular simulation.

Due to properties of Caputo derivative, the fractional system (1), (2) at boundary conditions (5), (6) has the same spatially homogeneous stationary solutions as the standard one ($\alpha = 1$). They can be obtained from the algebraic system

$$W(u,v,\mathrm{A}) = 0, \qquad Q(u,v,\mathrm{A}) = 0 \tag{7}$$

(intersection point of two null-clines, see Fig. 1). Nevertheless, the stability and dynamical properties of these solutions strongly depend on the order of the fractional derivative.

3. Linear stability analysis

The system (1), (2) can be linearized with respect to these spatially homogeneous and stationary solutions

$$\tau_u\frac{\partial^\alpha \delta u(x,t)}{\partial t^\alpha} = l^2\frac{\partial^2 \delta u(x,t)}{\partial x^2} + W_u'\delta u(x,t) + W_v'\delta v(x,t), \tag{8}$$

$$\tau_v\frac{\partial^\alpha \delta v(x,t)}{\partial t^\alpha} = L^2\frac{\partial^2 \delta v(x,t)}{\partial x^2} + Q_u'\delta u(x,t) + Q_v'\delta v(x,t). \tag{9}$$

The linearized system (8), (9) allows to determine the spectrum of small perturbations $\delta u(x,t)$, $\delta v(x,t)$ and to estimate the asymptotic behavior of the solution for great values of t.

Using the formula for the Fourier transform of fractional derivative [3, 27]

$$\int_0^\infty e^{-st}{}_0D_t^\alpha f(t)dt = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(m)}(0) \tag{10}$$

for small perturbations $\delta u(x,t) \sim \exp(ikx)$, $\delta v(x,t) \sim \exp(ikx)$ we obtain linear system of algebraic equations relative to Fourier transform of these perturbations, which has nontrivial solution when its determinant is equal to zero. The characteristic equation is quadratic polynomial relative to the power α leading to the next two roots:

$$(i\omega)^\alpha_\pm = \gamma(k) \pm i\omega_0(k), \tag{11}$$

where $\gamma(k) = trF(k)/2$, $\omega_0(k) = \sqrt{\det F(k) - tr^2F(k)/4}$. Here the linearized right-hand side of the system (8), (9) is given by the matrix

$$F(k) = \begin{pmatrix} (a_{11}(k)/\tau_u) & a_{12}/\tau_u \\ a_{21}/\tau_v & a_{22}(k)/\tau_v \end{pmatrix} \tag{12}$$

with

$$\begin{aligned} a_{11}(k) &= a_{11} - k^2l^2, & a_{22}(k) &= a_{22} - k^2L^2, \\ a_{11} &= W_u', & a_{12} &= W_v', \\ a_{21} &= Q_u', & a_{22} &= Q_v'. \end{aligned} \tag{13}$$

All derivatives are taken at homogeneous equilibrium state $(\overline{u}, \overline{v})$:

$$W(\overline{u},\overline{v}) = Q(\overline{u},\overline{v}) = 0, \qquad k = \pi j/\mathscr{L}_x, \qquad j = 1, 2, \ldots\ .$$

In a standard RDS ($\alpha = 1$) we obtain that for $k = 0$ at conditions

$$trF(0) > 0, \qquad \det F(0) > 0 \tag{14}$$

the equilibrium point, which corresponds to the null-cline intersection, is unstable and the system possesses spatially homogeneous oscillations due to Hopf bifurcation (a limit cycle in the case of local system ($l = L = 0$)). For the fractional one the situation is different [28, 29]. The instability conditions for homogeneous state depend on fractional order of the system and are determined by critical value [21]

$$\alpha_0 = \begin{cases} \left(\dfrac{2}{\pi}\right)\arctan\sqrt{\dfrac{4\det F}{tr^2F} - 1}, & trF > 0, \\ 2 - \left(\dfrac{2}{\pi}\right)\arctan\sqrt{\dfrac{4\det F}{tr^2F} - 1}, & trF < 0. \end{cases} \tag{15}$$

As a result, the stability region of the system is significantly other from the one determined by conditions (14) (for details, see [21]). It is defined for all values $\alpha_0 \in (0,2)$, and having the value of α_0, we can determine other system parameters, which together with the order of the fractional derivative define the region, where the homogeneous state $(\overline{u}, \overline{v})$ is unstable.

For conditions

$$trF(0) < 0, \qquad \det F(0) > 0 \tag{16}$$

the equilibrium point of the system is stable and corresponds to a stable focus or a node. In this case at sufficiently small $\varepsilon < 1$ and $L = 0$ (or $l/L \gg 1$) in standard RDS we can observe a traveling solitary wave solutions. Due the theorem of uniqueness and existence [27] fractional systems remain the properties their integer analogues in an appropriate range of the system parameters, including the order of fractional derivative. Therefore we can expect that such type solutions can also exist in fractional RDS.

The basic properties of static auto-wave solutions in FRDS are mainly determined by the spectrum of eigenvalues of the linearized system [23]. The traveling auto-waves have an additional characteristic – velocity of propagation. For estimation of it dependence on value of fractional order we will consider a group velocity of small perturbations. In this case at $|trF(k)| \ll 1$, the linearized dynamical system for amplitudes of the plane waves can be presented by the following differential equation

$$\ddot{\rho} - \dot{\rho} \cdot trF(k) + \rho \cdot \det F(k) = 0, \tag{17}$$

which describes a damping oscillator with damping factor $trF(k)/2$ and oscillatory frequency of about $\omega_0 = (\det F)^{1/2}$. In fact, the solution of this equation has the form

$$\rho = \rho_+ \exp(i\omega_+ t) + \rho_- \exp(i\omega_- t). \tag{18}$$

So, any perturbation in the stable region $trF(k) < 0$ will decay with time $2/|trF(k)|$ and oscillate with a frequency ω_0 which in the case of classical FHN system $(L = 0)$ can be represented as

$$\omega_0 = \left(\det F(0) - \frac{k^2 l^2 a_{22}}{\tau_u \tau_v} - \frac{1}{4}\left(trF(0) - \frac{k^2 l^2}{\tau_u} \right)^2 \right)^{1/2}. \tag{19}$$

Analyzing this term we can conclude that ω_0 depends on k and reaches the maximum value

$$\omega_0^{\max} = \left[\det F(0) - a_{11}a_{22} \,/\, \tau_u \tau_v \right]^{1/2}, \tag{20}$$

at

$$k_0^2 = \frac{\tau_u}{l^2}(trF(0) - 2a_{22}/\tau_v). \tag{21}$$

In this case, for a standard system with integer derivatives, the group velocity of the small perturbations in the form of plane waves is the following

$$V_g = \frac{d\omega_0(k)}{dk}. \tag{22}$$

Using the same considerations as for the standard system, we can obtain the tendency of the group velocity of small perturbations in the case of fractional system. Formally calculating this velocity for noninteger α we will have following relationship:

$$c_g^{\pm} = Re\left[\frac{\omega_{\pm}^{1-\alpha}(k)}{\alpha i^{\alpha-1}} \frac{d\omega_{\pm}(k)}{dk} \right] \tag{23}$$

or

$$c_g \simeq \frac{\omega(k)}{\alpha\omega_0(k)} \frac{d\omega_0(k)}{dk} = \frac{\omega(k)}{\alpha\omega_0(k)} V_g, \tag{24}$$

where $\omega(k)$ is determined by equation (11). Since the coefficient before the standard group velocity is inversely proportional to α, we can expect that the group velocity c_g will increase with decreasing α. This means that order of fractional derivative can change not only the stability and static properties of spatially-nonhomogeneous solutions but also their dynamical properties.

The aim of next section is to demonstrate how the fractional derivative order in a generalized van der Pol-FitzHugh-Nagumo system changes the main characteristics and nonlinear dynamics of the solitary traveling waves in comparison with standard systems.

4. Traveling pulses: qualitative analysis and computer simulation

First, we consider the standard van der Pol-FitzHugh-Nagumo model $(\alpha = 1)$. Calculation of the coefficients a_{ij} of matrix (12) for sources (3):

$$a_{11} = (1 - \overline{u}^2), \qquad a_{12} = -1, \qquad a_{21} = \beta, \qquad a_{22} = -1$$

makes it possible to qualitatively analyze the formation of a traveling pulse in FHN model and to explicitly obtain the main dependencies for this analysis. The linear analysis shows that at $\tau_u/\tau_v > 1$ the steady state solution corresponding to any intersections of null-clines is stable. Formally, the condition $trF > 0$ is reduced to the inequality

$$W_u' > (\tau_u/\tau_v)Q_v',$$

which in this case has a very simple form

$$1 - \overline{u}^2 > \tau_u/\tau_v.$$

The smaller is the ratio of τ_u/τ_v, the wider is the instability region. At $\tau_u/\tau_v \to 0$, the instability region for u coincides with the interval between the extremum points $(u_{\min}, v_{\min}) = (-1, -2/3)$, $(u_{\max}, v_{\max}) = (1, 2/3)$ where the null cline $W(u, v) = 0$ has its increasing part (Fig. 1).

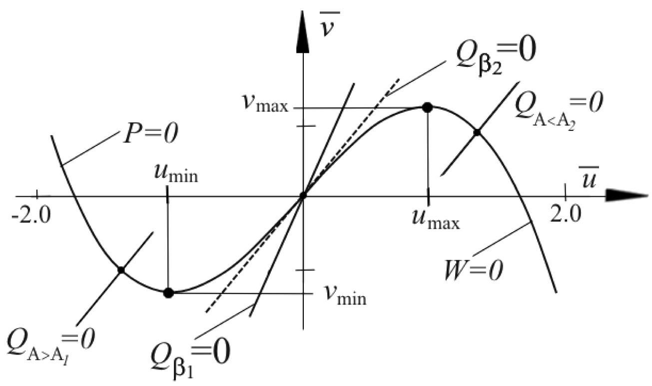

Fig. 1. The null-clines positions for different values of bifurcation parameter A. Results of computer simulation of the system (1), (2) with sources (3) at $\beta_1 = 2.0$ (solid line), $\beta_2 = 1.01$ (dashed line)

The null-cline $Q = 0$ intersects these extremum points at the values of the bifurcation parameter $A_1 = 2/3 - \beta$ and

$A_2 = -2/3 + \beta$ correspondingly (Fig. 1). For the values $A > A_1$ and $A < A_2$ stationary state $(\overline{u}, \overline{v})$ of the system is stable. In other words, decreasing parts of null-cline $P = 0$ correspond to stability region of standard FHN system.

The simple view of nonlinearities (3) allows to determine the intersection point of null-clines by the equation

$$u - u^3/3 - \beta u + A = 0. \tag{25}$$

In this case, the values of external parameters A, β determine the value of u and this makes it possible to consider the variable u as the main parameter for the system analysis and on the basis of formula (15) to build the instability domain for any relation between system parameters. The instability domains for the considered fractional system, in the coordinates $(u, \lg(\tau_u/\tau_v))$ for different values α at a given value β are presented in Fig. 2. For each particular value α in the region between the corresponding curve the system is unstable for homogeneous perturbation with the wave-number $k = 0$, and on the outside – it is stable. The curve corresponding to $\alpha = 1$ is denoted by a thicker line.

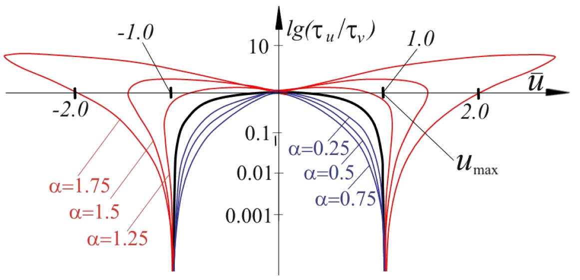

Fig. 2. Instability domains for the local system ($l = L = 0$) in coordinates $(u, \lg(\tau_u/\tau_v))$ for different values α. Results of computer simulation of the system (1), (2) with sources (3) at α changing from 0.25 to 1.75 with step 0.25 for $\beta = 1.01$

The system (1), (2) with the source terms (3) and boundary conditions (5) or (6) was investigated numerically [30–34]. The traveling pulses were initiated by disturbance of the local region $\mathscr{L}_p \ll \mathscr{L}_x$ in initial conditions by certain values $v = v_0 = \overline{v}$, $u = u_0 > \overline{u}$, (Fig. 1). In the rest of the simulation space domain the initial state corresponds to equilibrium point $(\overline{u}, \overline{v})$.

By computer simulation it was established that solitary traveling auto-waves can be excited also in fractional RDS. Moreover, the existence conditions of such solutions, as well as their main characteristics strongly depend on the fractional order of the system and are different than for the classical FHN model.

The results of numerical investigation of the fractional system for different values of system parameters and order of fractional derivative α are presented in Figs. 3–7. The plots on these figures demonstrate that the traveling pulse can exist in a wide range of system parameters, including the order of the fractional derivatives. We can also see that the fractional derivative order qualitatively changes the pulse width and relation between the activator and inhibitor variables. For $\alpha < 1$ the traveling pulse is wider than for $\alpha > 1$ (Figs. 3–5). This trend is of a general nature.

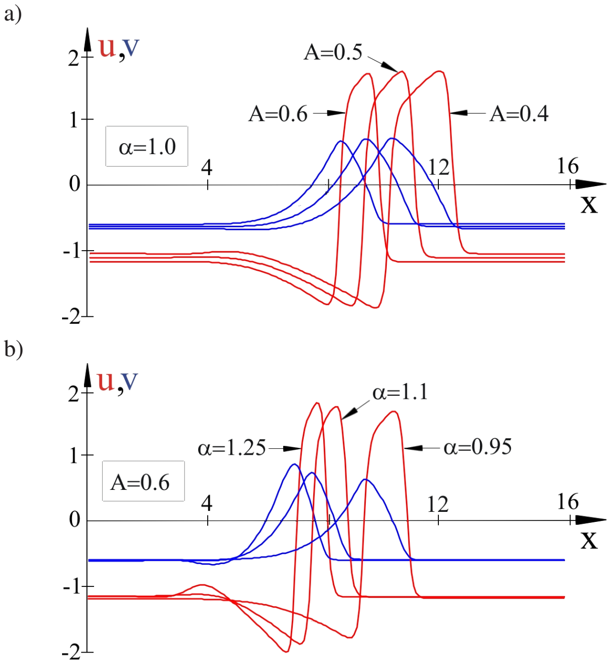

Fig. 3. Distribution of activator variable u (red) and inhibitor variable v (blue) in a traveling pulse. Evolution of pulse shape on dependence from value of bifurcation parameter A ($A_1 = 0.4$, $A_2 = 0.5$, $A = 0.6$) in classical FHN model ($\alpha = 1.0$) – (a). Evolution of pulse shape in fractional FHN system on dependence from order of fractional derivative α ($\alpha_1 = 0.95$, $\alpha_2 = 1.05$, $\alpha_3 = 1.25$) at the same value of bifurcation parameter $A = 0.6$ – (b). The results of computer simulation of the system (1), (2) at boundary condition (5) for the parameters: $\varepsilon = \tau_1/\tau_2 = 0.05$, $l^2 = 0.0125$, $\beta = 1.01$, $L = 0$

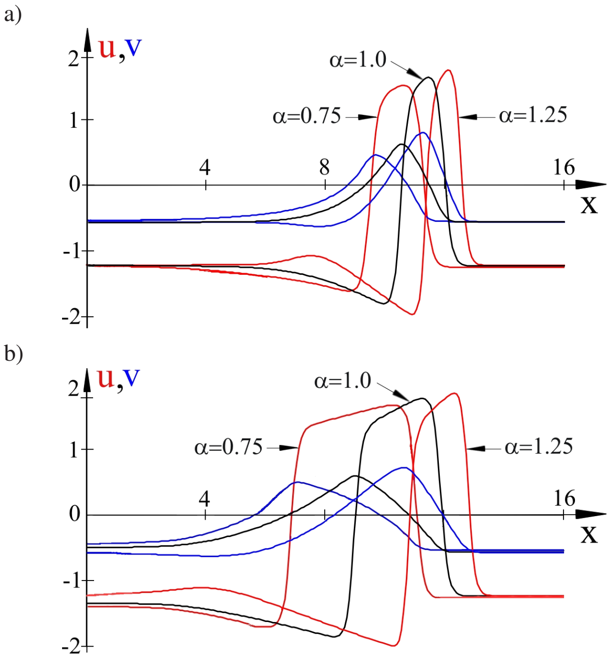

Fig. 4. Evolution of pulse shape on dependence from order of fractional derivative α ($\alpha_1 = 0.75$, $\alpha_2 = 1.0$, $\alpha_3 = 1.25$) for $\varepsilon = \tau_1/\tau_2 = 0.05$ – (a) and for $\varepsilon = \tau_1/\tau_2 = 0.025$ – (b). The results of computer simulation of the system (1), (2) at boundary condition (5) for the parameters: $A = 0.6$, $l^2 = 0.025$, $\beta = 1.01$, $L = 0$

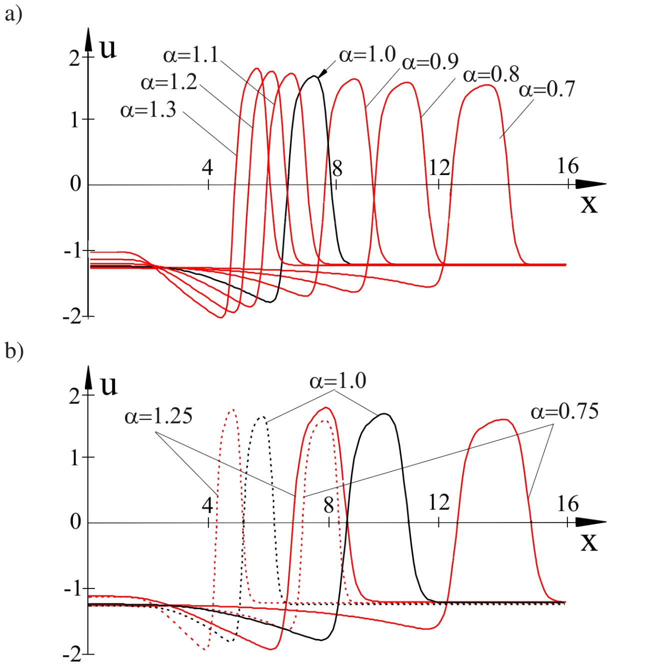

Fig. 5. Disposition of traveling pulses (activator variable) at the same moment of time ($t = 3$) for different values of fractional order α from $\alpha_1 = 0.7$ to $\alpha_7 = 1.3$ with step 0.1 and $l^2 = 0.025$ – (a). Location of the traveling pulses for different values α ($\alpha_1 = 0.75$, $\alpha_2 = 1.0$, $\alpha_3 = 1.25$) and characteristic length l ($l^2 = 0.05$ (solid lines), $l^2 = 0.0125$ (dashed lines) – (b)). The results of computer simulation of the system (1), (2) at boundary condition (5) for the parameters: $\varepsilon = \tau_1/\tau_2 = 0.05$, $A = 0.6$, $\beta = 1.01$

For any value $\varepsilon = \tau_u/\tau_v$ the interval, on which a traveling pulse exists, increases as the fractional derivative order α becomes smaller. It is worth noting that the stable solitary wave solutions in FRDS can exist even in the region where null-clines intersect on the increasing part of $W = 0$. It is not surprising since in contrast to the standard systems the fractional ones for $\alpha < 1$ become stable in the region where $dv/du = -W'_u/W'_v > 0$ [21]. For $\alpha > 1$ the region of existence of solitary traveling waves also changes but becomes smaller. There are two main explanations of this fact. First of all, the instability region for $\alpha > 1$ becomes more extended and the homogeneous state can be unstable even for $|\overline{u}| > 1$ [20, 21]. This means that the traveling pulse exists only for $|\overline{u}|$, which at a certain ratio between α and ε can be significantly greater than unity (Figs. 1, 2). On the other hand, increasing α or A makes the pulse narrower (Fig. 3) and at certain ratio between these parameters it completely vanishes.

In common, the domain of existing stable travelling pulses monotonously changes with the fractional derivative order at given parameters A and ε: for $\alpha < 1$ this domain is the greatest, the standard system ($\alpha = 1$) occupies the intermediate position, and for $\alpha > 1$ this domain is the smallest.

The plots presented in Figs. 3–7 demonstrate that the shape of the traveling pulse varies depending on all the system parameters τ_u, τ_v, β, A, l, as well as the order of fractional derivative. The competition between these system parameters determines the form of the traveling pulse and its velocity.

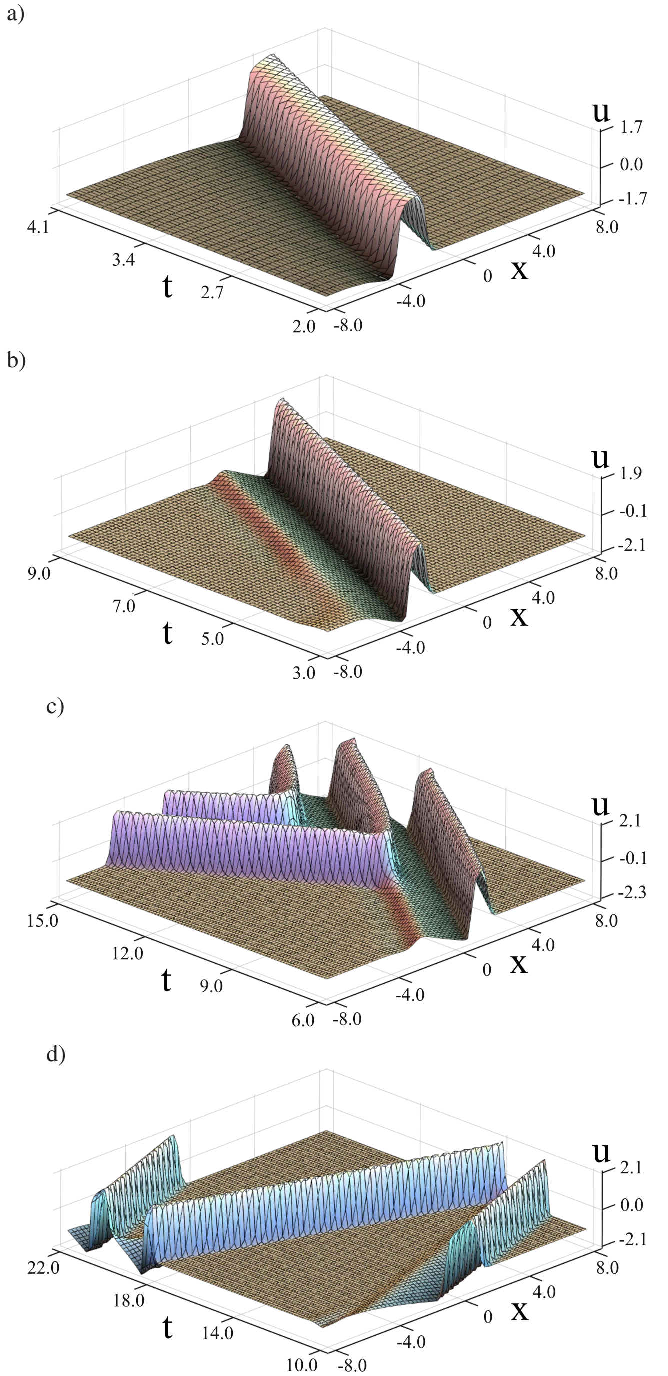

Fig. 6. The traveling pulse solution at ($\alpha = 0.75$) – (a) and ($\alpha = 1.25$) – (b); generation of leading center solution as a result of breakdown in the traveling pulse tail at the increasing fractional derivative order from $\alpha = 1.25$ to $\alpha = 1.3$ – (c); traveling pulse bouncing from the boundaries – (d). The results of computer simulation of the system (1), (2) at boundary condition (5) for the parameters: $\varepsilon = 0.05$, $l^2 = 0.0125$, $A = 0.6$, $L^2 = 0.0$, $\beta = 1.01$ – (a)–(c); $\alpha = 1.175$, $\varepsilon = 0.04$, $l^2 = 0.025$, $L^2 = 0.6$, $\beta = 0.85$, $A = 0.3$ – (d)

The conclusion which can be drawn from computer simulation is that the fractional order is dominant and for bigger values α the pulse has more contrast walls and is narrower. The smaller is value α the wider is the traveling pulse and with smoother walls. The standard system has parameters which are intermediate among the pulses with $\alpha < 1$ and $\alpha > 1$ (Figs. 3–5).

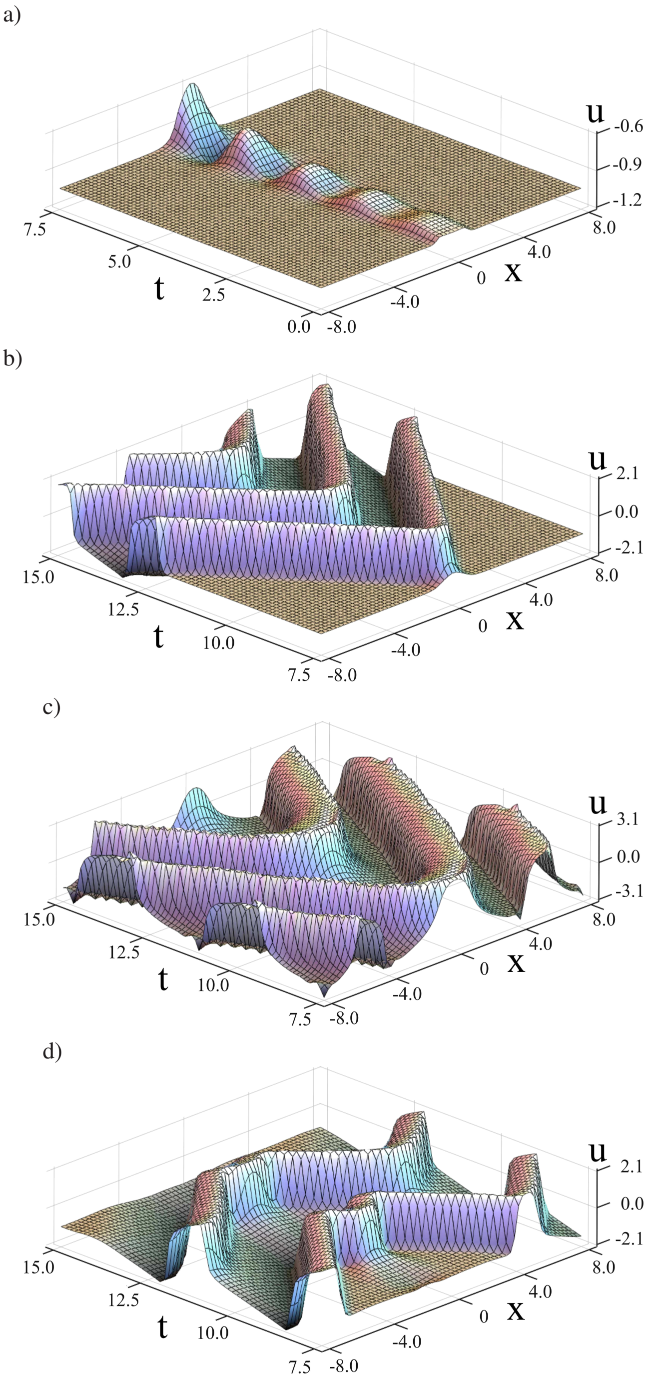

Fig. 7. Leading center solution as a result of a small inhomogeneity at $x = 0$ for $\alpha = 1.3$ – (a)–(b) and for $\alpha = 1.7$ – (c). Complex dynamics as a result of interplay two traveling pulses stimulated by local inhomogenities – (d). The results of computer simulation of the system (1), (2) at boundary condition (5) for the parameters: $\varepsilon = 0.05$, $l^2 = 0.02$, $L^2 = 0.0$, $\beta = 1.01$, $A = 0.4$ – (a)–(c); $\varepsilon = 0.05$, $l^2 = 0.0125$, $L^2 = 0.0$, $\beta = 1.01$, $A = 0.4$ – (d)

In Fig. 5 the positions of the sequence of pulses generated for different α for the same moment of time are presented. We can see that the speed of the pulse with the smallest order α is the highest and for the greatest α it is the smallest. This plot also demonstrates the change of the shape of the pulse by each value of α. The variation in the speed with the fractional order α can be explained by analyzing the group velocity of the traveling pulse. The matter is that these pulses exist in the stable region and the dispersion relation of the stable perturbation can be obtained from (22). In our case for any α for $trF(k) \sim 0$ we have the following dependence

$$c_g^{\pm} \sim Re\left[\frac{\left(i\sqrt{\det F(k)}\right)^{1-\alpha}}{\alpha}\frac{d\omega_{\pm}}{dk}\right]$$
$$\simeq Re\left[\frac{\left(i\sqrt{\beta+\overline{u}-1}\right)^{1-\alpha}}{(\tau_u\tau_v)^{(1-\alpha)/2}\alpha}V_g^{\pm}\right].$$

In other words, the group velocity is a function of the fractional order α. As a result, with decreasing α pulses increase and with increasing α pulses decrease their speed, that is what we see in the computer simulation experiments (Fig. 5a). By this formula we can also estimate the influence of other system parameters on pulse velocity. Figure 5b makes it possible to understand the system dynamics depending on the diffusion length l, which mainly determines the curvature of the front for travelling pulse. Numerical results presented in Fig. 5b demonstrate that the change of fractional order has more significant impact on the speed of pulse in comparison with the variation of the diffusion length.

The discussed above trends in the shape and velocity dependence are of a general nature and are typical for other basis FRDS of activator-inhibitor type. By compute experiments it was established that in fractional activator-inhibitor systems the stability domain of the traveling pulses is adjacent to the instability region of spatially homogeneous states and solitary wave solutions can be easier to generate very close to the bifurcation point of the system. This is true for $\alpha < 1$. In this case the tail of the pulse is monotonic and it reaches the equilibrium point $(\overline{u}, \overline{v})$ without oscillation (Figs. 3 and 4). The more stable is the system with respect to instability condition ($A < A_{cr}$, where A_{cr} is the critical value of bifurcation parameter A for instability of stationary homogeneous state) the narrower the pulse becomes. At $\alpha < 1$ the traveling pulse exists also at an intersection of null-clines on increasing part of the null-cline $W = 0$ and can lead to more complex system dynamics.

Now we consider a traveling pulse shape when the activator variable is in a refractor domain. In classical RDS the maximum in the traveling pulse tail appears only for a complete RDS with L not equal to zero $(L \gg l)$ and it increases by approaching the Turing bifurcation point. In FRDS the tail arises even for an FHN model. At a given bifurcation parameter A the trend of tail behavior is presented in Figs. 3 and 4. The higher is fractional derivative order the narrower is the pulse, and it generates a substantial maximum in the tail. At the same time, the homogeneous state is the same since the main bifurcation parameter A is the same (null-clines intersect at the same point).

This nontrivial phenomenon has a substantial effect for the system dynamics. It was shown that in a standard system it can lead to the formation of a leading center solution (or pacemaker) when the system is close to the instability point (see [35] for details). If the tail maximum amplitude is small enough we have a solitary traveling waves as reported above. In opposite case we can have a variety of pattern formation phenomena. The matter is that the tail part of traveling pulse contains maxima of a fast variable distribution (Figs. 3b, and 4a). As a result, when two pulses are generated at some place, they start to propagate in a different directions and there is a moment when two tail maxima coincide and their total maximum becomes greater than is needed for it to be stable. In this local domain there are two new pulses generated that start running in the opposite directions.

The same idea of a leading center (LC) formation is realized in FRDS. With an new additional parameter – fractional derivative order – we can design a leading center with sufficiently wide system parameters. A local excitation of a stable homogeneous system produces traveling pulses (Fig. 6a,b). When A is close to A_{cr}, the oscillatory distributions in a tail at some moment can lead to LC appearing (Fig. 6b,c). The resulting tail oscillatory distribution at some time moment may raise the value of u above the stability domain value, and the effect of a local breakdown will cause a further abrupt increase in an activator. This can lead to a spontaneous generation of two new pulses in the tail. One of them continues traveling at the same direction while the second one starts running in opposite direction to the opposite boundary (Fig. 6b,c). Two receding traveling pulses give rise to a pair of new receding pulses, and so the process continues. In other words, the point of initial local disturbance becomes a self-contained source of receding pulses. In two and three-dimensional systems this can be a source of diverging radially-symmetric autowaves.

It is a common understanding that the wave reaching the boundary vanishes at the neutral boundary condition (5). For periodic (a cyclic) boundary condition the wave is generated at the opposite boundary and runs in the same direction. It has also been known for many years that when two autowaves collide then, as a rule, they vanish. For FRDS, we establish that autowaves can have a more complex dynamics. The matter is that due to a fractional nature of the system it memorizes some previous system dynamics and tries to reproduce it with a greater success than the standard system. A phenomenon of bouncing autowaves from the boundary is presented in Fig. 6d. When an unique pulse reaches the end of the simulated domain it vanishes at the boundary at a Neumann boundary conditions. At the same time, when the bifurcation parameter is close to the instability point, the pulse vanishes, but its tail generates another pulse which can run in the opposite direction (Fig. 6d). This process will be continuous and can lead to a set of train patterns depending on the fractional derivative order.

The generation of the traveling nonlinear waves presented in Fig. 6 was done by the initial condition or external pulse in some moment of time $t = t_0$. Since we study the system properties close to bifurcation point, the spatial inhomogeneity of the distributed parameters also may play an essential role and the nonlinear regime may emerge spontaneously. Small inhomogeneity may lead the system parameters into an instability domain and, as a result, we can expect a generation of the auto-waves of a large amplitude despite the fact that the system steady state solution is presented inside the stability domain. These properties of activator-inhibitor systems were recognized from standard RDS when a local inhomogeneity can generate traveling or stationary auto-wave solutions [35]. Similar scenarios can take place also in fractional RDS. In Fig. 7a-d a formation of leading center solution in a fractional system, as a result of a small inhomogeneity, is presented (local additional source of a different size and amplitude in the equation (1) was considered).

At the same time, a fractional derivative order can cause a much more complicated dynamics than we meet in standard RDS. Increasing α leads to an enlargement of instability domain for the given values of the system parameters (see Fig. 2) and as a consequence, in fractional RDS a much more complicated nonlinear dynamics appears. In this case, we can have a variety of shape or velocity of traveling pulses during periodic pulse generation (Fig. 7c). Real systems may contain many inhomogeneities and each of it can generate traveling or stationary auto-waves. In this case, an interaction between such solitary autowaves can be also very complex and sometimes it reminds of chaotic dynamics. The result of such interplay of two traveling pulses stimulated by local inhomogeneities which demonstrates the processes of appearing and disappearing of solitary autowave solutions is presented in Fig. 7d. It shows that small local inhomogeneities in two-component fractional activator-inhibitor systems can lead to very complex dynamics which was earlier observed only in a system of bigger dimensionality.

5. Conclusions

In this article we studied nonlinear traveling auto-wave solutions, which appear in the stability region of the time fractional reaction-diffusion systems and can coexist with spatially homogeneous system states. By linear analysis and numerical simulations, it was shown that in FRDS, especially in fractional van der Pol-FitzHugh-Nagumo system, solitary auto-wave solutions can exist in a wide range of system parameters, including the order of the fractional derivatives. It was established that the system dynamics is very sensitive to fractional derivative order α, parameters of the inhomogeneity, bifurcation parameter etc. In many cases the effect of a local breakdown depends on the order of fractional derivative and leads to more complex auto-wave solutions in comparison to classical RDS.

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