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Single-server queueing system with external and internal customers

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Abstract. In the paper, we investigate a single-server queueing system with unlimited memory space and non-homogeneous customers (calls) of the two following types: 1) external customers that are served by the system under consideration, 2) internal customers that arrive and interrupt the service process only when an external customer is being served. The external customers appear according to a stationary Poisson process. Customers of each of the above-mentioned types are characterized by some random volume. The customer service time depends arbitrarily on its volume. Two schemes of customer service organization are analyzed. The non-stationary and stationary distributions of the total volume of customers present in the system are determined in terms of Laplace and Laplace-Stieltjes transforms. The stationary first and second moments of total customers volume are also calculated. The obtained results are used to approximate loss characteristics in analogous systems with limited buffer space. Numerical examples illustrating theoretical results are attached as well.

Key words: single-server queueing system with non-homogeneous customers, random volume customers, loss probability, total volume distribution, Laplace transform, Laplace-Stieltjes transform, RAM memory space.

1. Introduction

In late 1970s, it was noticed by engineers from different countries (see e.g. [1–4]) that the analytical methods of queueing theory are generally insufficient for solving the problem of communication nodes buffer space determination. Indeed, different messages in a network consist, as a rule, of different numbers of bites, in other words, they can be of different volumes.

Then, it is clear that the important characterisic of a queueing system with customers of random volume becomes the total volume of customers present in it at an arbitrary time instant. We also have to take account of the possible dependence between the volume of the customer and their service time. The value of the total customer volume can certainly be limited by some deterministic value, known as buffer space capacity, and its distribution depends on the joint distribution of the customer volume and service time. In [5, 6] examples demonstrating the influence of this dependence can be found.

Probably, the first paper devoted to queueing systems with random volume customers and service times dependent on their volumes was published by A.M. Alexandrov and B.A. Kaz in 1973 [7]. The first monography devoted to the theory of such systems was published in 1990 [5]. At present, these queueing systems are widely analyzed in literature (see e.g. [5–10]).

Examples of application of the theory of systems with random volume customers and service time depending on customer volume for the determination of buffer space capacity in real nodes of computer-communication networks can be found in [5, 6, 8]. Particularly, examples of buffer space capacity determination for different nodes of telecommunication networks (concentrator (router) and communication center) are discussed in [6]. In this monography, the example of communication center buffer space capacity determination, taking into account network environment, can also be found.

In the paper, we introduce some modifications of queueing systems with random volume customers. They involve dividing the arriving customers into two classes: an internal and external one.

Such division makes evident practical sense and can be confirmed by many interesting examples from computer science. Naturally, computer applications (processes) belong to one of two classes. The first class (internal customers) are operating system services, and the second one (external customers) are user applications. Evidently, some operating system services are necessary to start or continue user applications and thus have higher priority.

Many examples can also be found in internet networks. While a browser operates, many necessary extensions (plugins) are being downloaded. It is clear that, in such situations, the external customer needs additional memory space. We assume that, during internal customer service, servicing of the external customer is stopped, to be continued after the internal customer service termination.

The model analyzed in this paper is a generalization of $M/G/1/\infty$ queueing system with service time dependent on customer volume [5], in which the total volume of customers present in the system is unlimited.

As a most relevant example, we can consider the model of sharing processor with jobs switching. In this model, the computer system consists of a single computer having a single processor. RAM memory space is limited by the value V. During the service of user applications, the processor can

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initialize the additional system processes that are necessary to service these applications. While the service of a system process lasts, the service of user one is stopped and it is continued after the service of the system process termination. Then, the processor functioning is connected with permanent switching between user and system processes.

The above technical implementation can be analyzed using the introduced analytical model with unlimited memory space, but the main purpose of the modeling is to estimate information loss characteristics for the systems with a limited one. Obtained loss characteristics can be used to calculate the required RAM memory size. The main background of such model is described in [11].

The paper is organized as follows. In the next Sec. 2, we provide the mathematical description of the model and introduce some necessary notations. In Sec. 3, we define Markov process describing the evolution of the system under consideration and introduce the functions characterizing this process. In Sec. 4, we write out the system of equations for the introduced functions and give its solution that presents classical characteristics of the system. In Sec. 5, we obtain the non-stationary and stationary characteristics of total customers volume for two schemes of the system behavior. In Sec. 6, we estimate loss characteristics for the system with limited memory space. In this section, we also present sample of illustrative numerical computations for some special cases of this system. Section 7 contains concluding remarks.

2. The model and notation

We consider $M/G/1/\infty$ queueing system with customers of two following types: 1) external customers, 2) internal customers.

External customers appear according to a stationary Poisson process with parameter a. Each customer is characterized by some random volume ζ, his service time ξ can be dependent on the volume. Let $F(x,t)=\mathbf{P}\{\zeta<x,\xi<t\}$ be the joint distribution function of non-negative random variables ζ and ξ, $L(x)=F(x,\infty)$ and $B(t)=F(\infty,t)$ be the distribution functions of the random variables ζ and ξ, respectively.

Internal customers arrive only during external customers service. If T is an epoch of this service beginning and the service process is not completed in the time interval $[T;T+t)$, an internal customer appears in this interval with probability $E(t)=1-e^{-ct}$, $c>0$. Denote as γ and κ the internal customer volume and service time, respectively. Let $\Phi(x,t)=$ $=\mathbf{P}\{\gamma<x,\kappa<t\}$ be the joint distribution function of the random variables γ and κ, $R(x)=\Phi(x,\infty)$ and $H(t)=\Phi(\infty,t)$ be the distribution functions of the random variables γ and κ, respectively. An internal customer appearance interrupts the external customer service. After an internal customer arriving, his service begins immediately. After completing the internal customer service (at epoch T), the interrupted service of the external customer will continue. If this service is not completed during time t, an internal customer appears in the interval $[T;T+t)$ with probability $E(t)$ etc.

Denote by $\sigma(t)$ the total volume of external and internal customers present in the system at time instant t. Suppose that values of the process $\sigma(t)$ are not limited for all $t\geq 0$. If T is an epoch of external or internal customer appearance, then $\sigma(T)=\sigma(T^-)+x$, where x is the volume of the arriving customer. Denote by $\eta(t)$ the number of external customers present in the system at time instant t. Let $P_k(t)=\mathbf{P}\{\eta(t)=k\}$, $k=0,1,\ldots$. At the epoch t of external customer appearance, we have $\eta(t)=\eta(t^-)+1$. If external customer service completes at the epoch t, then $\eta(t)=\eta(t^-)-1$.

We shall analyze two following schemes of system behavior at the epoch T of service termination.

Scheme 1. If T is the epoch of the external or internal customer service termination, then $\sigma(T)=\sigma(T^-)-x$, where x is the volume of this customer.

Scheme 2. If T is the epoch of the internal customer service termination, then $\sigma(T)=\sigma(T^-)$. If T is the epoch of the external customer service termination, then $\sigma(T)=\sigma(T^-)-y$, where y is the total volume of the external customer and all internal customers arriving during his service.

So, in Scheme 1, each customer (external or internal) releases memory after his service termination, whereas, in Scheme 2, all internal customers that came during the service of an external one release memory only after the end of his service together with him.

We shall use the following notations. Let

$$\alpha(s,q)=\int_0^\infty\int_0^\infty e^{-sx-qt}dF(x,t)$$

and

$$\psi(s,q)=\int_0^\infty\int_0^\infty e^{-sx-qt}d\Phi(x,t)$$

be the double Laplace-Stieltjes transforms (LST) of the distribution functions $F(x,t)$ and $\Phi(x,t)$, respectively. Denote by

$$\alpha_{ij}=\mathbf{E}(\zeta\xi)=(-1)^{i+j}\frac{\partial^{i+j}\alpha(s,q)}{\partial s^i\partial q^j}\bigg|_{s=0,q=0},$$

$$\psi_{ij}=\mathbf{E}(\gamma\kappa)=(-1)^{i+j}\frac{\partial^{i+j}\psi(s,q)}{\partial s^i\partial q^j}\bigg|_{s=0,q=0}$$

the mixed $(i+j)$th moments of the distribution functions $F(x,t)$ and $\Phi(x,t)$, respectively, $i,j=1,2,\ldots$. Let $\varphi(s)=\alpha(s,0)$, $\beta(q)=\alpha(0,q)$, $g(s)=\psi(s,0)$, $h(q)=\psi(0,q)$ be the LST of the random variables ζ, ξ, γ and κ, respectively. Denote by φ_i, β_i, r_i, h_i the i-th moments of these random variables, respectively.

Let $D(x,t)=\mathbf{P}\{\sigma(t)<x\}$ be the distribution function of the total volume of external and internal customers present in the system at time instant t. It is known [12] that the stability condition for the system under consideration has the form $\rho=a\beta_1(1+ch_1)<1$. If this condition takes place, then the limits $p_k=\lim\limits_{t\to\infty}P_k(t)=\mathbf{P}\{\eta=k\}$, $k=0,1,\ldots,$

and $D(x) = \lim_{t\to\infty} D(x,t) = \mathbf{P}\{\sigma < x\}$ exist, where η and σ are the stationary number of external customers present in the system and the stationary total customers volume, respectively.

Let $\overline{\delta}(s,t) = \int_0^\infty e^{-sx} d_x D(x,t)$ be the LST of the distribution function $D(x,t)$ with respect to x. The Laplace transform of this function with respect to t is denoted by $\delta(s,q) = \int_0^\infty e^{-qt}\overline{\delta}(s,t)dt$. Our main purpose is to determine the function $\delta(s,q)$, from which all characteristics of the process $\sigma(t)$ can be determined. For example, stationary characteristics can be obtained from the function

$$\delta(s) = \int_0^\infty e^{-sx} dD(x) = \lim_{t\to\infty} \overline{\delta}(s,t) = \lim_{q\to 0} q\delta(s,q).$$

3. Random process and characteristics

First we introduce the following notation. Let $\nu(t)$ be the function taking two values: $\nu(t) = 0$, if the service of an external customer takes place at the moment t, and $\nu(t) = 1$, if the service of an internal customer takes place at this moment (this function is undefined at the moment t, if there are no customers in the system at this moment).

Suppose that external or internal customer service takes place at the moment t. Let $\xi^*_{(0)}(t)$ be the duration of an external customer service from the beginning to the moment t, if $\nu(t) = 0$ or $\nu(t) = 1$ (it is possible that this duration consists of more than one time intervals because of a possibility of service interruptions) and $\xi^*_{(1)}(t)$ be the time from the beginning of an internal customer service to the moment t, if $\nu(t) = 1$ (it is clear that the function $\xi^*_{(1)}(t)$ is undefined, if $\nu(t) = 0$). Then the Markov process

$$\begin{cases} \eta(t), & \eta(t) = 0, \\ (\eta(t), \nu(t), \xi^*_{(0)}(t)), & \eta(t) > 0,\ \nu(t) = 0, \\ (\eta(t), \nu(t), \xi^*_{(0)}(t), \xi^*_{(1)}(t)), & \eta(t) > 0,\ \nu(t) = 1 \end{cases} \quad (1)$$

describes the system behavior. We shall characterize this process by the functions having the following probability sense:

$$P_0(t) = \mathbf{P}\{\eta(t) = 0\}; \quad (2)$$

$$\Theta_k(0,x,t)dx = \mathbf{P}\{\eta(t) = k, \ \nu(t) = 0, \quad \xi^*_{(0)}(t) \in [x; x+dx)\}, \quad k = 1,2,\dots; \quad (3)$$

$$\Theta_k(1,x,y,t)dx\,dy = \mathbf{P}\{\eta(t) = k, \quad \nu(t) = 1, \ \xi^*_{(0)}(t) \in [x; x+dx), \quad \xi^*_{(1)}(t) \in [y; y+dy)\}, \quad k = 1,2,\dots. \quad (4)$$

Let us assume for simplicity that the densities $b_{(0)}(t)$ and $b_{(1)}(t)$ of random variables ξ and κ exist. Note that all results of the paper can be obtained without this assumption. Suppose that $P_0(0) = 0$ (the system is empty when $t = 0$, this is identified as zero initial condition). Denote by $\mu_{(0)}(t) = \frac{b_{(0)}(t)}{1-B(t)}$ and $\mu_{(1)}(t) = \frac{b_{(1)}(t)}{1-H(t)}$ the rates of external and internal customers service, respectively.

4. Equations for the system characteristics and their solution

Let $\delta_{i,j}$ be Kronecker's symbol: $\delta_{i,j} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$

Using the method of auxiliary variables [13], we can write out partial differential equations for the functions defined by the relations (2)–(4):

$$\frac{\partial P_0(t)}{\partial t} = -aP_0(t) + \int_0^t \Theta_1(0,x,t)\mu_{(0)}(x)dx; \quad (5)$$

$$\frac{\partial \Theta_k(0,x,t)}{\partial t} + \frac{\partial \Theta_k(0,x,t)}{\partial x} = -(a+c+\mu_{(0)}(x))\Theta_k(0,x,t) + \int_0^t \Theta_k(1,x,y,t)\mu_{(1)}(y)dy + (1-\delta_{k,1})a\Theta_{k-1}(0,x,t), \quad k = 1,2,\dots; \quad (6)$$

$$\Theta_k(0,0^+,t) = \int_0^t \Theta_{k+1}(0,x,t)\mu_{(0)}(x)dx + \delta_{k,1}aP_0(t), \quad k = 1,2,\dots; \quad (7)$$

$$\frac{\partial \Theta_k(1,x,y,t)}{\partial t} + \frac{\partial \Theta_k(1,x,y,t)}{\partial y} = -(a+\mu_{(1)}(y))\Theta_k(1,x,y,t) + (1-\delta_{k,1})a\Theta_{k-1}(1,x,y,t), \quad k = 1,2,\dots; \quad (8)$$

$$\Theta_k(1,x,0^+,t) = c\Theta_k(0,x,t). \quad (9)$$

Denote by

$$r_0(q) = \int_0^\infty e^{-qt}P_0(t)dt$$

the Laplace transform of the function $P_0(t)$ with respect to t.

Passing to Laplace transform in the equation (5) and taking into account the initial condition, we obtain:

$$qr_0(q) - 1 = -ar_0(q) + \int_0^\infty e^{-qt}\left(\int_0^t \Theta_1(0,x,t)\mu_{(0)}(x)dx\right)dt,$$

where

$$\int_0^\infty e^{-qt}\left(\int_0^t \Theta_1(0,x,t)\mu_{(0)}(x)dx\right)dt = \int_0^\infty \left(\int_x^\infty e^{-qt}\Theta_1(0,x,t)dt\right)\mu_{(0)}(x)dx.$$

Using the notation $p_1^*(0,x,q) = \int\limits_x^\infty e^{-qt}\Theta_1(0,x,t)dt$, we obtain the following equation:

$$(q+a)r_0(q) = \int\limits_0^\infty p_1^*(0,x,q)\mu_{(0)}(x)dx + 1. \tag{10}$$

Let us introduce the following generation functions:

$$\Pi_{(0)}(z,x,t) = \sum_{k=1}^\infty \Theta_k(0,x,t)z^k,$$

$$\Pi_{(1)}(z,x,y,t) = \sum_{k=1}^\infty \Theta_k(1,x,y,t)z^k.$$

Then, from the equations (6), we obtain the following equation for the function $\Pi_{(0)}(z,x,t)$:

$$\begin{aligned}&\frac{\partial\Pi_{(0)}(z,x,t)}{\partial t} + \frac{\partial\Pi_{(0)}(z,x,t)}{\partial x}\\ &= -(a+c+\mu_{(0)}(x) - az)\Pi_{(0)}(z,x,t)\\ &+ \int\limits_0^t \Pi_{(1)}(z,x,y,t)\mu_{(1)}(y)dy.\end{aligned} \tag{11}$$

Let us introduce the following Laplace transforms:

$$\pi_{(0)}(z,x,q) = \int\limits_0^\infty e^{-qt}\Pi_{(0)}(z,x,t)dt,$$

$$\pi_{(1)}(z,x,y,q) = \int\limits_0^\infty e^{-qt}\Pi_{(1)}(z,x,y,t)dt.$$

Then, passing to Laplace transform in the equation (11), we have:

$$\begin{aligned}&q\pi_{(0)}(z,x,q) + \frac{\partial\pi_{(0)}(z,x,q)}{\partial x}\\ &= -(c+a-az+\mu_{(0)}(x))\pi_{(0)}(z,x,q)\\ &+ \int\limits_0^\infty e^{-qt}\left(\int\limits_0^t \Pi_{(1)}(z,x,y,t)\mu_{(1)}(y)dy\right)dt,\end{aligned}$$

where

$$\begin{aligned}&\int\limits_0^\infty e^{-qt}\left(\int\limits_0^t \Pi_{(1)}(z,x,y,t)\mu_{(1)}(y)dy\right)dt\\ &= \int\limits_0^\infty \pi_{(1)}(z,x,y,q)\mu_{(1)}(y)dy,\end{aligned}$$

i.e. we obtain the following equation:

$$\begin{aligned}\frac{\partial\pi_{(0)}(z,x,q)}{\partial x} &= -(q+c+a-az+\mu_{(0)}(x))\pi_{(0)}(z,x,q)\\ &+ \int\limits_0^\infty \pi_{(1)}(z,x,y,q)\mu_{(1)}(y)dy.\end{aligned} \tag{12}$$

Passing to generation function in the equation (7), we have:

$$\begin{aligned}\Pi_{(0)}(z,0,t) &= \frac{1}{z}\int\limits_0^t \Pi_{(0)}(z,x,t)\mu_{(0)}(x)dx\\ &- \int\limits_0^t \Theta_1(0,x,t)\mu_{(0)}(x)dx + azP_0(t),\end{aligned}$$

whereas we obtain, passing to Laplace transforms:

$$\begin{aligned}\pi_{(0)}(z,0,q) &= \frac{1}{z}\int\limits_0^\infty \pi_{(0)}(z,x,q)\mu_{(0)}(x)dx\\ &- \int\limits_0^\infty p_1^*(0,x,q)\mu_{(0)}(x)dx + azr_0(q).\end{aligned}$$

But, from the equation (10) we have:

$$\int\limits_0^\infty p_1^*(0,x,q)\mu_{(0)}(x)dx = (q+a)r_0(q) - 1.$$

Then, we obtain the equation:

$$\begin{aligned}\pi_{(0)}(z,0,q) &= \frac{1}{z}\int\limits_0^\infty \pi_{(0)}(z,x,q)\mu_{(0)}(x)dx\\ &+1 - (q+a-az)r_0(q).\end{aligned} \tag{13}$$

Passing to generation functions in the equation (8), we have:

$$\begin{aligned}&\frac{\partial\Pi_{(1)}(z,x,y,t)}{\partial t} + \frac{\partial\Pi_{(1)}(z,x,y,t)}{\partial y}\\ &= -(a-az+\mu_{(1)}(y))\Pi_{(1)}(z,x,y,t).\end{aligned}$$

In terms of Laplace transforms, we obtain the following equation:

$$\begin{aligned}&\frac{\partial\pi_{(1)}(z,x,y,q)}{\partial y}\\ &= -(q+a-az+\mu_{(1)}(y))\pi_{(1)}(z,x,y,q).\end{aligned} \tag{14}$$

Passing to generation functions in the equation (9), we have:

$$\Pi_{(1)}(z,x,0,t) = c\Pi_{(0)}(z,x,t),$$

or (in the terms of Laplace transforms):

$$\pi_{(1)}(z,x,0,q) = c\pi_{(0)}(z,x,q). \tag{15}$$

The solution of the equation (14) has the form (taking into account the form of the function $\mu_{(1)}(y)$):

$$\pi_{(1)}(z,x,y,q) = [1-H(y)]e^{-(q+a-az)y}\pi_{(1)}(z,x,0,q)$$

or, as it follows from the equation (15),

$$\pi_{(1)}(z,x,y,q) = c[1-H(y)]e^{-(q+a-az)y}\pi_{(0)}(z,x,q). \tag{16}$$

By substitution the relation (16) into the equation (12), taking into account the form of the function $\mu_1(y)$, we obtain:

$$\begin{aligned}\frac{\partial\pi_{(0)}(z,x,q)}{\partial x} &= -(q+c-ch(q+a-az)+a-az\\ &+\mu_{(0)}(x))\pi_{(0)}(z,x,q)\end{aligned}$$

or, if we introduce the notation $\chi(z,q) = q + c - ch(q + a - az) + a - az$,

$$\frac{\partial \pi_{(0)}(z,x,q)}{\partial x} = -(\chi(z,q) + \mu_{(0)}(x))\pi_{(0)}(z,x,q).$$

The solution of the last equation has the following form:

$$\pi_0(z,x,q) = [1 - B(x)]e^{-\chi(z,q)x}\pi_0(z,0,q). \tag{17}$$

Then, from the equation (13) we obtain:

$$\pi_0(z,0,q) = \frac{z[1-(q+a-az)]r_0(q)}{z-\beta(\chi(z,q))},$$

and from the relation (17) we have:

$$\pi_0(z,x,q) = \frac{z[1-(q+a-az)]r_0(q)}{z-\beta(\chi(z,q))}[1-B(x)]e^{-\chi(z,q)x}. \tag{18}$$

From the relation (16), we obtain:

$$\pi_{(1)}(z,x,y,q) = \frac{cz[1-(q+a-az)]r_0(q)}{z-\beta(\chi(z,q))} \times [1-B(x)][1-H(y)]\exp[-\chi(z,q)x - (q+a-az)y]. \tag{19}$$

Denote by $\pi(q)$ the LST of the busy period of the system under consideration. It is known [12] that $r_0(q) = (q + a - a\pi(q))^{-1}$.

Let us determine the function $\pi(q)$. Let $\omega(q)$ be the LST of the random time τ from the beginning to the termination of an arbitrary external customer service. Then we have (as it follows from the theory of usual $M/G/1/\infty$ queue [5]) that

$$\pi(q) = \omega(q + a - a\pi(q)). \tag{20}$$

Let us determine the function $\omega(q)$. Let $\omega(q|\xi = u)$ be the conditional LST of the random variable τ under condition that the service time of the customer is equal to u. It is obvious that

$$\omega(q|\xi = u) = e^{-qu}\sum_{k=0}^{\infty}\frac{(cu)^k}{k!}e^{-cu}(h(q))^k = e^{-(q+c-ch(q))u},$$

whereas it follows that

$$\omega(q) = \int_0^{\infty}\omega(q|\xi=u)dB(u) = \int_0^{\infty}e^{-(q+c-ch(q))u}dB(u) = \beta(q+c-ch(q)),$$

i.e.

$$\omega(q+a-a\pi(q)) = \beta(q+a-a\pi(q)+c-ch(q+a-a\pi(q))),$$

and the equation (20) takes the form:

$$\pi(q) = \beta(q+a-a\pi(q)+c-ch(q+a-a\pi(q))).$$

From the last relation, we can determine the moments of the busy period. For example, for the first moment, we have (if $\rho = a\beta_1(1+ch_1) < 1$):

$$\pi_1 = -\pi'(q)|_{q=0} = \frac{\beta_1(1+ch_1)}{1-a\beta_1(1+ch_1)}.$$

5. Non-stationary and stationary characteristics of the total customers volume

In this section, we obtain the total volume characteristics for two above schemes of system behavior.

Scheme 1. For the scheme 1, we have obviously:

$$\begin{aligned}D(x,t) &= \mathbf{P}\{\sigma(t)<x\} = P_0(t) \\ &+ \sum_{k=1}^{\infty}\int_0^t \mathbf{P}\{\sigma(t)<x \,|\, \eta(t)=k,\ \nu(t)=0,\ \xi^*_{(0)}(t)=y\}\Theta_k(0,y,t)dy \\ &+ \sum_{k=1}^{\infty}\int_{y=0}^{t}\int_{u=0}^{t}\mathbf{P}\{\sigma(t)<x \,|\, \eta(t)=k,\ \nu(t)=1,\ \xi^*_{(0)}(t)=y,\ \xi^*_{(1)}(t)=u\}\Theta_k(1,y,u,t)dy\,du.\end{aligned} \tag{21}$$

Denote by $A*B(x)$ the Stieltjes convolution of the distribution functions $A(x)$ and $B(x)$ of non-negative random variables, i.e. $A*B(x) = \int_0^x A(x-u)dB(u)$. Denote by $A_*^{(n)}(x)$ the n-fold Stieltjes convolution of the distribution function $A(x)$, $n = 0,1,\ldots$, i.e.

$$A_*^{(0)}(x) \equiv 1,$$

$$A_*^{(n)}(x) = \int_0^x A_*^{(n-1)}(x-u)dA(u), \quad n = 1,2,\ldots.$$

Then, as it follows from [6],

$$\mathbf{P}\{\sigma(t)<x|\eta(t)=k,\ \ \nu(t)=0,\ \ \xi^*_{(0)}(t)=y\} = L_*^{(k-1)} * E_y^{(0)}(x),$$

where $E_y^{(0)}(x) = \dfrac{L(x)-F(x,y)}{1-B(y)}$.

Analogously, we obtain:

$$\mathbf{P}\{\sigma(t)<x \,|\, \eta(t)=k, \nu(t)=1, \xi^*_{(0)}(t)=y, \xi^*_{(1)}(t)=u\} = L_*^{(k-1)} * E_y^{(0)} * E_u^{(1)}(x),$$

where $E_u^{(1)}(x) = \dfrac{R(x)-\Phi(x,u)}{1-H(u)}$.

Then, we have from the relation (21) that

$$D(x,t) = P_0(t) + \sum_{k=1}^{\infty}\int_0^t L_*^{(k-1)} * E_y^{(0)}(x)\Theta_k(0,y,t)dy + \sum_{k=1}^{\infty}\int_{y=0}^{t}\int_{u=0}^{t} L_*^{(k-1)} * E_y^{(0)} * E_u^{(1)}(x)\Theta_k(1,y,u,t)dy\,du.$$

Passing to LST of the function $D(x,t)$ with respect to x, we obtain:

$$\overline{\delta}(s,t)=P_0(t)+\sum_{k=1}^{\infty}\int_0^t(\varphi(s))^{k-1}e_y^{(0)}(s)\Theta_k(0,y,t)dy$$
$$+\sum_{k=1}^{\infty}\int_{y=0}^{t}\int_{u=0}^{t}(\varphi(s))^{k-1}e_y^{(0)}(s)e_u^{(1)}(s)\Theta_k(1,y,u,t)dy\,du, \tag{22}$$

where (see again [6])

$$e_y^{(0)}(s)=[1-B(y)]^{-1}\int_{x=0}^{\infty}e^{-sx}\int_{w=y}^{\infty}dF(x,w),$$

$$e_u^{(1)}(s)=[1-H(u)]^{-1}\int_{x=0}^{\infty}e^{-sx}\int_{w=u}^{\infty}d\Phi(x,w).$$

Passing in the relation (22) to the Laplace transform with respect to t, we obtain the following relation for the function $\delta(s,q)$:

$$\delta(s,q)=\int_0^{\infty}e^{-qt}\overline{\delta}(s,t)dt$$
$$=r_0(q)+\sum_{k=1}^{\infty}\int_0^{\infty}e^{-qt}\left(\int_0^t(\varphi(s))^{k-1}e_y^{(0)}(s)\Theta_k(0,y,t)dy\right)dt$$
$$+\sum_{k=1}^{\infty}\int_0^{\infty}e^{-qt}\left(\int_{y=0}^{t}\int_{u=0}^{t}(\varphi(s))^{k-1}e_y^{(0)}(s)e_u^{(1)}(s)\right.$$
$$\left.\times\Theta_k(1,y,u,t)dy\,du\right)dt. \tag{23}$$

It can be easily shown that

$$S_1=\sum_{k=1}^{\infty}\int_0^{\infty}e^{-qt}\left(\int_0^t(\varphi(s))^{k-1}e_y^{(0)}(s)\Theta_k(0,y,t)dy\right)dt$$
$$=(\varphi(s))^{-1}\int_0^{\infty}e_y^{(0)}(s)\left(\int_y^{\infty}\Pi_{(0)}(\varphi(s),y,t)e^{-qt}dt\right)dy$$
$$=(\varphi(s))^{-1}\int_0^{\infty}\pi_{(0)}(\varphi(s),y,q)e_y^{(0)}(s)dy.$$

Then, as it follows from the relation (18),

$$S_1=\frac{1-(q+a-a\varphi(s))r_0(q)}{\varphi(s)-\beta(\chi(\varphi(s),q))}$$
$$\times\int_0^{\infty}\left(\int_{x=0}^{\infty}e^{-sx}\int_{w=y}^{\infty}dF(x,w)\right)e^{-\chi(\varphi(s),q)y}dy.$$

For the integral in the last relation we have:

$$\int_0^{\infty}\left(\int_{x=0}^{\infty}e^{-sx}\int_{w=y}^{\infty}dF(x,w)\right)e^{-\chi(\varphi(s),q)y}dy$$
$$=\int_{x=0}^{\infty}e^{-sx}\int_{w=0}^{\infty}dF(x,w)\int_{y=0}^{w}e^{-\chi(\varphi(s),q)y}dy$$
$$=\frac{1}{\chi(\varphi(s),q)}\int_{x=0}^{\infty}e^{-sx}\int_{w=0}^{\infty}\left(1-e^{-\chi(\varphi(s),q)w}\right)dF(x,w)$$
$$=\frac{\varphi(s)-\alpha(s,\chi(\varphi(s),q))}{\chi(\varphi(s),q)}.$$

Taking into account that $r_0(q)=(q+a-a\pi(q))^{-1}$, we finally obtain:

$$S_1=\frac{a[\varphi(s)-\pi(q)][\varphi(s)-\alpha(s,\chi(\varphi(s),q))]}{\chi(\varphi(s),q)[q+a-a\pi(q)][\varphi(s)-\beta(\chi(\varphi(s),q))]}.$$

In similar way, we obtain that

$$S_2=\sum_{k=1}^{\infty}\int_0^{\infty}e^{-qt}\left(\int_{y=0}^{t}\int_{u=0}^{t}(\varphi(s))^{k-1}e_y^{(0)}(s)e_u^{(1)}(s)\right.$$
$$\left.\times\Theta_k(1,y,u,t)dy\,du\right)dt=\frac{ac[\varphi(s)-\pi(q)]}{\chi(\varphi(s),q))[q+a-a\pi(q)]}$$
$$\times\frac{[\varphi(s)-\alpha(s,\chi(\varphi(s),q))][g(s)-\psi(s,q+a-a\varphi(s))]}{[\varphi(s)-\beta(\chi(\varphi(s),q))][q+a-a\varphi(s))]}.$$

From the relation (23), after some calculation we obtain:

$$\delta(s,q)=[q+a-a\pi(q)]^{-1}$$
$$\times\left\{1+\frac{a[\pi(q)-\varphi(s)][\varphi(s)-\alpha(s,\chi(\varphi(s),q))]}{\chi(\varphi(s),q)[\beta(\chi(\varphi(s),q))-\varphi(s)]}\right.$$
$$\left.\times\left[1+\frac{c(g(s)-\psi(s,q+a-a\varphi(s)))}{q+a-a\varphi(s)}\right]\right\}. \tag{24}$$

Now, let us suppose that the stability condition takes place. Then, for the function $\delta(s)$, after some calculation we have:

$$\delta(s)=\lim_{q\to 0}q\delta(s,q)=p_0\left\{1+\frac{\varphi(s)-\alpha(s,\epsilon(\varphi(s)))}{\epsilon(\varphi(s))}\right.$$
$$\left.\times\frac{a[1-\varphi(s)]+c[g(s)-\psi(s,a-a\varphi(s))]}{\beta(\epsilon(\varphi(s)))-\varphi(s)}\right\}, \tag{25}$$

where $p_0=1-a\beta_1(1+ch_1)$, $\epsilon(z)=c-ch(a-az)+a-az$.

We can calculate arbitrary order stationary moments (if they exist) of total customers volume using the relation (25). For example, if we introduce the notation $C=1+ch_1$, then for the first moment after rather complicated calculation we obtain:

$$\delta_1=\mathbf{E}\sigma=-\delta'(0)$$
$$=a(C\alpha_{1\,1}+c\beta_1\psi_{1\,1})+\frac{a^2\varphi_1\left(C^2\beta_2+c\beta_1h_2\right)}{2(1-\rho)}. \tag{26}$$

The relation for the second moment for the system under consideration we obtain using the *Mathematica* enviroment:

$$\delta_2 = \mathbf{E}\sigma^2 = \delta''(0) = \frac{a^3\beta_1^3 c\psi_{21}C^2}{(1-\rho)^2} - \frac{2a^3\alpha_{12}\beta_1\varphi_1 C^3}{(1-\rho)^2} + \frac{a^4\alpha_{12}\beta_1^2\varphi_1 C^4}{(1-\rho)^2} - \frac{a^4\alpha_{11}\beta_1\beta_2\varphi_1 C^4}{(1-\rho)^2} + \frac{a^4\beta_2^2\varphi_1^2 C^4}{2(1-\rho)^2} - \frac{a^4\beta_1\beta_3\varphi_1^2 C^4}{3(1-\rho)^2} + \frac{a(\alpha_{21}C + 2\alpha_{11}c\psi_{11} + \beta_1 c\psi_{21})}{(1-\rho)^2} - \frac{a^3\beta_1\beta_2 C^2(\varphi_2 C + 2c\varphi_1\psi_{11})}{2(1-\rho)^2} + \frac{a^3\varphi_1^2 C(\beta_3 C^2 + 3\beta_2 ch_2)}{3(1-\rho)^2} + \frac{a^4\beta_1^3 c\varphi_1 C(\psi_{12}C - c\psi_{11}h_2)}{(1-\rho)^2} + \frac{a^4\beta_1^2 c^2\varphi_1^2 h_2^2}{2(1-\rho)^2} + \frac{a^3\alpha_{11}C(\beta_2\varphi_1 C^2 + \beta_1 c(2\beta_1\psi_{11}C - \varphi_1 h_2))}{(1-\rho)^2} - \frac{a^2(-\beta_2\varphi_2 + 8\alpha_{11}\beta_1 c\psi_{11} + 4\alpha_{21}\beta_1 C^2 - 2\alpha_{12}\varphi_1 C^2)}{2(1-\rho)^2} + \frac{a^2 c(-4\beta_1^2\psi_{21}C + \beta_2(\varphi_2 h_1(1+C) + 2\varphi_1\psi_{11}C) + 2\alpha_{11}\varphi_1 h_2)}{2(1-\rho)^2} + \frac{a^2 c\beta_1(2\varphi_1\psi_{12} - 8\alpha_{11}c\psi_{11}h_1 + \varphi_2 h_2)}{2(1-\rho)^2} + \frac{a^3\beta_1^2(2\alpha_{21}C^3 - c(\varphi_2 Ch_2 + \varphi_1(4\psi_{12}C - 2c\psi_{11}h_2)))}{2(1-\rho)^2} + \frac{a^3\beta_1 c\varphi_1^2 h_3}{3(1-\rho)}. \tag{27}$$

Scheme 2. Suppose that an external customer service begins at the moment $t = 0$. Denote by $\eta_1(t)$ the number of internal customers present in the system under consideration at the moment t, which are not served at this moment unless the external customer service has already been completed. Then, for the scheme 2 we have:

$$D(x,t) = \mathbf{P}\{\sigma(t) < x\} = P_0(t) + \sum_{k=1}^{\infty}\int_0^t \sum_{l=0}^{\infty} \mathbf{P}\{\sigma(t) < x \,|\, \eta(t) = k,\ \eta_1(y) = l,\ \nu(t) = 0, \xi^*_{(0)}(t) = y\} \times \mathbf{P}\{\eta_1(y) = l\}\Theta_k(0,y,t)dy \tag{28}$$
$$+ \sum_{k=1}^{\infty}\int_0^t\int_0^t \sum_{l=0}^{\infty} \mathbf{P}\{\sigma(t) < x \,|\, \eta(t) = k, \eta_1(y) = l,\ \nu(t) = 1,\ \xi^*_{(0)}(t) = y,\ \xi^*_{(1)}(t) = u\} \times \mathbf{P}\{\eta_1(y) = l\}\Theta_k(1,y,u,t)dy\,du,$$

where, obviously, $\mathbf{P}\{\eta_1(y) = l\} = \dfrac{(cy)^l}{l!}e^{-cy}$.

It is clear that (see [6])

$$\mathbf{P}\{\sigma(t) < x \,|\eta(t) = k, \eta_1(y) = l, \nu(t) = 0, \xi^*_{(0)}(t) = y\} = L_*^{(k-1)} * E_y^{(0)} * R_*^{(l)}(x)$$

and

$$\mathbf{P}\{\sigma(t) < x \,|\, \eta(t) = k, \eta_1(y) = l,\ \nu(t) = 1,\ \xi^*_{(0)}(t) = y,\ \xi^*_{(1)}(t) = u\} = L_*^{(k-1)} * E_y^{(0)} * E_u^{(1)} * R_*^{(l)}(x),$$

whereas the relation (28) takes the form:

$$D(x,t) = P_0(t) + \sum_{k=1}^{\infty}\int_0^t \sum_{l=0}^{\infty} L_*^{(k-1)} * E_y^{(0)} * R_*^{(l)}(x)\frac{(cy)^l}{l!}e^{-cy} \times \Theta_k(0,y,t)dy \tag{29}$$
$$+ \sum_{k=1}^{\infty}\int_0^t\int_0^t \sum_{l=0}^{\infty} L_*^{(k-1)} * E_y^{(0)} * E_u^{(1)} * R_*^{(l)}(x)\frac{(cy)^l}{l!}e^{-cy} \times \Theta_k(1,y,u,t)dy\,du.$$

Passing in the relation (29) to LST with respect to x, we have after some calculation:

$$\overline{\delta}(s,t) = P_0(t) + \frac{1}{\varphi(s)}\int_0^t e^{-(1-g(s))cy} e_y^{(0)}(s)\Pi_{(0)}(\varphi(s),y,t)dy + \frac{1}{\varphi(s)g(s)}\int_0^t e^{-(1-g(s))cy} e_y^{(0)}(s) \times \left(\int_0^t e_u^{(1)}(s)\Pi_{(1)}(\varphi(s),y,u,t)du\right)dy.$$

Now, passing to Laplace transform with respect to t, after some calculation we obtain (as it was done for scheme 1):

$$\delta(s,q) = [q + a - a\pi(q)]^{-1} \times \left\{1 + \frac{a[\pi(q) - \varphi(s)][\varphi(s) - \alpha(s, c - cg(s) + \chi(\varphi(s),q))]}{[\beta(\chi(\varphi(s),q)) - \varphi(s)][c - cg(s) + \chi(\varphi(s),q)]} \times \left[1 + \frac{c(g(s) - \psi(s, q + a - a\varphi(s)))}{g(s)(q + a - a\varphi(s))}\right]\right\}. \tag{30}$$

In this case, we have for the function $\delta(s)$:

$$\delta(s) = \lim_{q\to 0} q\delta(s,q) = p_0\left\{1 + \frac{\varphi(s) - \alpha(s, c - cg(s) + \epsilon(\varphi(s)))}{c - cg(s) + \epsilon(\varphi(s))} \times \frac{ag(s)[1 - \varphi(s)] + c[g(s) - \psi(s, a - a\varphi(s))]}{g(s)(\beta(\epsilon(\varphi(s))) - \varphi(s))}\right\}. \tag{31}$$

For the first stationary moment of total customers volume, we obtain:

$$\delta_1 = \mathbf{E}\sigma = -\delta'(0) = a(C\alpha_{1\,1} + c\beta_1(\psi_{1\,1} - h_1 r_1)) + \frac{a^2C^2\varphi_1\beta_2}{2(1-\rho)} + \frac{ac}{2}\left(C\beta_2 r_1 + \frac{a\beta_1 h_2\varphi_1}{1-\rho}\right). \tag{32}$$

For the second moment we obtain (with the help of *Mathematica*):

$$\delta_2 = \mathbf{E}\sigma^2 = \delta''(0) = \frac{a^3\beta_1^3 c\psi_{21}C^2}{(1-\rho)^2} - \frac{2a^3\alpha_{12}\beta_1\varphi_1C^3}{(1-\rho)^2}$$
$$+\frac{a^4\alpha_{12}\beta_1^2\varphi_1C^4}{(1-\rho)^2} - \frac{a^4\alpha_{11}\beta_1\beta_2\varphi_1C^4}{(1-\rho)^2}$$
$$+\frac{a^4\beta_2^2\varphi_1^2C^4}{2(1-\rho)^2} - \frac{a^4\beta_1\beta_3\varphi_1^2C^4}{3(1-\rho)^2} + \frac{a^4\beta_1^2c^2\varphi_1^2h_2^2}{2(1-\rho)^2}$$
$$+\frac{a(\alpha_{21}C + 2\alpha_{11}c\psi_{11} + \beta_1c\psi_{21})}{(1-\rho)^2} - \frac{a^3\beta_1\beta_2C^2(\varphi_2C + 2c\varphi_1\psi_{11})}{2(1-\rho)^2}$$
$$+\frac{a^3\varphi_1^2C(\beta_3C^2 + 3\beta_2ch_2)}{3(1-\rho)^2} + \frac{a^4\beta_1^3c\varphi_1C(\psi_{12}C - c\psi_{11}h_2)}{(1-\rho)^2}$$
$$+\frac{a^3\alpha_{11}C(\beta_2\varphi_1C^2 + \beta_1c(2\beta_1\psi_{11}C - \varphi_1h_2))}{(1-\rho)^2}$$
$$-\frac{a^2(-\beta_2\varphi_2 + 8\alpha_{11}\beta_1c\psi_{11} + 4\alpha_{21}\beta_1C^2 - 2\alpha_{12}\varphi_1C^2)}{2(1-\rho)^2}$$
$$+\frac{a^2c(-4\beta_1^2\psi_{21}C + \beta_2(\varphi_2h_1(1+C) + 2\varphi_1\psi_{11}C) + 2\alpha_{11}\varphi_1h_2}{2(1-\rho)^2}$$
$$+\frac{a^2c\beta_1(2\varphi_1\psi_{12} - 8\alpha_{11}c\psi_{11}h_1 + \varphi_2h_2)}{2(1-\rho)^2}$$
$$-\frac{a^2\beta_3cC^2r_1(-2\varphi_1 + \beta_1cr_1)}{3(1-\rho)}$$
$$+\frac{a^3\beta_1^2(2\alpha_{21}C^3 - c(\varphi_2Ch_2 + \varphi_1(4\psi_{12}C - 2c\psi_{11}h_2)))}{2(1-\rho)^2}$$
$$+a\alpha_{12}cCr_1 + \frac{a^3c\varphi_1((3\beta_2^2 - 4\beta_1\beta_3)C^3 + 6\beta_1^2h_2)r_1}{6(1-\rho)}$$
$$-\frac{acr_1(3\beta_2c\psi_{11} - 6\alpha_{11}h_1 + c(\beta_3C - 3\beta_2h_1)r_1 - 6\beta_1(\psi_{11} - h_1r_1))}{3(1-\rho)}$$
$$-\frac{a^2cr_1(\beta_2\varphi_1(2h_1C - ch_2) + 4\beta_1^2C(-\psi_{11} + h_1r_1))}{2(1-\rho)}$$
$$-\frac{2a^2cr_1\beta_1(-2\alpha_{11}h_1C + \varphi_1h_2 + c\beta_2C(\psi_{11} - h_1r_1))}{2(1-\rho)}$$
$$+\frac{1}{2}ac(\beta_2C - 2\beta_1h_1)r_2 + \frac{a^3\beta_1c\varphi_1^2h_3}{3(1-\rho)}. \tag{33}$$

The obtained analytical results seem to be rather complicated and the usage of Laplace-Stieltjes transforms causes some computation problems even on the level of computing of two first moments. It is also clear that if we substitute $c = 0$ to the relations (25) and (31), we get the classical model of the system $M/G/1/\infty$ without introduced modifications [5].

6. Estimation of loss characteristics

It is clear that there are no losses in the system under consideration. We denote it by QS_∞ and assume that the stationary mode exists for the system. In this section, we show that the obtained analytical results can be used to estimate loss characteristics for the system QS_V, which differs from the system QS_∞ in limitation (by V, $V < \infty$) of memory space only.

Let us consider loss characteristics conformably to external customers.

It is clear that the most familiar characteristic of losing is the loss probability P_{loss} [9]. Intuitively, it is a part of losing customers. If there are no other limitations in the system, except of the system memory space one (as in our case), we can calculate P_{loss} using the relation [6, 9]:

$$P_{loss} = 1 - \int_0^V D_V(V-x)dL(x),$$

where $D_V(x)$ is the steady-state distribution function of the total customers volume in the system QS_V.

Unfortunately, P_{loss} is less informative loss characteristic of the system with limited memory space, because it is a part of losing customers, not a part of losing information. Therefore, more informative loss characteristic is the loss probability of a unit of customers volume (a part of losing volume) Q_{loss}. If there are no other limitations in the system, this characteristic has the form [6, 9]:

$$Q_{loss} = 1 - \frac{1}{\varphi_1}\int_0^V xD_V(V-x)dL(x),$$

where $\varphi_1 = \mathbf{E}\zeta$. It can be proved (see [9]) that $Q_{loss} \geq P_{loss}$.

As it was shown in [9], the following inequalities take place:

$$P_{loss} = 1 - \int_0^V D_V(V-x)dL(x) \leq 1 - \int_0^V D_\infty(V-x)dL(x) = P^*_{loss}.$$

$$Q_{loss} = 1 - \frac{1}{\varphi_1}\int_0^V xD_V(V-x)dL(x) \leq 1 - \frac{1}{\varphi_1}\int_0^V xD_\infty(V-x)dL(x) = Q^*_{loss},$$

where $D_\infty(x)$ is the steady-state distribution function of the total customers volume in the system QS_∞. As it follows from these inequalities, the values P^*_{loss} and Q^*_{loss} can be interpreted as an upper boundaries for P_{loss} and Q_{loss}, respectively, when the distribution function $D_\infty(x)$ is known.

Unfortunately, calculation of this function is often difficult, because we have it in exact form very rarely. To estimate P^*_{loss}, we can approximate the convolution

$G(x) = \int_0^x D_\infty(x-u)dL(u)$, which is the distribution function of the random variable $\theta = \zeta + \sigma$, by the function $G^*(x) = \frac{\gamma(p,bx)}{\Gamma(p)}$, where $\gamma(p,bx) = \int_0^{bx} t^{p-1}e^{-t}\, dt$ is incomplete Gamma function and $\Gamma(p) = \gamma(p,\infty)$ is Gamma function.

It is known that the first and the second moments of the random variable having the distribution function $G^*(x)$ can be calculated as: $f_1^* = p/b$, $f_2^* = p(p+1)/b^2$. The first and the second moment of the random variable θ have the following form: $f_1 = \delta_1 + \varphi_1$, $f_2 = \delta_2 + \varphi_2 + 2\delta_1\varphi_1$, because the random variables ζ and σ are independent. The values of parameters p and b we choose so that the first and the second moment f_1^* and f_2^* of the approximate distribution are equal to f_1 and f_2, respectively, whereas we obtain:

$$p = \frac{f_1^2}{f_2 - f_1^2}, \qquad b = \frac{f_1}{f_2 - f_1^2}.$$

A good quality of this approximation has been confirmed by simulation. So, we can use the approximate relation: $P^*_{loss} \approx 1 - G^*(V)$.

In the second inequality, we approximate $D_\infty(x)$ by the function $D^*_\infty(x) = p_0 + (1-p_0)\frac{\gamma(d,gx)}{\Gamma(d)}$ [14]. Parameters d and g are also calculated by the same way – assuming that first two moments of the functions $D_\infty(x)$ and $D^*_\infty(x)$ are the same. Then, we have:

$$d = \frac{\delta_1^2}{(1-p_0)\delta_2 - \delta_1^2}, \qquad g = \frac{(1-p_0)\delta_1}{(1-p_0)\delta_2 - \delta_1^2}.$$

Additionally, if the random variable ζ is absolute continuous, we obtain

$$\int_0^V xD_\infty(V-x)dL(x) = \int_0^V xD_\infty(V-x)l(x)dx,$$

where $l(x)$ is the density of the random variable ζ. The above facts allow to present the value Q^*_{loss} in the following form:

$$Q^*_{loss} \approx 1 - \frac{1}{\varphi_1}\int_0^V x\left[p_0 + (1-p_0)\frac{\gamma(d, g(V-x))}{\Gamma(d)}\right] l(x)dx.$$

To calculate P^*_{loss} and Q^*_{loss}, we use the methods of numerical integration and *Mathematica* enviroment (see [15]).

Summarising, the obtained formulas (26), (27), (32), (33) for the stationary first and second moments of the total customers volume in the analyzed systems without losses can be used for estimation of loss characteristics in analogous systems with limited memory space.

Below, we present analytical and simulation results of this estimation for some special cases of such systems.

6.1. Scheme 1. Customer volume and its service time are independent. Assume that the volumes of external and internal customers have an exponential distribution with the same parameter f and their service times are independent on their volumes and have an exponential distribution with the same parameter μ. Then, the first and second moment of the total customers volume, as it follows from the relation (26), (27), are equal to:

$$\delta_1 = \frac{a(ac^2 - \mu^2(2c+\mu))}{f\mu^2(a(c+\mu) - \mu^2)}. \tag{34}$$

$$\delta_2 = \frac{2a(a^3c^2 - 3ac^2\mu^2 + a^2c^2(c+\mu) + \mu^4(3c+\mu))}{f^2\mu^2(\mu^2 - a(c+\mu))^2}. \tag{35}$$

Now, using the relations (34), (35), we can estimate the loss characteristics P^*_{loss} and Q^*_{loss} with the help of approximation functions $G^*(x)$ and $D^*_\infty(x)$, respectively.

In Table 1 and Table 2 we compare loss characteristics P^*_{loss} and Q^*_{loss} with estimators P^{SIM}_{loss} and Q^{SIM}_{loss} (of P_{loss} and Q_{loss}, respectively) obtained by simulation.

Table 1
Loss characteristics in the case of $\rho = 0.75$ (service time and customer volume are independent)

V	P^{SIM}_{loss}	Q^{SIM}_{loss}	P^*_{loss}	Q^*_{loss}
1	0.4862	0.8088	0.8066	0.9214
2	0.2982	0.5896	0.6530	0.7859
3	0.2001	0.4285	0.5292	0.6535
4	0.1412	0.3150	0.4292	0.5380
5	0.1034	0.2356	0.3481	0.4407
6	0.0777	0.1791	0.2825	0.3599
7	0.0590	0.1372	0.2293	0.2932
8	0.0456	0.1063	0.1861	0.2386
9	0.0358	0.0835	0.1511	0.1938
10	0.0282	0.0659	0.1227	0.1573

Table 2
Loss characteristics in the case of $\rho = 0.11$ (service time and customer volume are independent)

V	P^{SIM}_{loss}	Q^{SIM}_{loss}	P^*_{loss}	Q^*_{loss}
1	0.3943	0.7525	0.4127	0.7538
2	0.1662	0.4435	0.1722	0.4399
3	0.0718	0.2376	0.0720	0.2291
4	0.0312	0.1204	0.0302	0.1125
5	0.0136	0.0588	0.0126	0.0533
6	0.0059	0.0280	0.0053	0.0246
7	0.0026	0.0131	0.0022	0.0112
8	0.0011	0.0060	0.0009	0.0050
9	0.0005	0.0027	0.0004	0.0022
10	0.0002	0.0012	0.0002	0.0010

In Table 1 we present the obtained results for the following parameters: $a = 1$, $f = 1$, $c = 1$, $\mu = 2$ (hence $\rho = 0.75$); and in Table 2 – for parameters $a = 1$, $f = 1$, $c = 1$, $\mu = 10$ (then $\rho = 0.11$). The appropriate graphs, presenting the dependence of loss characteristics on the memory volume V, are shown in Figs. 1 and 2.

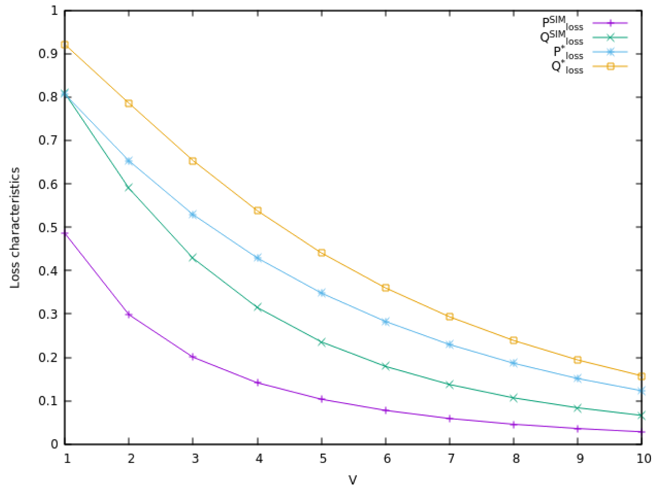

Fig. 1. Loss characteristics – scheme 1, service time and customer volume are independent, $\rho = 0.75$

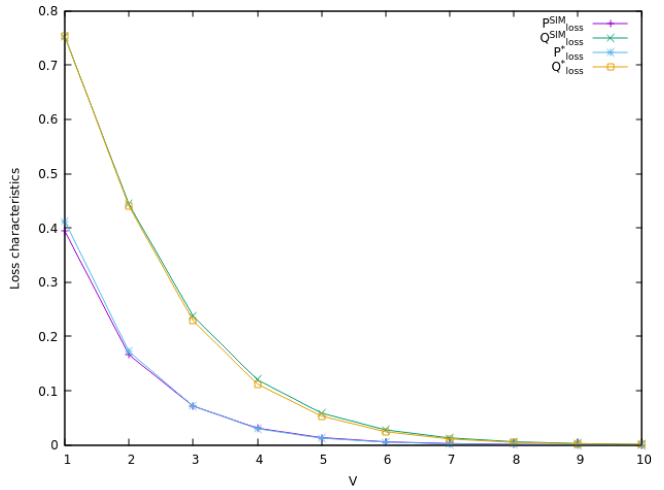

Fig. 2. Loss characteristics – scheme 1, service time and customer volume are independent, $\rho = 0.11$

6.2. Scheme 1. Service time is proportional to customer volume. Assume that volumes of external and internal customers have an exponential distribution with the same parameter f and their service times are proportional to their volumes with the same proportionality coefficient l. Then, the first two stationary moments of the total customers volume are equal to:

$$\delta_1 = \frac{al(2f^3 - 3acfl^2 - 3ac^2l^3 + f^2l(4c-a))}{f^3(f^2 - al(f+cl))}. \tag{36}$$

$$\begin{aligned}\delta_2 = {} & \frac{2(4a^2+6ac-3c^2)}{c^2f^2} - \frac{2a(3a+2c)l}{cf^3} \\ & + \frac{6a(a+2c)l^2}{f^4} + \frac{6a^2cl^3}{f^5} + \frac{2a^2c^2l^4}{f^6} \\ & + \frac{2(a^3f^2 + 3a^2cf^2 + c^3f^2 + 2ac^3fl)}{c^3(acl^2 + afl - f^2)^2} \\ & - \frac{2(ac^2f^2 + a^4fl + 4a^3cfl + a^2c^2fl)}{c^3(acl^2 + afl - f^2)^2} \\ & - \frac{2(2c^3f + 5a^2cl + 11a^2c^2l - a^3f - 7a^2cf - 5ac^2f - ac^3l)}{c^3f(acl^2 + afl - f^2)}.\end{aligned} \tag{37}$$

Using the relations (36), (37), we can estimate loss characteristics P^*_{loss} and Q^*_{loss} with the help of above approximate functions. We also estimate P_{loss} and Q_{loss} by simulation. In Table 3, we present the results for the following parameters: $a = 1$, $f = 2$, $c = 1$, $l = 1$ (hence $\rho = 0.75$), and in Table 4 – for parameters $a = 0.2$, $f = 1$, $c = 1$, $l = 1$ (then $\rho = 0.4$). The proper graphs are presented in Figs. 3 and 4.

Table 3

Loss characteristics in the case of $\rho = 0.75$ (service time is proportional to customer volume)

V	P^{SIM}_{loss}	Q^{SIM}_{loss}	P^*_{loss}	Q^*_{loss}
1	0.2755	0.5570	0.7355	0.8193
2	0.1389	0.3060	0.5176	0.6190
3	0.0768	0.1763	0.3595	0.4528
4	0.0448	0.1052	0.2479	0.3206
5	0.0273	0.0649	0.1702	0.2215
6	0.0171	0.0489	0.1164	0.1504
7	0.0110	0.0264	0.0795	0.1009
8	0.0071	0.0169	0.0541	0.0671
9	0.0046	0.0110	0.0368	0.0443
10	0.0030	0.0072	0.0250	0.0291

Table 4

Loss characteristics in the case of $\rho = 0.4$ (service time is proportional to customer volume)

V	P^{SIM}_{loss}	Q^{SIM}_{loss}	P^*_{loss}	Q^*_{loss}
1	0.4490	0.7830	0.6396	0.8387
2	0.2798	0.5631	0.4355	0.6201
3	0.1835	0.4016	0.3007	0.4520
4	0.1195	0.2801	0.2093	0.3298
5	0.0770	0.1914	0.1463	0.2396
6	0.0495	0.1282	0.1026	0.1721
7	0.0317	0.0850	0.0721	0.1219
8	0.0207	0.0569	0.0508	0.0850
9	0.0133	0.0375	0.0359	0.0585
10	0.0087	0.0246	0.0253	0.0397

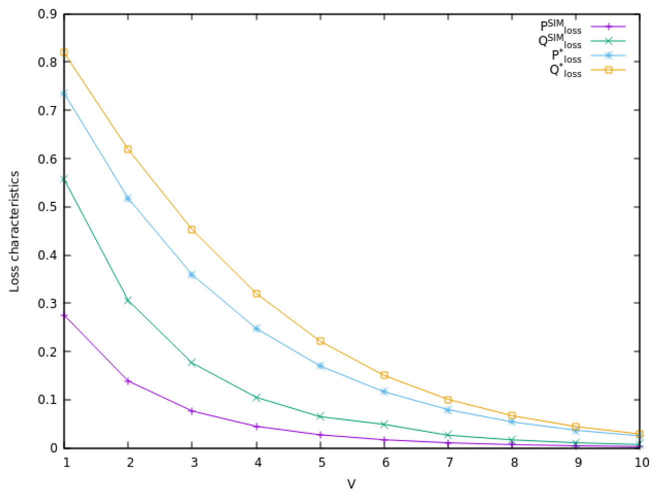

Fig. 3. Loss characteristics – scheme 1, service time is proportional to customer volume, $\rho = 0.75$

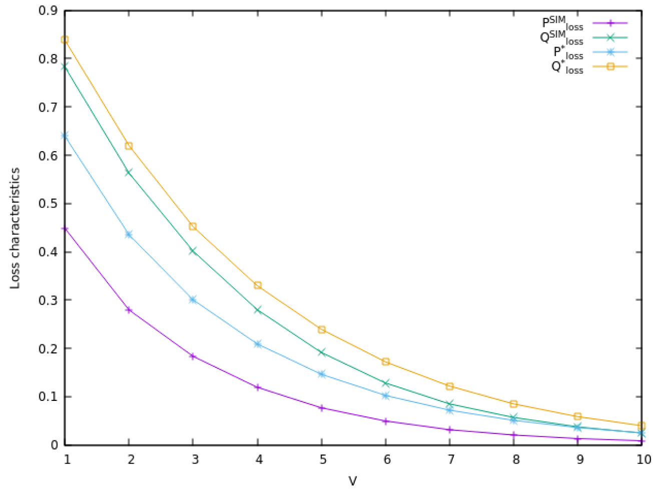

Fig. 4. Loss characteristics – scheme 1, service time is proportional to customer volume, $\rho = 0.4$

6.3. Scheme 2. Customer volume and its service time are independent. Here, we also assume that volumes of the external and internal customers have an exponential distribution with the same parameter f and their service times are independent on their volumes having an exponential distribution with the same parameter μ. Then, first two moments of the stationary total customers volume (as it follows from the relations (32), (33)) are equal to:

$$\delta_1 = \frac{a(ac^2(c+2\mu) - \mu^2(c+\mu)^2)}{f\mu^3(a(c+\mu) - \mu^2)}. \tag{38}$$

$$\delta_2 = \frac{2a^2c^3}{f^2\mu^5} + \frac{2(2a^2c^2+ac^3)}{f^2\mu^4} + \frac{2(a^2c+2ac^2)}{f^2\mu^3} - \frac{2(a^2-ac+c^2)}{cf^2\mu} + \frac{2(a^4+2a^3c+a^2c^2+4a^2c\mu)}{f^2(\mu^2-a\mu-ac)^2}. \tag{39}$$

Using the relations (38), (39) and above approximations, we can obtain estimators of loss characteristics. Table 5 contains the results for the following parameters: $a = 1$, $f = 1$, $c = 1$, $\mu = 2$ (hence $\rho = 0.75$), and Table 6 – for parameters: $a = 1$, $f = 1$, $c = 1$, $\mu = 10$ (then $\rho = 0.11$). The proper graphs are shown in Figs. 5 and 6.

Table 5
Loss characteristics in the case of $\rho = 0.75$ (service time and customer volume are independent)

V	P^{SIM}_{loss}	Q^{SIM}_{loss}	P^*_{loss}	Q^*_{loss}
1	0.4964	0.8125	0.8109	0.9221
2	0.3198	0.6025	0.6600	0.7890
3	0.2257	0.4483	0.5378	0.6593
4	0.1671	0.3391	0.4384	0.5458
5	0.1271	0.2605	0.3576	0.4498
6	0.0982	0.2027	0.2917	0.3696
7	0.0769	0.1592	0.2380	0.3029
8	0.0607	0.1260	0.1943	0.2479
9	0.0481	0.1000	0.1586	0.2025
10	0.0385	0.0800	0.1294	0.1653

Table 6
Loss characteristics in the case of $\rho = 0.11$ (service time and customer volume are independent)

V	P^{SIM}_{loss}	Q^{SIM}_{loss}	P^*_{loss}	Q^*_{loss}
1	0.3950	0.7528	0.4127	0.7573
2	0.1684	0.4455	0.1726	0.4399
3	0.0740	0.2400	0.0724	0.2292
4	0.0331	0.1234	0.0304	0.1127
5	0.0148	0.0609	0.0128	0.0535
6	0.0068	0.0300	0.0054	0.0248
7	0.0030	0.0142	0.0026	0.0113
8	0.0014	0.0067	0.0010	0.0051
9	0.0006	0.0031	0.0004	0.0023
10	0.0003	0.0015	0.0002	0.0010

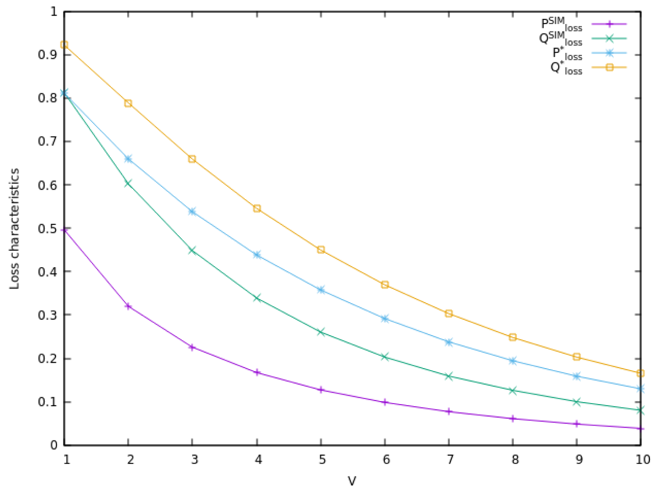

Fig. 5. Loss characteristics – scheme 2, service time and customer volume are independent, $\rho = 0.75$

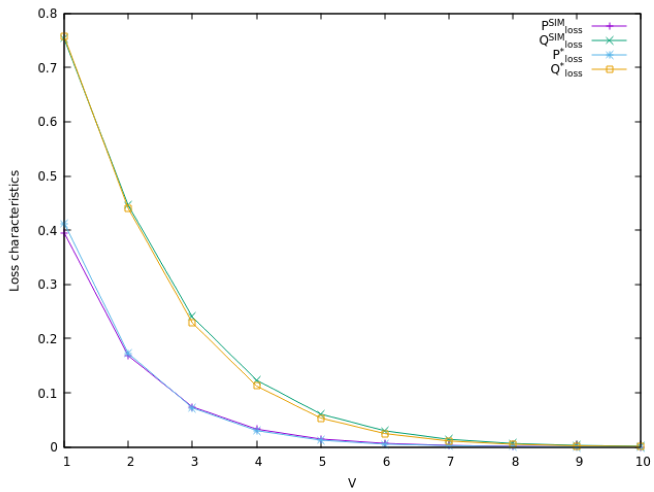

Fig. 6. Loss characteristics – scheme 2, service time and customer volume are independent, $\rho = 0.11$

6.4. Scheme 2. Service time is proportional to customer volume. Assume that the volumes of external and internal customers have an exponential distribution with the same parameter f and their service times are proportional to their volumes with the same proportionality coefficient l. Then, the first two stationary moments of the total customers volume are equal to:

$$\delta_1 = \frac{al(2f^4-4ac^2fl^3-ac^3l^4+f^3(4cl-al)+cf^2(cl^2-3al^2))}{f^4(f^2-afl-acl^2)}. \quad (40)$$

$$\begin{aligned}\delta_2 = {} & \frac{6(a^2+ac-c^2)}{c^2f^2} - \frac{2(2a^2+c^2)l}{cf^3} + \frac{2a(2a+5c)l^2}{f^4} \\ & + \frac{2ac(4a+5c)l^3}{f^5} + \frac{2ac^2(3a+c)l^4}{f^6} + \frac{2a^2c^3l^5}{f^7} \\ & + \frac{2(a^3f^2+3a^2cf^2-ac^2f^2+c^3f^2-a^4fl-4a^3cfl-a^2c^2fl+2ac^3fl)}{c^3(acl^2+afl-f^2)^2} \\ & - \frac{2(-a^3f-6a^2cf-2ac^2f+2c^3f+4a^3cl+7a^2c^2l-3ac^3l+c^4l)}{c^3f(acl^2+afl-f^2)}. \quad (41)\end{aligned}$$

Using the relations (40), (41) and above approximations, we obtain analogous estimators of information loss characteristics. Table 7 contains the results for the following parameters: $a = 1$, $f = 2$, $c = 1$, $l = 1$ (then $\rho = 0.75$) and Table 8 – for parameters $a = 0.2$, $f = 1$, $c = 1$, $l = 1$ (then $\rho = 0.4$). The proper graphs are presented in Figs. 7 and 8.

Table 7
Loss characteristics in the case of $\rho = 0.75$ (service time is proportional to customer volume)

V	P_{loss}^{SIM}	Q_{loss}^{SIM}	P_{loss}^*	Q_{loss}^*
1	0.2828	0.5609	0.7376	0.8197
2	0.1600	0.3238	0.5234	0.6221
3	0.0994	0.1992	0.3672	0.4588
4	0.0622	0.1240	0.2560	0.3284
5	0.0389	0.0771	0.1778	0.2298
6	0.0242	0.0477	0.1231	0.1582
7	0.0149	0.0294	0.0851	0.1076
8	0.0092	0.0182	0.0587	0.0726
9	0.0057	0.0112	0.0404	0.0487
10	0.0035	0.0068	0.0278	0.0325

Table 8
Loss characteristics in the case of $\rho = 0.4$ (service time is proportional to customer volume)

V	P_{loss}^{SIM}	Q_{loss}^{SIM}	P_{loss}^*	Q_{loss}^*
1	0.4512	0.7838	0.6243	0.8377
2	0.2960	0.5726	0.4398	0.6174
3	0.2171	0.4285	0.3176	0.4514
4	0.1640	0.3244	0.2322	0.3350
5	0.1247	0.2462	0.1712	0.2516
6	0.0952	0.1875	0.1269	0.1899
7	0.0723	0.1420	0.0945	0.1431
8	0.0552	0.1084	0.0706	0.1074
9	0.0421	0.0823	0.0529	0.0801
10	0.0322	0.0625	0.0397	0.0595

Comparison of obtained results (see e.g. Figs. 1 and 2) shows that the estimation of loss characteristics P_{loss} and Q_{loss} by using the functions $G^*(x)$ and $D_\infty^*(x)$ is useful in the cases when these characteristics are very small (near zero). If the loss characteristics are relatively large, then the obtained estimations of them P_{loss}^* and Q_{loss}^* are much more large, than real results, e.g. obtained by simulation, (compare P_{loss}^*, Q_{loss}^* and P_{loss}^{SIM}, Q_{loss}^{SIM}). It may lead to situation, when we choose the needful memory volume with big excess. So, in the case of small loss characteristics we should use obtained estimators to calculate needful memory space, but, in the other case, we should use simulation methods as more effective.

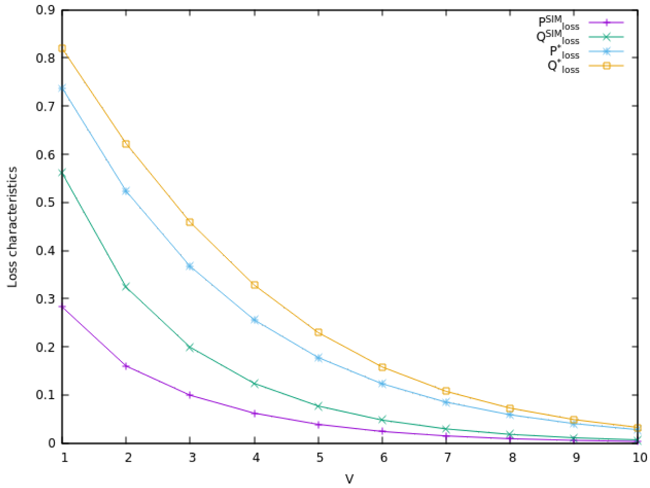

Fig. 7. Loss characteristics – scheme 2, service time is proportional to customer volume, $\rho = 0.75$

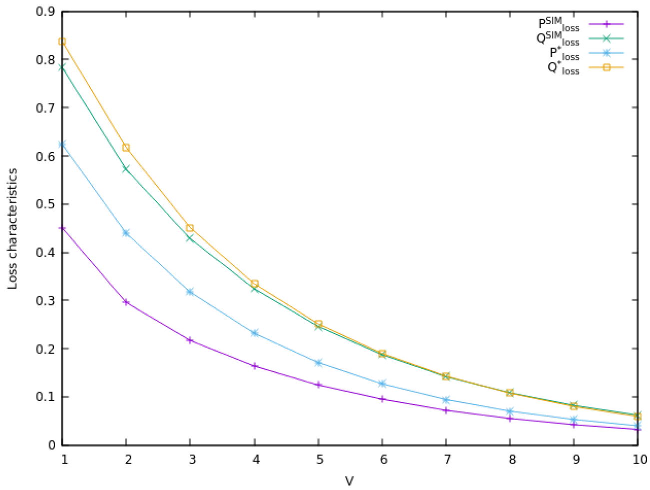

Fig. 8. Loss characteristics – scheme 2, service time is proportional to customer volume, $\rho = 0.4$

7. Conclusions

In the paper, we investigated a one-server queueing system with random volume external and internal customers under the assumption that service time of the customer depends of its volume and the total customers volume is unlimited. The non-stationary and stationary customers total volume distribution was obtained in terms of Laplace and Laplace-Stieltjes transforms. The first and second stationary moments of the total volume of customers present in the system were calculated for two schemes of system behavior.

The analytical results obtained were applied in estimation of information loss characteristics in the system with limited memory space.

The results obtained in the paper can be used for estimating the required memory volume in the nodes of computer and communication networks.

REFERENCES

[1] V.N. Roginsky, A.D. Kharkevich, and M.A. Shneps, *Communication Networks Theory*, Radio i Svyaz, Moscow, 1981 (in Russian).

[2] M. Schwartz, *Computer-Communication Network Design and Analysis*, Prentice-Hall, New Jersey, 1977.

[3] M. Schwartz, *Telecommunication Networks: Protocols, Modeling and Analysis*, Addison-Wesley Publishing Company, 1987.

[4] N.B. Zeliger, O.S. Chougreev, and G.G. Yanovsky, *Design of Networks and Systems of Discrete Messages Transmission*, Radio i Svyaz, Moscow, 1984 (in Russian).

[5] O.M. Tikhonenko, *Queueing Models in Information Systems*, Universitetskoe, Minsk, 1990 (in Russian).

[6] O. Tikhonenko, *Probability Methods of Information Systems Analysis*, Akademicka Oficyna Wydawnicza EXIT, Warszawa, 2006 (in Polish).

[7] A.M. Aleksandrov and B.A. Kaz, "Non-homogeneous demands arrival servicing", *Izv. AN SSSR. Tekhnicheskaya Kibernetika* N 2, 47–53 (1973) (in Russian).

[8] B. Sengupta, "The spatial requirement of M/G/1 queue or: how to design for buffer space", *Lect. Notes Contr. Inf. Sci.* 60, 547–562 (1984).

[9] O. Tikhonenko, "Determination of loss characteristics in queueing systems with demands of random space requirement", *Communications in Comp. and Inf. Sci.* 564, 209–215 (2015).

[10] O. M. Tikhonenko, "The problem of determination of the summarized messages volume in queueing systems and its applications", *J. Inf. Process. Cybern. EIK* 23(7), 339–352 (1987).

[11] V.B. Iversen, *Teletraffic Engineering Handbook*, Geneva, 2005.

[12] G.P. Klimov, *Stochastic Service Systems*, Nauka, Moscow, 1966 (in Russian).

[13] P.P. Bocharov, C. D'Apice, A.V. Pechinkin, and S. Salerno, *Queueing Theory*, VSP, Utrecht-Boston, 2004.

[14] Y.L. Luke, *Mathematical Functions and their Approximations*, Academic Press Inc. New York, 1975.

[15] M.L. Abell and J.P. Braselton, *The Mathematica Handbook*, Elsevier, 1992.