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Global convergence analysis of impulsive fractional order difference systems

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Abstract. This paper is designed to deal with the convergence and stability analysis of impulsive Caputo fractional order difference systems. Using the Lyapunov functions, the $\mathscr{Z}$-transforms of Caputo difference operators, and the properties of discrete Mittag-Leffler functions, some effective criteria are derived to guarantee the global convergence and the exponential stability of the addressed systems.

Key words: impulsive fractional difference systems, discrete Mittag-Leffler function, $\mathscr{Z}$-transforms, convergence.

1. Introduction

During the past decades, fractional order systems, including fractional order difference systems and fractional order differential systems, have been paid much attention due to their significant applications in the fields such as biology, physics, aerodynamics, electrical circuits, nonlinear oscillation of earthquake. Many significant results on the theory and application of fractional order systems have been obtained, see [1–20]. The basic theory of the fractional calculus are given in [1, 2]. The existence of solutions for fractional differential systems has been investigated in [6–9]. The stability of fractional differential systems has been considered in [5, 19]. The applications of fractional order differential systems in HIV model, SIR model, multi-agent systems and chaotic systems have been discussed in [3, 4, 10] and [11], respectively. The the initial value problem of fractional difference systems has been investigated in [12, 13, 16]. The stability of fractional difference systems has been considered in [14, 16, 18]. The observability of fractional difference systems has been studied in [15, 17]. The controllability and stabilising model predictive control of fractional difference systems are discussed in [15] and [20], respectively.

In addition, impulse effect exists in many evolution processes in which the states exhibit abrupt changes at certain moments. In recent years, some scholars try their efforts to introduce impulses into fractional order differential systems, and the dynamical behaviors of impulsive fractional order differential systems have become an active research topic. Many results are now available in the literature concerning stability [21–23], convergence [24–26] and existence and uniqueness [27, 28] of impulsive fractional order differential systems. However, the corresponding theory for impulsive fractional order difference systems has not been developed. Therefore, it is necessary and urgent to do research on the theory of impulsive fractional order difference systems. There is no doubt that stability is the main concern for dynamical systems. However, under perturbation of impulses, the equilibrium point probably does not exist in many practical systems. Therefore, the study of convergence is far more meaningful than the study of stability for impulsive systems. Meanwhile, convergence is an important asymptotic property of dynamical systems, which plays a key role in investigating the basic properties of the solutions such as stability, existence, persistence, and boundedness.

Motivated by the above discussion, this paper is mainly focused on the convergence of impulsive fractional order difference systems. Several sufficient criteria of the global convergence are obtained by using the Lyapunov functions, the $\mathscr{Z}$-transforms of Caputo difference operators, and the properties of discrete Mittag-Leffler functions.

2. Preliminaries

To begin with, we recall some useful notations, definitions and facts. For more details, one can see [14–16].

For any $a \in \mathbb{R}$, $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$. The family of binomial functions on $\mathbb{Z}$ parameterized by $\mu > 0$ and given by values: $\bar{\varphi}_\mu(n) = \binom{n+\mu-1}{n}$ for $n \in \mathbb{N}_0$ and $\bar{\varphi}_\mu(n) = 0$ for $n < 0$.

Definition 1. [14] For a function $y : \mathbb{N}_a \to \mathbb{R}$ the fractional sum of order $\alpha > 0$ is given by

$$\big({}_a\Delta^{-\alpha} y\big)(t) := \sum_{k=0}^{n} \binom{n-s+\alpha-1}{n-s} \bar{y}(s), \tag{1}$$

where $\bar{y}(s) := y(a+s)$ and $t \in \mathbb{N}_{a+\alpha}$.

Definition 2. [14] The Caputo-type difference operator ${}_a\Delta_*^\alpha$ of order α for a function $y : \mathbb{N}_a \to \mathbb{R}$ is defined by $\big({}_a\Delta_*^\alpha y\big)(t) := \big({}_a\Delta^{-(1-\alpha)}(\Delta y)\big)(t)$, where $t \in \mathbb{N}_{a+1-\alpha}$ and $\alpha \in (0, 1]$.

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Definition 3. [15] Let $n \in \mathbb{N}_0$, $\lambda \in (-1, 1)$, and $\alpha, \beta \in \mathbb{R}_+$. The one and two parameter discrete Mittag-Leffler functions are defined by

$$\begin{aligned} E_{\alpha,\beta}(\lambda, n) &:= \sum_{k=0}^{\infty} \lambda^k \bar{\varphi}_{k\alpha+\beta}(n-k) \\ &= \sum_{k=0}^{n} \lambda^k \bar{\varphi}_{k\alpha+\beta}(n-k), \end{aligned} \tag{2}$$

where the second equation only claim that for $k > n$ we have values of $\bar{\varphi}_{k\alpha+\beta}(n-k) = 0$,

$$\begin{aligned} E_{\alpha}(\lambda, n) := E_{\alpha,1}(\lambda, n) &= \sum_{k=0}^{\infty} \lambda^k \bar{\varphi}_{k\alpha+1}(n-k) \\ &= \sum_{k=0}^{\infty} \lambda^k \binom{n-k+k\alpha}{n-k}. \end{aligned} \tag{3}$$

Lemma 1. [16] Let $E_{\alpha,\beta}(\lambda, \cdot)$ be defined by (2). Then the $\mathscr{Z}$-transform of $E_{\alpha,\beta}(\lambda, \cdot)$ is given by

$$\mathscr{Z}[E_{\alpha,\beta}(\lambda, \cdot)](z) = \left(\frac{z}{z-1}\right)^{\beta}\left[1 - \frac{\lambda}{z}\left(\frac{z}{z-1}\right)^{\alpha}\right]^{-1}, \tag{4}$$

where $|z| < 1$ and $|z-1|^{\alpha}|z|^{1-\alpha} > |\lambda|$.

Lemma 2. [16] Let $a \in \mathbb{R}$, $\alpha \in (0, 1]$ and define $y(n) := ({}_a\Delta_*^{\alpha} y)(t)$, where $t \in \mathbb{N}_{a+1-\alpha}$. Then

$$\mathscr{Z}[y](z) = \left(\frac{z}{z-1}\right)^{1-\alpha}[(z-1)Y(z) - zy(a)] \tag{5}$$

where $Y(z) = \mathscr{Z}[\bar{y}](z)$ and $\bar{y}(n) := y(a+n)$.

Consider the following impulsive fractional order difference systems:

$$\begin{cases} {}_a\Delta_*^{\alpha} y(n) = f(n, y(a+n)), n \neq n_k, \;\; n \geq n_0, \\ y(n_k + a) = I_k(n_k - 1, y(n_k - 1 + a)), \;\; k \in \mathbb{N}_1 \\ y(a+n_0) = y_0 \end{cases} \tag{6}$$

where $0 < \alpha < 1$, $a = \alpha - 1$, $f : \mathbb{N}_{n_0} \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{N}_0 \times \mathbb{R}^n \to \to \mathbb{R}^n$, and $n_k (k \in \mathbb{N}_1)$ satisfy $n_1 < n_2 < \cdots < n_k < \cdots$ and $\lim_{k\to\infty} n_k = \infty$. In this paper, we always suppose that f and I_k satisfy the necessary conditions for the global existence and uniqueness of solutions for all $n \geq n_0$.

Definition 4. System (6) is said to be globally convergent to the ball

$$\mathscr{S} = \{y \in \mathbb{R}^n : \|y(n)\| \leq \mathscr{R}\}$$

if for any initial value $y_0 \in \mathbb{R}^n$, the solution $y(a+n; n_0, y_0)$ converges to $\mathscr{S}$ as $n \to \infty$.

Definition 5. System (6) is said to be globally exponentially stable with the exponential convergence rate λ, if there exist positive constants q, λ and K such that for any initial value $y_0 \in \mathbb{R}^n$,

$$\|y(a+n)\| \leq K\|y_0\|^q e^{-\lambda(n-n_0)}, \;\; n \geq n_0.$$

3. Convergence and stability analysis

Lemma 3. Let $n \in \mathbb{N}_0$, $\lambda \in (-1, 1)$ and $\alpha \in (0, 1)$. The discrete Mittag-Leffler functions have the following properties:

(a) $E_{\alpha}(\lambda, n) \geq 0$ and $E_{\alpha,\alpha+1}(\lambda, n) \geq 0$;

(b) $E_{\alpha}(\lambda, n)$ and $E_{\alpha,\alpha+1}(\lambda, n)$ are monotonically increasing on $\mathbb{N}_0$.

Proof. (a): The proof of (a) follows from (2) and (3).
(b): Let $m, n \in \mathbb{N}_0$ such that $m > n$. Then $m - n = s \in \mathbb{N}_1$. Using (2) and (3), we have

$$\begin{aligned} &E_{\alpha}(\lambda, m) - E_{\alpha}(\lambda, n) = \\ &= \sum_{k=0}^{\infty} \lambda^k \bar{\varphi}_{k\alpha+1}(m-k) - \sum_{k=0}^{\infty} \lambda^k \bar{\varphi}_{k\alpha+1}(n-k) \\ &= \sum_{k=0}^{\infty} \lambda^k \binom{m-k+k\alpha}{m-k} - \sum_{k=0}^{\infty} \lambda^k \binom{n-k+k\alpha}{n-k} \\ &= \begin{cases} \displaystyle\sum_{k=0}^{\infty} \lambda^k \frac{\Gamma(a+s-k+k\alpha+1)}{\Gamma(k\alpha+1)\Gamma(n+s-k+1)} \\ \displaystyle-\sum_{k=0}^{\infty} \lambda^k \frac{\Gamma(n-k+k\alpha+1)}{\Gamma(k\alpha+1)\Gamma(n-k+1)}, \;\; n \geq k; \\ 0, \; n < k. \end{cases} \\ &= \begin{cases} \displaystyle\sum_{k=0}^{\infty} \lambda^k \frac{\Gamma(n-k+k\alpha+1)}{\Gamma(k\alpha+1)\Gamma(n-k+1)}\left[\prod_{j=1}^{s}\left(1+\frac{k\alpha}{n+j-k}\right)-1\right] \geq 0, \; n \geq k; \\ 0, \; n < k, \end{cases} \end{aligned} \tag{7}$$

and

$$\begin{aligned} &E_{\alpha,\alpha+1}(\lambda, m) - E_{\alpha,\alpha+1}(\lambda, n) \\ &= \sum_{k=0}^{\infty} \lambda^k \binom{m-k+(k+1)\alpha}{m-k+\alpha} - \sum_{k=0}^{\infty} \lambda^k \binom{n-k+(k+1)\alpha}{n-k+\alpha} \\ &= \sum_{k=0}^{\infty} \lambda^k \frac{\Gamma(n+s-k+(k+1)\alpha+1)}{\Gamma(k\alpha+1)\Gamma(n+s-k+\alpha+1)} \\ &\quad - \sum_{k=0}^{\infty} \lambda^k \frac{\Gamma(n-k+(k+1)\alpha+1)}{\Gamma(k\alpha+1)\Gamma(n-k+\alpha+1)} \\ &= \begin{cases} \displaystyle\sum_{k=0}^{\infty} \lambda^k \frac{\Gamma(n-k+(k+1)\alpha+1)}{\Gamma(k\alpha+1)\Gamma(n-k+\alpha+1)}\left[\prod_{j=1}^{s}\left(1+\frac{k\alpha}{n+\alpha+j-k}\right)-1\right] \geq 0, \\ n+\alpha \geq k; \\ 0, \; n+\alpha < k. \end{cases} \end{aligned} \tag{8}$$

Therefore, for any $m, n \in \mathbb{N}_0$, if $m > n$, then $E_\alpha(\lambda, m) \geq E_\alpha(\lambda, n)$ and $E_{\alpha,\alpha+1}(\lambda, m) \geq E_{\alpha,\alpha+1}(\lambda, n)$. □

Theorem 1. Assume that there exists a function $V(n, y(a+n))$ $: \mathbb{N}_{n_0} \times \mathbb{R}^n \to \mathbb{R}_+ = [0, \infty)$ and several constants $\lambda_2 \geq 0$, $c_1 > 0$, $c_2 > 0$, $\mu_k > 0$ and $\lambda_1 \in (-1, 1)$ such that

(i) for all $(n, y) : \mathbb{N}_{n_0} \times \mathbb{R}^n$,

$$c_1 \|y(a+n)\|^2 \leq V(n, y(a+n)) \leq c_2 \|y(a+n)\|^2; \tag{9}$$

(ii) for all $k \in \mathbb{N}_1$ and $y \in \mathbb{R}^n$,

$$V(n_k, y(n_k + a)) \leq \mu_k V(n_k - 1, y(n_k - 1 + a)); \tag{10}$$

(iii) for all $k \in \mathbb{N}_1$, $n \in \Omega_k \triangleq [n_{k-1}, n_k)$, and $y \in \mathbb{R}^n$,

$${}_a\Delta_*^\beta V(n, y(n)) \leq -\lambda_1 V(n, y(a+n)) + \lambda_2; \tag{11}$$

(iv) $0 < N_k = n_k - n_{k-1} < \infty$, $k \in \mathbb{N}_1$,

$$\mu E_\beta(-\lambda_1, N_S) < 1; \tag{12}$$

where $\beta \in (0, 1)$, $a = \beta - 1$, $\mu = \sup_{k \in \mathbf{N}}\{\mu_k\}$ and $N_S = \sup_{k \in \mathbb{N}_1}\{N_k\}$. Then system (6) is globally convergent to the ball

$$\mathscr{S} = \left\{ y \in \mathbb{R}^n : \|y(n)\| \leq \sqrt{\frac{E_{\beta,\beta+1}(-\lambda_1, N_S - 1)\lambda_2}{c_1(1 - \mu E_\beta(-\lambda_1, N_S))}} \right\}. \tag{13}$$

Proof. It follows from (12) that there exists a nonnegative function $\mathscr{M}(n)$ such that

$$\begin{gathered} {}_a\Delta_*^\beta V(n, y(n)) + \lambda_1 V(n, y(a+n)) + \mathscr{M}(n) = \lambda_2, \\ n \in \Omega_k, \quad k \in \mathbb{N}_1 \end{gathered} \tag{14}$$

Taking the $\mathscr{Z}$-transform of equation (15) yields

$$\begin{gathered} \left(\frac{z}{z-1}\right)^{1-\beta} \left[(z-1)V(z) - zV(n_{k-1}, y(n_{k-1} + a))\right] \\ + \lambda_1 V(z) + \mathscr{M}(z) = \lambda_2 \frac{z}{z-1}, \end{gathered} \tag{15}$$

where $V(z) = \mathscr{Z}\{V(n, y(a+n))\}$ and $\mathscr{M}(z) = \mathscr{Z}\{\mathscr{M}(n)\}$.

Writing $V(z)$ in the form

$$\begin{aligned} V(z) = {} & \frac{z}{z-1} \frac{1}{\left(1 + \lambda_1 \frac{1}{z}\left(\frac{z}{z-1}\right)^\beta\right)} V(n_{k-1}, y(n_{k-1} + a)) \\ & - \frac{1}{z}\left(\frac{z}{z-1}\right)^\beta \frac{1}{\left(1 + \lambda_1 \frac{1}{z}\left(\frac{z}{z-1}\right)^\beta\right)} \mathscr{M}(z) \\ & + \frac{1}{z}\left(\frac{z}{z-1}\right)^{\beta+1} \frac{1}{\left(1 + \lambda_1 \frac{1}{z}\left(\frac{z}{z-1}\right)^\beta\right)} \lambda_2 . \end{aligned} \tag{16}$$

Taking inverse $\mathscr{Z}$-transform of (17) yields

$$\begin{aligned} & V(n, y(a+n)) = \\ & = V(n_{k-1}, y(n_{k-1} + a)) E_\beta(-\lambda, n - n_{k-1}) \\ & - \left[E_{\beta,\beta}(-\lambda, n - n_{k-1} - 1)\right] * \mathscr{M}(n) \\ & + \lambda_2 E_{\beta,\beta+1}(-\lambda, n - n_{k-1} - 1), \\ & t \in \Omega_k, \quad k \in \mathbb{N}_1, \end{aligned} \tag{17}$$

where $*$ denotes the convolution operator. Using (18) and noting that $E_{\beta,\beta}(-\lambda, n - n_{k-1} - 1) * \mathscr{M}(n)$ is nonnegative function, we obtain

$$\begin{aligned} & V(n, y(a+n)) \leq \\ & \leq V(n_{k-1}, y(n_{k-1} + a)) E_\beta(-\lambda_1, n - n_{k-1}) \\ & + \lambda_2 E_{\beta,\beta+1}(-\lambda_1, n - n_{k-1} - 1), \\ & n \in \Omega_k, \quad k \in \mathbb{N}_1, \end{aligned} \tag{18}$$

Taking $k = 1$ in (19), we get

$$\begin{aligned} V(n, y(a+n)) \leq {} & V(n_0, y(n_0 + a)) E_\beta(-\lambda_1, n - n_0) \\ & + \lambda_2 E_{\beta,\beta+1}(-\lambda_1, n - n_0 - 1), n \in \Omega_1. \end{aligned} \tag{19}$$

By (11) and (20),

$$\begin{aligned} & V(n_1, y(n_1 + a)) \leq \mu_1 V(n_1 - 1, y(a + n_1 - 1)) \\ & \leq \mu_1 V(n_0, y(n_0 + a)) E_\beta(-\lambda_1, n_1 - n_0 - 1) \\ & + \mu_1 \lambda_2 E_{\beta,\beta+1}(-\lambda_1, n_1 - n_0 - 2). \end{aligned} \tag{20}$$

Using (19) and (21),

$$\begin{aligned} & V(n, y(a+n)) \leq V(n_1, y(n_1 + a)) E_\beta(-\lambda_1, n - n_1) \\ & \qquad + \lambda_2 E_{\beta,\beta+1}(-\lambda_1, n - n_1 - 1) \\ & \leq \mu_1 V(n_0, y(n_0 + a)) E_\beta(-\lambda_1, n - n_1) \\ & \quad E_\beta(-\lambda_1, n_1 - n_0 - 1) \\ & + \mu_1 \lambda_2 E_\beta(-\lambda_1, n - n_1) E_{\beta,\beta+1}(-\lambda_1, n_1 - n_0 - 2) \\ & + \lambda_2 E_{\beta,\beta+1}(-\lambda_1, n - n_1 - 1), n \in \Omega_2. \end{aligned} \tag{21}$$

Further, we can get the following inequality

$$\begin{aligned} & V(n, y(a+n)) \leq \prod_{i=1}^{k-1} \mu_i E_\beta(-\lambda_1, n_i - n_{i-1})) E_\beta(-\lambda_1, \\ & \qquad n - n_{k-1}) \times V(n_0, y(n_0 + a)) \\ & + \sum_{j=2}^{k-1} \left\{ \left[\prod_{i=j}^{k-1} \mu_i E_\beta(-\lambda_1, n_i - n_{i-1})) \right] \mu_{j-1} E_{\beta,\beta+1}(-\lambda_1, \right. \\ & \quad \left. n_{j-1} - n_{j-2} - 2)) \right\} \lambda_2 E_\beta(-\lambda_1, n - n_{k-1}) \\ & + \mu_{k-1} E_{\beta,\beta+1}(-\lambda_1, n_{k-1} - n_{k-2} - 1)) \lambda_2 E_\beta(-\lambda_1, \\ & \quad n - n_{k-1}) + \lambda_2 E_{\beta,\beta+1}(-\lambda_1, n - n_{k-1} - 1)), \\ & n \in \Omega_k, \quad k \geq 3. \end{aligned} \tag{22}$$

Using (13, 23) and Lemma 3, we get

$$\begin{aligned}
&V(n,y(a+n))\\
&\leq(\mu E_\beta(-\lambda_1,N_S))^{k-1}E_\beta(-\lambda_1,N_S)V(n_0,y(n_0+a))\\
&\quad+\sum_{j=2}^{k-1}(\mu E_\beta(-\lambda_1,N_S))^{k-j}\mu E_{\beta,\beta+1}(-\lambda_1,N_S-2))\\
&\quad\times\lambda_2 E_\beta(-\lambda_1,N_S)\\
&\quad+\mu E_{\beta,\beta+1}(-\lambda_1,N_S-1))\lambda_2 E_\beta(-\lambda_1,N_S)\\
&\quad+\lambda_2 E_{\beta,\beta+1}(-\lambda_1,N_S-1))\\
&\leq(\mu E_\beta(-\lambda_1,N_S))^{k-1}E_\beta(-\lambda_1,N_S)V(n_0,y(n_0+a))\\
&\quad+\frac{(\mu E_\beta(-\lambda_1,N_S))^2}{1-\mu E_\beta(-\lambda_1,N_S)}E_{\beta,\beta+1}(-\lambda_1,N_S-2))\lambda_2\\
&\quad+\mu E_{\beta,\beta+1}(-\lambda_1,N_S-1))\lambda_2 E_\beta(-\lambda_1,N_S)\\
&\quad+\lambda_2 E_{\beta,\beta+1}(-\lambda_1,N_S-1))\\
&\leq(\mu E_\beta(-\lambda_1,N_S))^{k-1}E_\beta(-\lambda_1,N_S)V(n_0,y(n_0+a))\\
&\quad+\frac{E_{\beta,\beta+1}(-\lambda_1,N_S-1))\lambda_2}{1-\mu E_\beta(-\lambda_1,N_S)},n\in\Omega_k,k\geq 3.
\end{aligned}\tag{23}$$

From (20) and (22), we derive that

$$\begin{aligned}
V(n,y(a+n))\leq V(n_0,y(n_0+a))E_\beta(-\lambda_1,N_S)+\\
+\lambda_2 E_{\beta,\beta+1}(-\lambda_1,N_S-1),\quad n\in\Omega_1,
\end{aligned}\tag{24}$$

and

$$\begin{aligned}
&V(n,y(a+n))\leq\\
&\leq\mu_1 V(n_0,y(n_0+a))E_\beta(-\lambda_1,N_S)E_\beta(-\lambda_1,N_S-1)+\\
&\quad+\mu_1\lambda_2 E_\beta(-\lambda_1,N_S)E_{\beta,\beta+1}(-\lambda_1,N_S-2)+\\
&\quad+\lambda_2 E_{\beta,\beta+1}(-\lambda_1,N_S-1),\quad n\in\Omega_2,
\end{aligned}\tag{25}$$

respectively. Combining with (24–26) yields

$$\begin{aligned}
&V(n,y(a+n))\leq\\
&\leq(\mu E_\beta(-\lambda_1,N_S))^{k-1}E_\beta(-\lambda_1,N_S)V(n_0,y(n_0+a))+\\
&\quad+\frac{E_{\beta,\beta+1}(-\lambda_1,N_S-1)\lambda_2}{1-\mu E_\beta(-\lambda_1,N_S)},\quad n\in\Omega_k,\quad k\geq 1.
\end{aligned}\tag{26}$$

This, together with Condition (i), implies that

$$\|y(a+n)\|\leq\sqrt{\Theta(\mu E_\beta(-\lambda_1,N_S))^k+\Xi},n\in\Omega_k,k\geq 1,\tag{27}$$

where $\Theta=\frac{c_2}{\mu c_1}\|y(n_0+a)\|^2$ and $\Xi=\frac{E_{\beta,\beta+1}(-\lambda_1,N_S-1)\lambda_2}{c_1(1-\mu E_\beta(-\lambda_1,N_S))}$. The proof is completed. □

Corollary 1. Suppose that Conditions (i)-(iv) of Theorem 1 with $\lambda_2=0$ hold. Then system (6) is globally exponentially stable with the exponential convergence rate

$$\lambda=\frac{1}{2N_S}\ln\left(\frac{1}{\mu E_\beta(-\lambda_1,N_S)}\right).\tag{28}$$

Proof. If $\lambda_2=0$, then from (28) we have

$$\begin{aligned}
&\|y(a+n)\|\leq\sqrt{\frac{c_2}{\mu c_1}}\|y(n_0+a)\|(\mu E_\beta(-\lambda_1,N_S))^{k/2},\\
&n\in\Omega_k,\ k\geq 1.
\end{aligned}\tag{29}$$

For $n\in\Omega_k$, $k\geq 1$, we have

$$n-n_0\leq n_k-n_0\leq kN_S.\tag{30}$$

Using (30) and (31)

$$\begin{aligned}
&\|y(a+n)\|\leq\sqrt{\frac{c_2}{\mu c_1}}\|y(n_0+a)\|(\mu E_\beta(-\lambda_1,N_S))^{\frac{n-n_0}{2N_S}}=\\
&=\sqrt{\frac{c_2}{\mu c_1}}\|y(n_0+a)\|e^{-\left[\frac{1}{2N_S}\ln\left(\frac{1}{\mu E_\beta(-\lambda_1,N_S)}\right)\right](n-n_0)},\ n\geq n_0.
\end{aligned}\tag{31}$$

which ends the proof of Corollary 1. □

Theorem 2. Assume that there exists a function $V(n,y(a+n)):\mathbb{N}_{n_0}\times\mathbb{R}^n\to\mathbb{R}_+=[0,\infty)$ and several constants $\lambda_2\geq 0$, $c_1>0$, $c_2>0$, $p>0$, $q>0$, $\mu_k>0$ and $0<\lambda_3<1$ such that

(i) for all $(n,y)\in\mathbb{N}_{n_0}\times\mathbb{R}^n$,
$$c_1\|y(a+n)\|^p\leq V(n,y(a+n))\leq c_2\|y(a+n)\|^{pq};\tag{32}$$

(ii) for all $k\in\mathbb{N}_1$ and $y\in\mathbb{R}^n$,
$$V(n_k,y(n_k+a))\leq\mu_k V(n_k-1,y(n_k-1+a));\tag{33}$$

(iii) for all $k\in\mathbb{N}_1$, $n\in\Omega_k\triangleq[n_{k-1},n_k)$, and $y\in\mathbb{R}^n$,
$${}_a\Delta_*^\beta V(n,y(n))\leq-\lambda_3\|y(a+n)\|^{pq}+\lambda_2;\tag{34}$$

(iv) $0<N_k=n_k-n_{k-1}<\infty,\ k\in\mathbb{N}_1$,
$$\mu E_\beta\left(-\frac{\lambda_3}{c_2},N_S\right)<1;\tag{35}$$

where $\beta\in(0,1)$, $a=\beta-1$, $\mu=\sup_{k\in\mathbb{N}_1}\{\mu_k\}$ and $N_S=\sup_{k\in\mathbb{N}_1}\{\mu_k\}$. Then system (6) is globally convergent to the ball

$$\mathscr{S}=\left\{y\in\mathbb{R}^n:\|y(n)\|\leq\left[\frac{E_{\beta,\beta+1}\left(-\frac{\lambda_3}{c_2},N_S-1\right)\lambda_2}{c_1\left(1-\mu E_\beta\left(-\frac{\lambda_3}{c_2},N_S\right)\right)}\right]^{\frac{1}{p}}\right\}.\tag{36}$$

Proof. From inequalities (33) and (35) the following inequality holds

$$\begin{aligned}
&{}_a\Delta_*^\beta V(n,y(n))\leq-\frac{\lambda_3}{c_2}V(n,y(a+n))+\lambda_2,\\
&n\in\Omega_k\triangleq[n_{k-1},n_k),\ k\in\mathbb{N}_1.
\end{aligned}\tag{37}$$

Similar to the proof of Theorem 1, we have

$$\begin{aligned} &V(n, y(a+n)) \\ &\leq \left(\mu E_\beta\left(-\frac{\lambda_3}{c_2}, N_S\right)\right)^{k-1} E_\beta\left(-\frac{\lambda_3}{c_2}, N_S\right) V(n_0, y(n_0+a)) \\ &\quad+ \frac{E_{\beta,\beta+1}\left(-\frac{\lambda_3}{c_2}, N_S - 1\right)\lambda_2}{1-\mu E_\beta\left(-\frac{\lambda_3}{c_2}, N_S\right)}, \ n \in \Omega_k, \ k \geq 1. \end{aligned} \tag{38}$$

This, together with Condition (i), implies that

$$\begin{aligned} \|y(a+n)\| \leq &\left[\frac{c_2}{\mu c_1}\|y(n_0+a)\|^{pq}\left(\mu E_\beta\left(-\frac{\lambda_3}{c_2}, N_S\right)\right)^k\right. \\ &\left.+ \frac{E_{\beta,\beta+1}\left(-\frac{\lambda_3}{c_2}, N_S-1\right)\lambda_2}{c_1\left(1-\mu E_\beta\left(-\frac{\lambda_3}{c_2}, N_S\right)\right)}\right]^{\frac{1}{p}}, n \in \Omega_k, k \geq 1. \end{aligned} \tag{39}$$

The proof is completed. □

Corollary 2. Suppose that Conditions (i)-(iv) of Theorem 2 with $\lambda_2 = 0$ hold. Then system (6) is globally exponentially stable with the exponential convergence rate

$$\lambda = \frac{1}{pN_S} \ln\left(\frac{1}{\mu E_\beta\left(-\frac{\lambda_3}{c_2}, N_S\right)}\right). \tag{40}$$

Proof. If $\lambda_2 = 0$, then from (40) we have

$$\begin{aligned} &\|y(a+n)\| \leq \left(\frac{c_2}{\mu c_1}\right)^{\frac{1}{p}} \|y(n_0+a)\|^q \left(\mu E_\beta\left(-\frac{\lambda_3}{c_2}, N_S\right)\right)^{k/p}, \\ &n \in \Omega_k, \ k \geq 1. \end{aligned} \tag{41}$$

Using (42) and (31)

$$\begin{aligned} &\|y(a+n)\| \\ &\leq \left(\frac{c_2}{\mu c_1}\right)^{\frac{1}{p}} \|y(n_0+a)\|^q \left(\mu E_\beta\left(-\frac{\lambda_3}{c_2}, N_S\right)\right)^{\frac{n-n_0}{pN_S}} \\ &= \left(\frac{c_2}{\mu c_1}\right)^{\frac{1}{p}} \|y(n_0+a)\|^q e^{-\left[\frac{1}{pN_S}\ln\left(\frac{1}{\mu E_\beta\left(-\frac{\lambda_3}{c_2}, N_S\right)}\right)\right](n-n_0)}, n \geq n_0, \end{aligned} \tag{42}$$

which ends the proof of Corollary 2. □

4. Illustrative example

Example 1. Consider the impulsive fractional order difference system

$$\begin{cases} {}_{-0.5}\Delta_*^{0.5}\|y(n)\| = -b\|y(n-0.5)\| + d, \ n \neq n_k, \ n \geq n_0, \\ y(n_k - 0.5) = h_k y(n_k - 1 - 0.5)), \ k \in \mathbb{N}_1 \\ y(-0.5) = (0.3, 0.2)^T, \end{cases} \tag{43}$$

where $y \in \mathbb{R}^2$, $b > 0$, $d \geq 0$, $n_k = n_{k-1} + 2$, $k \in \mathbb{N}_1$, $h_k = [2E_{0.5}(-b, 2)]^{-0.5}$.

Let $V(n, y) = \|y\|$. Then by Theorem 2 for $c_1 = c_2 = p = q = 1$, $\mu_k = \mu = [2E_{0.5}(-b, 2)]^{-1}$, $N_k = N_S = 2$, $\lambda_3 = b$ and $\lambda_2 = d$, system (44) is globally convergent to the ball $\mathscr{S} = \{y \in \mathbb{R}^2 : \|y(n)\| \leq 2E_{0.5,1.5}(-b, 1)d\}$. Furthermore, if $d = 0$, then by Corollary 2, system (6) is globally exponentially stable with the exponential convergence rate $\lambda = \frac{1}{2}\ln 2$.

5. Conclusions

We have investigated the convergence and stability problem for a class of impulsive Caputo fractional order difference systems. Sufficient conditions for the global convergence and the exponential stability of the addressed systems have been presented based on the Lyapunov functions, the $\mathscr{Z}$-transforms of Caputo difference operators, and the properties of discrete Mittag-Leffler functions. The obtained results can be used to discuss the convergence of more complicated systems such as neural networks, multi-agent systems, and switching systems. We will do some further research in this direction.

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