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On a variational approach to the problem of singular geological structures

Abstract: Recognition of geological structures often requires understanding the causes of diverse kinetic phenomenon and its underlying foundations. This pertains, e.g., to the phenomenon of mass movement within a rock formation leading to fault formation. We discuss here the possibility that variational calculus may be an important tool for investigating this problem. Analysis of variations may yield important information concerning a physical phenomenon. Here we will neglect the best known problems of extremals in the analysis of variations and will focus our attention on electromagnetic and physico-mechanical problems.

Adaptation of a Hamiltonian as an entropy operator may serve, not only for the problems of singular crystalline structures, but also geological singularities such as faults, oleate impermeabilities, deep-seated eruptions as well as in problems of seismology, vulcanology and earthquakes.

This paper is an attempt to initiate a discussion about the possible development of the ideas presented. It might be that the formulae presented will be useful for the solution of other geophysical problems in future.

Keywords: variations of functional, fault formation

O wariacyjnym podejściu do problemu osobliwych struktur geologicznych

Streszczenie: Poznanie geologicznych struktur często wymaga zrozumienia przyczyn kinetyki zjawiska przy uwzględnieniu jego zasadniczych podstaw. Takie ujęcie umożliwia nam powiązanie skutków procesów z przyczynami je warunkującymi. Dotyczy to wielu zjawisk, na przykład transportu masy wewnątrz górotworu prowadzącego do powstania uskoku. Rozważamy możliwość zastosowania rachunku wariacyjnego w rozwiązywaniu wspomnianego zagadnienia, zachęcając jednocześnie do krytycznego oglądu i własnych przemyśleń. Przy pomocy analiz wariacyjnych uzyskać możemy wiele informacji o zjawisku fizycznym. Wyłączeniu ulegną najbardziej znane problemy ekstremali, które zostaną w znacznej części pominięte, natomiast całkowicie skupimy się na zagadnieniach elektromagnetycznych i mechanicznych.

Adaptacja hamiltonianu jako operatora entropijnego służyć może nie tylko problematyce osobliwych struktur krystalicznych, ale również problematyce osobliwych struktur geologicznych, takich jak uskoki, pułapki naftowe, erupcje wgłębne czy w zagadnieniach sejsmologii, wulkanologii czy trzęsień ziemi.

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Niniejsza praca jest próbą zachęty do dyskusji nad rozwinięciem tego zagadnienia.
Trzeba też zauważyć, że rachunek wariacyjny może okazać się przydatny w rozwiązywaniu innych problemów geofizycznych, co nastąpi w nieodległej przyszłości.

Słowa kluczowe: rachunek wariacyjny, tworzenie uskoku

Introduction

The search for the optimal structures, such as the best trajectories, the shortest distances, surfaces of stable potentials, constant speed, depth &c. that accomplish some physical processes, resulted within the last 200 years in development of calculus of variations.

Hamiltonian H_a was the result of transformation of Newton's equations. The equations transforming Newton's equations were invented by Lagrange. Hamilton, in turn, transformed Lagrange's equations, obtaining the equations called later Hamilton's equations. They are equivalent to Euler's equations for variance problems, and concern the point dynamics. Later on, the variance calculus was used for description of other phenomena, not related to point dynamics, and the function yielding results was called "Hamiltonian" without relations with dynamic phenomena. At this point, the contact between the problems of variance analysis and point dynamics became loosened.

The present epistemological questions create new problems to be solved by variation calculus. The problems discussed in this paper offer possibilities of prediction of the course of faults, so important in the prognosis of earthquakes, the problem particularly investigated by professor Teisseyre (Teisseyre 2006).

Let us begin from relatively simple problems, creating the ideology of the investigation.

1. Variational principle

Let us consider a functional ϕ. For a certain curve $x(t)$ the functional ϕ can reach its extremum value (Elsgolc 1960)

$$\phi = \int_a^b H_a\left(x(t),t\right)dt = extremum \tag{1}$$

and the variation calculus will be equal to zero;

$$\delta\varphi = 0 \tag{2}$$

This notation denotes the search for such trajectory $x(t)$ for which variance of the functional ϕ becomes zero. In such situation this function H_a must fulfill the known Euler's equation

$$\frac{\partial H_a}{\partial x} - \frac{d}{dt}\frac{\partial H_a}{\partial_x} = 0 \tag{3}$$

Generally the functional problem is expanded to multiparameter problem, because of the problem is connected with many derivatives of the optimal function. In the cases of variable areas in the space, the original Euler's equation can become other equation (e.g., Euler – Ostrogradsky equation).

To elucidate the ideology of the functional variation we shall quote simple examples of variation calculus from the Elsgolc textbook "Calculus of Variations" (Elsgolc 1951).

Examples

Example 1. A material point of unitary mass slides down from the point $A(0, 0)$ to point $B(x_1, y_1)$. Find the shortest time and trajectory of sliding.

As the potential energy is transformed without loss into kinetic energy, the velocity of the point at the height of y equals

$$\frac{ds}{dt} = \sqrt{2gy} \tag{4}$$

where:

- s – path,
- t – time,
- g – gravity constant.

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{5}$$

$$dt = \frac{1}{\sqrt{2gy}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{6}$$

After bilateral integration the time of sliding is given by a functional with border conditions $y(0) = 0$, $y(x_1) = y_1$

$$t = \frac{1}{\sqrt{2g}}\int_0^{x_1}\frac{1}{\sqrt{y}}\sqrt{1+y'^2}\,dx \tag{7}$$

where:

$\frac{1}{\sqrt{y}}\sqrt{1+y'^2}$ – is the so-called Hamiltonian.

Writing the Euler's equation (3) for the problem we receive differential equation

$$\frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'^2}{y\left(1+y'^2\right)} = C \tag{8}$$

The solution of this equation yields trajectory $y = y(x)$.

Example 2:

On which curves the functional

$$F\left[y(x)\right]=\int_{0}^{\frac{\pi}{2}}\left(\left(y'\right)^{2}-y^{2}\right)dx \tag{9}$$

may obtain extremal.

The border conditions for the curve solving the problem are

$$y\,(0)=0, \qquad y\left(\frac{\pi}{2}\right)=1$$

Answer: The form of Euler's equation is $y'' + y = 0$. Its $y = C_1\cos x + C_2\sin x$. From the border conditions, we receive $C_1 = 0$, $C_2 = 1$, and hence the extremal may be reached only on the curve: $y = \sin x$.

Example 3:

Find the shortest line connecting two given points on the surface Z

$$\varphi(x,y,z)=0 \tag{10}$$

According to the literature we should find the minimum of functional L

$$L=\int_{x_1}^{x_2}\sqrt{1+y^{2}+z^{2}}\,dx \tag{11}$$

This problem was solved in 1697 by Jan Bernoulli, but the general method was presented only in the works of Euler and Lagrange.

Time $t(y, x)$ of traveling along the each of the curves $y(x)$ is a functional of type

$$t(y,x)=\int_{x_1}^{x_2}\frac{\sqrt{1+y^{2}}}{v(y)}dx \tag{12}$$

Reasons for the shape of the functional

$$\frac{ds}{dt}=v(y)\rightarrow dt=\frac{ds}{v(y)}=\frac{\sqrt{1+y^{2}}}{v(y)}dx \tag{13}$$

Calculation permits to find extremals, and the speed of point movement depends only on the y. The extremals are the straight lines.

1. Application of variational methods to description of some geological phenomena

Analysis by means of variational methods may yield important information about geological structures or phenomena. We shall not consider here the most classical problems using variational principles, but shall focus our attention on electromagnetic and mechanical problems.

Thus, let us consider the general functional ϕ (Hestens 1966)

$$\varphi = \int_D H_a(x, y_i(x), y_i'(x), y_i''(x) \ldots) dx = extremum \tag{14}$$

$$x \in R^{n+1}$$

where D is a certain domain in which the variation $\delta\phi$ becomes zero

$$\delta\phi = 0 \tag{15}$$

where the difference of the functional ϕ at y_i and $y_i + h_i$ is

$$\phi(y_i + h_i) - \phi(y_i) = \delta\phi \tag{16}$$

h_i is a variation of y_i ($h_i = \delta h$).

For a function H_a which is:

a) Hamiltonian H_a

$$H_a = E_{kin} - E_{pot} \tag{17}$$

where:

E_{kin} and E_{pot} are the kinetic and potential energy respectively of a material particle, from the variation of functional of the Euler-Lagrange equations we obtain

$$H_{y_r} - \sum_{k=1}^{n+1} \frac{\partial}{\partial x_k} H_{y_r'} = 0 \tag{18}$$

leading eventually to the equations of dynamics for a material particle.

b) The Noether operator (unfortunately, often also called "Hamiltonian") H_a

$$\mathrm{H_a} = m\frac{v^2}{2} - eV - e\mathbf{Av} \tag{19}$$

where:

$\boldsymbol{A}$ – vector potential,
V – scalar potential,
e – electron charge,
$\boldsymbol{v}$ – particle speed,
m – particle mass

we obtain Maxwell's equations (Morse and Feshbach 1953)

$$\frac{\partial \boldsymbol{H}}{\partial t} + rot\ \boldsymbol{E} = 0 \tag{20}$$

$$\boldsymbol{E}: R^3 \times R^1 \rightarrow R^3$$

$$-\frac{\partial \boldsymbol{E}}{\partial t} + rot\ \boldsymbol{H} = J \tag{21}$$

$$\boldsymbol{H}: R^3 \times R^1 \rightarrow R^3$$

where:

$\boldsymbol{E}$ – vector of the electric intensity,
$\boldsymbol{H}$ – vector of the magnetic intensity,
$\boldsymbol{J}$ – vector of current density.

Maxwell's equations describe the electromagnetic field that is used in geophysics. These equations are used for propagation of the electromagnetic wave in the medium.

c) Brunacci's operator

$$H_a = (\nabla V)^2 \tag{22}$$

where V is a scalar potential, we obtain the Laplace equation (Lauwerier 1966)

$$\nabla V = 0 \tag{23}$$

d) Operator

$$H_a = -\frac{h^2}{2m} \nabla\psi \cdot \nabla\psi - \frac{h}{2i}(\psi \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi}{\partial t}) - \psi V \psi \tag{24}$$

where:

h – Planck's constant,
m – particle mass,
V – potential, acting on the particle,
ψ – Born's function

we obtain a Schrödinger equation (Ławrynowicz 1977)

$$\frac{\partial \psi}{\partial t} = i\Delta\psi + V\psi \tag{25}$$

e) Wójcik's operator of entropy density

$$H_a = s = f\ln f \tag{26}$$

where:

- s – entropy density,
- f – stationary solution of Boltzmann's equation. The entropy density is to be considered as entropy. At first approximation it is the distribution density beta B: $6E(1 - E)$, where: E denotes the total energy.

Thus, one obtains differential equations for singular crystalline structures that are important in the area of nanomechanics (Wójcik 2005). Only Wójcik's Hamiltonian has a physical justification, as it is the movement of a particle in the field of energy probability given by Boltzman's distribution.

2. Considering geological problems

Stresses in rock masses produce different kinds of deformations. Thanks to geophysical measurements such as for example electricity survey, one can predict a certain of rock mass deformations. The apparent resistivity changes are considered to be precursors related to earthquakes (XueBin Du 2011). Changes in electrical resistivity were observed as a function of stress in a variety of rocks before earthquake (Brace et al. 1968).

In order to investigate the relationship between the evolution of strain induced by external stress and electrical resistivity laboratory tests were made. The samples were compressed triaxially until destruction occurred. The dependence of the electrical resistivity and differential stress on the strain force were carried out at the laboratory of Geophysical Department AGH (Fig. 1) (Tomecka-Suchoń 1997).

Despite the field experiments are in approximate agreement with laboratory measurements, field experiments have had much less success. In some cases, no precursors occurred prior to earthquakes (Park et al. 1993). In predicting events such as the formation of sinkholes or faults creation, in addition to the geophysical methods used so far, it could be used variation calculus. Variation calculus will require obtaining precise data about deformations in the area under investigation. Geodetic measurements of the solid Earth integrated with interferometric synthetic aperture radar provide spatially continuous observations of deformation with sub-centimeter accuracy (Tralli et al. 2005).

In order to be able to predict fault creation by measuring electrical resistivity changes it would be useful to know the rock-mass strain. This can be calculating using variations calculus.

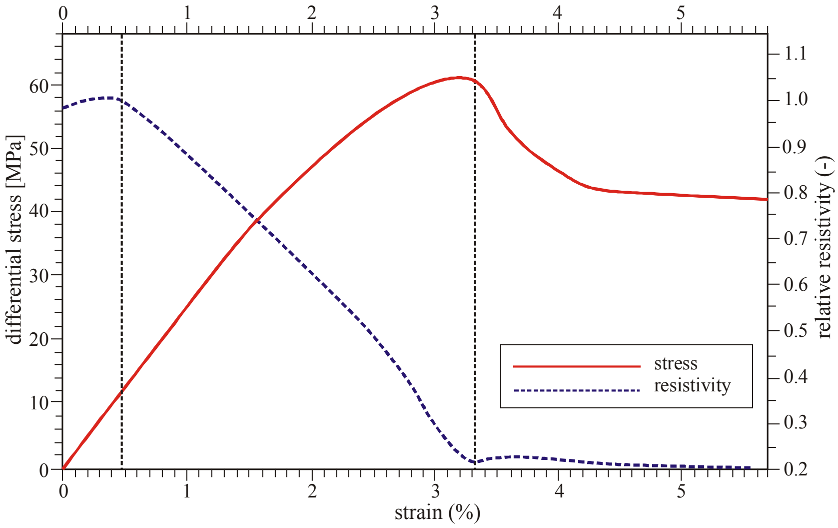

Fig. 1. Changes of resistivity in specimen under triaxial stress (Tomecka-Suchoń et al. 1997)

Rys. 1. Zmiany rezystancji w próbce w trójosiowym stanie naprężenia (Tomecka-Suchoń et. al. 1997)

Theoretical seismology has been used to develop ground motion simulation methods that can predict these ground motion variations (Somerville 2004).To reduce the uncertainty in the estimation of seismic hazard ground motion models are needed that can predict these site-to site variations.

Models can explain variations in ground motions that depend on the orientation and geometry of the fault. Seismological ground motion prediction can reduce the uncertainty in ground motion prediction.

Adaptation of a Hamiltonian as an entropy operator may serve, not only for the problems of singular crystalline structures, but also geological singularities such as faults, plate impermeabilities, deep-seated eruptions as well as in problems of seismology and volcanology.

By means of geological considerations, we may obtain the structure of a fault by analysis of the Hamiltonian considering the set of lines forming it. For each line we can find a variation of the respective functional ϕ set by the Hamiltonian H as a difference between the kinetic energy and an appropriately chosen potential energy $V(x,a)$. The Hamiltonian then takes the form

$$H = \frac{m}{2}\left(x^2 + y^2\right) - V\left(x, y, a\right) \tag{27}$$

e.g., for $(x, y) \in R^2$

where:

V – potential of interaction of masses in the orogen,
m – the mass of the element,
a – parameter.

Let us write Euler's equation (3) for the above presented problem

$$\frac{\partial \mathrm{H_a}}{\partial x} - \frac{d}{dt}\frac{\partial H_a}{\partial \dot{x}} = 0 \tag{28}$$

$$\frac{\partial H_a}{\partial y} - \frac{d}{dt}\frac{\partial H_a}{\partial \dot{y}} = 0 \tag{29}$$

or

$$-\frac{\partial V}{\partial x} + \frac{d}{dt}(m\dot{x}) = 0 \tag{30}$$

$$-\frac{dV}{\partial y} + \frac{d}{dt}(m\dot{y}) = 0 \tag{31}$$

where:

$x(t)$, $y(t)$ – coordinates of the trajectory of a particle,
m – particle mass.

We therefore may derive a dynamic equation of the second order

$$m\ddot{x} - \frac{\partial V}{\partial x} = 0 \tag{32}$$

$$m\ddot{y} - \frac{\partial V}{\partial y} = 0 \tag{33}$$

where: $\ddot{x} = \frac{\partial^2 x}{\partial t^2}$, $\ddot{y} = \frac{\partial^2 y}{\partial t^2}$

Let us consider a case

$$V(x, y; a) = V_1(x; a) + V_2(y; a) \tag{34}$$

The Euler equation for $x(t)$ is

$$2m\dot{x}\ddot{x} = 2\dot{x}\frac{dV_I}{dx} \tag{35}$$

Hence

$$\dot{x} = \pm\sqrt{\frac{2}{m}\int \frac{dV_I(x)}{dx}\dot{x}dt} \tag{36}$$

or

$$\dot{x} = \pm\sqrt{\frac{2}{m}}\sqrt{V_1(x)} \tag{37}$$

Therefore

$$\frac{dx}{\pm\sqrt{\frac{2}{m}V_1(x)}} = dt \tag{38}$$

for which we have two solutions (therefore, depending on the boundary conditions, the trajectory may not be smooth

$$t + C_1 = \sqrt{\frac{m}{2}\int\frac{dx}{\sqrt{V_1(x)}}} \tag{39}$$

and

$$t + C_1 = -\sqrt{\frac{m}{2}\int\frac{dx}{\sqrt{V_1(x)}}} \tag{40}$$

As seen, we obtained the solution for the inverse function.

We did not consider here the possibility that the potential V may depend on a parameter a, but we must remember that the presence of such a parameter may appreciably influence the solution.

In some cases and for certain boundary conditions, the functions $\phi(t)$ are singular. Therefore, we may obtain a description of functions corresponding to geomorphic changes such as faults.

It is known (from the appropriate theorem) that if Hamiltonian depends only on $\dot{x}$, the trajectories are straight lines of different slopes dependent on border conditions (and thus on the speed of the process).

Conclusions

This paper is an attempt to initiate a discussion about the possible development of the application of the variation calculus functional in the geology. We expect that the formulae presented here will be useful for the solution of diverse geophysical problems in future.

If the Hamiltonian H depends only on one dimension, the solution are straight lines that determine the singularity, in which the potential V is discontinuous.

The fundamental premise of calculations is determination of border conditions that describe the situation of the border of the area. They may depend, among other, on compactness of the ground, mass distribution, gravitational interactions, and viscosity. Some considerations presented by Jose M. Carcione may serve as a background for analysis of potential distribution in geological medium (Carcione 2000).

After proper determination of border conditions and solving Laplace's equation $\Delta V = 0$ we obtain the answer concerning the potential within the area.

The paper was the review of theoretical basics of the problem. In real situation we must examine, using geological and geophysical methods, all occurring physical parameters leading to solution of Laplace's equation against singularities of potentials.

Following the determination of border conditions we solve the Laplace's equation which is a prerequisite of finding of potential as a discontinuous function's. The final form of the answer on the fundamental question is the result of the solution of Euler's differential equation, which yields the description of the trajectory of the event.

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