Bayesian Inference for a Deterministic Cycle with Time-Varying Amplitude: The Case of the Growth Cycle in European Countries

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Abstract

The main goal of this paper is to propose the probabilistic description of cyclical (business) fluctuations. We generalize a fixed deterministic cycle model by incorporating the time-varying amplitude. More specifically, we assume that the mean function of cyclical fluctuations depends on unknown frequencies (related to the lengths of the cyclical fluctuations) in a similar way to the almost periodic mean function in a fixed deterministic cycle, while the assumption concerning constant amplitude is relaxed. We assume that the amplitude associated with a given frequency is time-varying and is a spline function. Finally, using a Bayesian approach and under standard prior assumptions, we obtain the explicit marginal posterior distribution for the vector of frequency parameters. In our empirical analysis, we consider the monthly industrial production in most European countries. Based on the highest marginal data density value, we choose the best model to describe the considered growth cycle. In most cases, data support the model with a time-varying amplitude. In addition, the expectation of the posterior distribution of the deterministic cycle for the considered growth cycles has similar dynamics to cycles extracted by standard bandpass filtration methods.

Keywords: deterministic cycle with time-varying amplitude, Bayesian inference, almost periodic function, growth cycle, industrial production

JEL Classification: C11, E32

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1 Introduction

The concept of stochastic cycle is very well known (see Harvey and Jaeger (1993), Harvey and Trimbur (2003), Harvey (2004), Azevedo, Koopman, and Rua (2006), Trimbur (2006), Harvey, Trimbur, and Dijk (2007), Koopman and Shephard (2015), Pelagatti (2016) and many others). This concept assumes the stationarity of cyclical fluctuations with a zero mean function. In the univariate case, the aforementioned construct assumes the following for observed process Y_t:

$$\begin{cases} Y_t = \mu_t + \psi_{n,t} + \epsilon_t \\ \mu_t = \mu_{t-1} + \beta_{t-1} \\ \beta_t = \beta_{t-1} + \zeta_t, \end{cases} \tag{1}$$

where ϵ_t and ζ_t are white noise and the stochastic component $\Delta\mu_t$ is a random walk. The component $\psi_{n,t}$ is a generalization of the cycle presented in Harvey and Jaeger (1993). This generalization is called the nth-order cycle and is characterized by the concentration of the spectral density function around the frequency λ_c (see illustrative example in Trimbur (2006)). The multivariate case was considered in Azevedo, Koopman, and Rua (2006) and Harvey, Trimbur, and Dijk (2007) where the trivariate example was considered. In Koopman and Azevedo (2008) the multivariate case was also considered. It was shown that the phase shifts incorporated in this stationary model are flexible and allow for increasing or diminishing.

So far, no other concept has been developed to compete with the stochastic cycle $\psi_{n,t}$. Recently, some preliminary results concerning a new concept of a stationary nonlinear stochastic cycle model were presented in Lenart and Wróblewska (2018). They combine the idea of the linear innovations state space model with the properties of the *sine* function in the following way:

$$\begin{cases} Y_t = (A + A_{t-1})\sin[\lambda(t + T + T_{t-1})] + \mu + \epsilon_t & \\ A_t = \psi_A A_{t-1} + \alpha_A \epsilon_t & \text{deviations from amplitude } A \\ T_t = \psi_T T_{t-1} + \alpha_T \epsilon_t & \text{deviations from phase shift } T \end{cases} \tag{2}$$

where $A, T, \mu, \lambda, \alpha_A, \alpha_T, \psi_A, \psi_T \in \mathbb{R}$, $|\psi_A| \leq 1$, $|\psi_T| \leq 1$, $\lambda \in (0, \pi)$ and ϵ_t is Gaussian white noise.

The models with a deterministic cycle are not as popular as models with a stochastic cycle. Following Harvey (2004), the concept of a fixed deterministic cycle is based on an almost periodic function at time $t \in \mathbb{Z}$ with one frequency $\lambda \in (0, \pi)$ of the form

$$f(t) = a\sin(\lambda t) + b\cos(\lambda t),$$

where $a, b \in \mathbb{R}$. It is widely known that the above function is not flexible enough to describe the variable in time dynamics of the business cycle. Therefore the more flexible concepts of the deterministic cycle were recently considered. In Lenart and Pipień (2013), the nonparametric inference was considered under the

assumption that the conditional expectation of observed process contains an almost periodic component with more than one frequency. In Lenart, Mazur, and Pipień (2016), the parametric and nonparametric inference were considered under the assumption that the mean function of a cyclical process is an almost periodic function with few frequencies. Finally, in Lenart and Pipień (2017), the authors consider the nonparametric test based on a subsampling approach to test the common deterministic cycles in industrial production in selected European countries. The concept of the deterministic cycle has been used many times for macroeconomic data using parametric inference and a Bayesian approach (see Mazur (2016), Mazur (2017a), Mazur (2017b), Mazur (2018)). Some preliminary results concerning the modelling of cyclical fluctuations using both deterministic and stochastic cycle concepts were considered in Lenart and Mazur (2017).
In all of the above approaches the amplitude of the considered deterministic cycle is assumed to be constant in time. This assumption seems to be too strong, taking into consideration the variable nature of the business cycle. In Lenart (2018), the preliminary results concerning time-varying amplitude in a deterministic cycle was developed. The illustrative example was considered using a growth rate cycle for industrial production in Poland (for monthly data).
In this paper, we significantly develop an approach introduced in Lenart (2018). Concurrently, we investigate the time-varying amplitude by considering the following function

$$g(\lambda, t) = a(t)\sin(\lambda t) + b(t)\cos(\lambda t) \tag{3}$$

of integer $t \in \mathbb{Z}$, where $a(\cdot)$ and $b(\cdot)$ are functions of integers. Note that $g(\lambda, t)$ is an alternative for the stochastic cycle component $\psi_{n,t}$ in (1). If $a(\cdot)$, $b(\cdot)$ are constant functions, then we obtain the usual almost periodic function. For the functions $a(\cdot)$, $b(\cdot)$ we make the following natural assumptions.

Assumption 1. *The functions $a(t)$ and $b(t)$ are bounded, which means that there exists $K \in \mathbb{R}$ such that $|a(t)| < K$ and $|b(t)| < K$, uniformly for $t \in \mathbb{Z}$.*

Assumption 2. *The functions $a(t)$ and $b(t)$ can be represented as $a(t) = a_0 + \tilde{a}(t)$ and $b(t) = b_0 + \tilde{b}(t)$, where $a_0, b_0 \in \mathbb{R}$ and the functions $\tilde{a}(t)$ and $\tilde{b}(t)$ are the functions with empty spectrum, that is for any $\kappa \in [0, 2\pi)$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \tilde{a}(t) e^{-it\kappa} = 0, \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \tilde{b}(t) e^{-it\kappa} = 0. \tag{4}$$

Assumption 1 means that the time varying pseudo-amplitude of the function (3)

$$\mathrm{amp}(t) = \sqrt{a^2(t) + b^2(t)} \tag{5}$$

is bounded, $\mathrm{amp}(t) < \sqrt{2}K$ uniformly at $t \in \mathbb{Z}$. Condition (4) guarantees that there is only one source of the cyclical fluctuation in function (3). This source is related

only to frequency λ (see Napolitano (2012) for more details).
Assume that we have a sample path $\{X_1, X_2, \ldots, X_n\}$ from time series $\{X_t : t \in \mathbb{Z}\}$ with mean function $g(\lambda, t)$, i.e.,

$$E(X_t) = a(t)\sin(\lambda t) + b(t)\cos(\lambda t).$$

In such a case, from the purely nonparametric point of view, under Assumption 2 we have

$$E\left(\frac{1}{n}\sum_{t=1}^{n} X_t \sin(\omega t)\right) \to \begin{cases} \frac{a}{2} & \text{if } \omega = \lambda \\ 0 & \text{if } \omega \neq \lambda \end{cases}, \tag{6}$$

$$E\left(\frac{1}{n}\sum_{t=1}^{n} X_t \cos(\omega t)\right) \to \begin{cases} \frac{b}{2} & \text{if } \omega = \lambda \\ 0 & \text{if } \omega \neq \lambda \end{cases}. \tag{7}$$

Hence, from the nonparametric point of view, the statistical inference concerning constants a and b seems possible. However, from the practical point of view, we are interested in approximating functions $a(t)$ and $b(t)$ rather than just constants a and b. Therefore, we propose using a parametric approach rather than the nonparametric approach. Moreover, in using the parametric approach we can propose some parametric forms for $a(t)$ and $b(t)$. In this paper, we assume that functions $a(t)$ and $b(t)$ are linear splines or are related to the Bézier curve. As will be shown later in this work, such a parametric approach based on a Bayesian approach allows the formation of a fully probabilistic inference concerning cyclical fluctuations.
The paper is organized as follows. In Section 2 we present the monthly data concerning industrial production in European economies from 2001 to 2017 and we formulate the main problems related to the cyclical behavior of such data. This type of behavior creates a real challenge for modelling. In Section 3 we introduce the model with time-varying amplitude of the deterministic cycle. In Section 4 we present the Bayesian inference for such model. In particular we show the closed form for the marginal posterior distribution for the frequency vector related to the length of the cycle for cyclical fluctuations. In the last section we analyze the empirical results.

2 Data presentation and the main hypothesis

We consider the growth cycle for industrial production (mining and quarrying; manufacturing; electricity, gas, steam and air conditioning supply) with a monthly frequency (calendar adjusted data, not seasonally adjusted; percentage change compared to the same period in the previous year, source: Eurostat). The data cover the period January 2001 to December 2017. We consider aggregate production for the European Union (28 countries) and for the Euro area (19 countries) and individual industrial production for 32 countries (Belgium; Bulgaria; Czech Republic; Denmark; Germany; Estonia; Ireland; Greece; Spain; France; Croatia; Italy; Cyprus; Latvia; Lithuania; Luxembourg; Hungary; Malta; the Netherlands; Austria; Poland;

Portugal; Romania; Slovenia; Slovakia; Finland; Sweden; the United Kingdom; Norway; the Former Yugoslav Republic of Macedonia or Macedonia; Serbia and Turkey).

Figures 2-7 present (among others) the growth cycles (gray line) for considered data sets. To formulate the model correctly, in the next section we will try to discuss the basic dynamic properties of the data analyzed below.

First, the recent world crisis in 2009 is clearly visible in the considered data as a period with extremely low values for the production index for most European economies. This is an important feature of the analyzed data, because without proper modelling, the final conclusions concerning the business cycle may be strongly influenced by the recent world crisis. Let us emphasize that the last crisis occurred after the dynamic economic growth in period of 2006-2007. It can therefore be concluded that during the period 2006-2007, the economic slowdown was expected to occur in near future. However, the scale of this slowdown was a surprise for most economies. In most European economies the intensification of the economic slowdown during the recent global crisis was manifested, among others, by a very low value of the industrial production index. This intensification of the economic slowdown in most European economies took place in a very similar period of time. For the growth cycle in production, the lowest values were recorded in the months of the first, second and third quarter of 2009 (most often in April 2009 for the considered economies or groups of countries). From a purely econometric point of view, the problem is not only with the decline in the production during the recent world crisis but also with the dynamic increase of the amplitude of these fluctuations for most of the considered economies. The purpose of this article is not, however, to analyse the scale of the recent crisis related to this amplitude. However, such a dynamic increase of the amplitude in industrial production may influence the results of statistical inference if this problem is ignored. Therefore, it will be considered during the construction of the model.

Second, for some countries the growth rate cycle is not smooth enough to visually identify the phase of the cycle (see, for example, Norway), while for other countries the phase of the cycle is much easier to identify (see, for example, Hungary, Poland, Finland and many others). For some countries, cyclical fluctuations in industrial production are of high amplitude, while for some the amplitude is much lower. However, in most cases the amplitude of cyclical fluctuations is not stable over time. It seems obvious that, despite some differences in the dynamics of analysed growth cycles, a common cyclical pattern should exist for most economies. Based on visual observation of the growth rate cycle over the period from 2001 to 2017, one can conclude that the Kitchen cycle (approximately 3-5 years) is observable in many European economies (see, for example, Belgium, Bulgaria, Czech Republic, Hungary, Malta, Poland, Romania, Slovakia). Hence, the hypothesis that will be subject to verification in the empirical section is that (in the considered period - the beginning of the XXI century) one common Kitchen inventory cycle dominates in European economies. To verify this hypothesis, we construct a model in the next section, which takes into account the above described features of the analysed data.

3 Model proposition

For the time series Y_t we propose the following model:

$$Y_t = g(\lambda, t) + \mu(t) + \epsilon_t, \tag{8}$$

with a time-varying mean function $g(\lambda, t) + \mu(t)$, where $g(\lambda, t)$ is of the form (3), $\mu(t)$ is a polynomial of order f

$$\mu(t) = p_0 + p_1 t + p_2 t^2 + \ldots + p_f t^f$$

and ϵ_t is a white noise. The function $g(\lambda, t)$ is used to model cyclical fluctuations (with period $2\pi/\lambda$) with time-varying amplitude, while $\mu(t)$ is responsible for long-term fluctuations (longer then 3-5 years). Note that no known stochastic cycle component is included in the model. All cyclical fluctuations during the considered period are assumed to be modelled by the time-varying function $g(\lambda, t)$.
Let us clearly note that in the above model we face a problem with the interpretation of parameters related to the length of cyclical fluctuations and time-varying amplitude. The problem is that it is difficult to assess the impact of the trend function $\mu(t)$ on cyclical fluctuations. Polynomials of the second, third or higher degree can have a significant impact on the estimation of not only long-term fluctuations but also cyclical fluctuations that we assign to the $g(\lambda, t)$ function. On the other hand, polynomials of low degree f may be supported by data, although it seems that such polynomials are not sufficient to describe the dynamics of the trend, understood (commonly) as all fluctuations longer than 10-12 years. Therefore, the obtained empirical results should be interpreted with caution.
The above model can be generalized in a natural way to multi-frequency case:

$$Y_t = \sum_{j=1}^{m} g_j(\lambda_j, t) + \mu(t) + \epsilon_t, \tag{9}$$

where $g_j(\lambda_j, t) = a_j(t)\sin(\lambda_j t) + b_j(t)\cos(\lambda_j t)$, for $j = 1, 2, \ldots, m$ and the functions $a_j(t)$ and $b_j(t)$ meet Assumptions 1 and 2.

3.1 The case of linear splines

For a given time interval $[1, n]$ we assume that $a(t)$ and $b(t)$ are linear splines (in (3)) with $r + 1$ knots $\{(t_i, a_i) \in \mathbb{Z} \times \mathbb{R}, i = 0, 1, \ldots, r\}$ for $a(t)$ and $\{(t_i, b_i) \in \mathbb{Z} \times \mathbb{R}, i = 0, 1, 2, \ldots, r\}$ for $b(t)$. We assume $1 = t_0 < t_1 < t_2 < \ldots < t_r = n$. Hence,

$$a(t) = \sum_{i=1}^{r} \mathrm{I}_{\{t_{i-1} \le t < t_i\}} \left[a_{i-1} \frac{(t_i - t)}{t_i - t_{i-1}} + a_i \frac{(t - t_{i-1})}{t_i - t_{i-1}} \right], \quad t \in [t_0, t_r), \quad a(t_r) = a_r,$$

$$b(t) = \sum_{i=1}^{r} \mathrm{I}_{\{t_{i-1} \le t < t_i\}} \left[b_{i-1} \frac{(t_i - t)}{t_i - t_{i-1}} + b_i \frac{(t - t_{i-1})}{t_i - t_{i-1}} \right], \qquad t \in [t_0, t_r), \quad b(t_r) = b_r.$$

Let us consider a linear function $s(t) : \mathbb{Z} \to \mathbb{R}$ passing through the points $(x, z_s) \in \mathbb{Z} \times \mathbb{R}$ and $(y, w_s) \in \mathbb{Z} \times \mathbb{R}$. In such case we have

$$s(t)\sin(\lambda t) = w_s \frac{(x-t)\sin(\lambda t)}{x-y} + z_s \frac{(t-y)\sin(\lambda t)}{x-y}.$$

In the same way we can decompose $c(t)\cos(\lambda t)$, where $c(t) : \mathbb{Z} \to \mathbb{R}$ is a linear function, passing through the points $(x, z_c) \in \mathbb{Z} \times \mathbb{R}$ and $(y, w_c) \in \mathbb{Z} \times \mathbb{R}$. Hence, (after elementary algebra), there exists functions $\alpha_i(\lambda, t)$ and $\gamma_i(\lambda, t)$, for $i = 0, 1, 2, \ldots, r$ such that

$$a(t)\sin(\lambda t) + b(t)\cos(\lambda t) = \sum_{i=0}^{r} a_i \alpha_i(\lambda, t) + \sum_{i=0}^{r} b_i \gamma_i(\lambda, t), \tag{10}$$

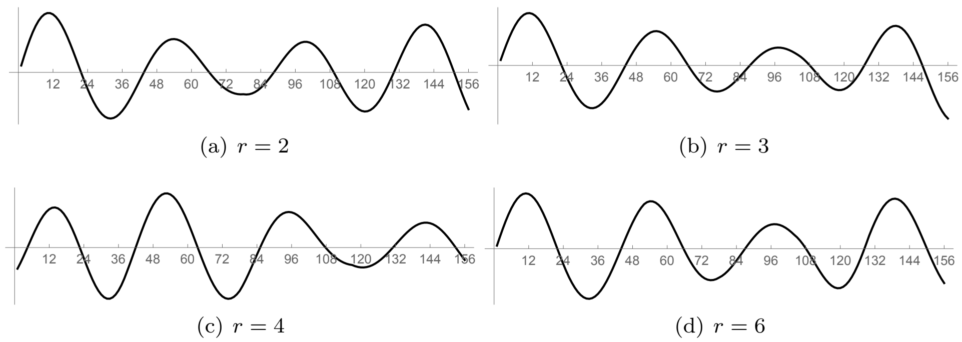
for $t \in \{1, 2, \ldots, n\}$.
Below we show an example where the function $a(t)\sin(\lambda t) + b(t)\cos(\lambda t)$ will be illustrated.

Example 1. *We consider $n = 157$, one frequency $\lambda = 0.15$, $r \in \{2, 3, 4, 6\}$ and equally spaced knots, i.e., $1 = t_0 < t_1 < t_2 < \ldots < t_r = n$, $t_i = \lfloor i(n-1)/r + 1 \rfloor$, $i = 1, 2, \ldots, r-1$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x \in \mathbb{R}$. For fixed r we draw each $a_0, a_1, a_2, \ldots, a_r$ from uniform distribution on the interval $(2, 15)$ and $b_0, b_1, b_2, \ldots, b_r$ from uniform distribution on the interval $(-5, 0)$ (see Table 1). The main finding from the presented example (see Figure 1) is that the cycle based on (3) with time-varying amplitude and with one frequency is much more flexible than the deterministic cycle with one frequency and a constant amplitude. Hence, the proposed deterministic cycle model with a time-varying amplitude may be more useful from a practical point of view in statistical inference concerning cyclical fluctuations.*

Table 1: Parameters used in example

r	$\{a_0, a_1, a_2, \ldots, a_r\}$	$\{b_0, b_1, b_2, \ldots, b_r\}$
$r = 2$	$\{8.9, 2.2, 5.8\}$	$\{-0.4, -2., -4.5\}$
$r = 3$	$\{14.7, 8.3, 3.2, 14.\}$	$\{-0.8, -3.9, -2.4, -0.7\}$
$r = 4$	$\{4.7, 10., 8.9, 2.6, 2.9\}$	$\{-4.5, 0., -1.4, -2.2, -4.1\}$
$r = 6$	$\{10.2, 8.8, 8., 4.7, 3.4, 8.5, 6.8\}$	$\{-1., -1.4, -3.4, -2.3, -2.1, -3.6, -4.2\}$

Figure 1: Paths for $g(\lambda, t)$ for different r and $\{a_0, a_1, a_2, \ldots, a_r\}$, $\{b_0, b_1, b_2, \ldots, b_r\}$ form Table 1



(a) $r = 2$ (b) $r = 3$

(c) $r = 4$ (d) $r = 6$

3.2 The case of Bézier curves

We assume that $a(t)$ and $b(t)$ are first coordinates of Bézier curves with $r+1$ control points $\{P_i = (t_i, a_i), i = 0, 1, \ldots, r\}$ for $a(t)$ and $\{Q_i = (t_i, b_i), i = 0, 1, 2, \ldots, r\}$ for $b(t)$. We fix $t_0 = 1$ and $t_r = n$. For equally spaced knots $1 = t_0 < t_1 < t_2 < \ldots <$ $< t_r = n$, $t_i = 1 + i(n-1)/r$ we have

$$\begin{aligned} a(t) &= \sum_{i=0}^{r} \binom{r}{i} \left(1 - \frac{t-1}{n-1}\right)^{r-i} \left(\frac{t-1}{n-1}\right)^{i} a_i = \\ &= \left(1 - \frac{t-1}{n-1}\right)^{r} a_0 + \binom{r}{1} \left(1 - \frac{t-1}{n-1}\right)^{r-1} \left(\frac{t-1}{n-1}\right) a_1 + \\ &+ \cdots + \binom{r}{r-1} \left(1 - \frac{t-1}{n-1}\right) t^{r-1} a_{r-1} + \left(\frac{t-1}{n-1}\right)^{r} a_r. \end{aligned} \tag{11}$$

where $t \in [1, n]$. This is analogical for $b(t)$. Note that $a(t)$ reduces to a constant if $r = 0$ and to a linear function if $r = 1$. The representation

$$a(t)\sin(\lambda t) + b(t)\cos(\lambda t) = \sum_{i=0}^{r} a_i \zeta_i(\lambda, t) + \sum_{i=0}^{r} b_i \eta_i(\lambda, t) \tag{12}$$

is straightforward for $t \in [1, n]$, where the functions $\zeta_i(\lambda, t)$ and $\eta_i(\lambda, t)$ can be evaluated using (11).

4 Bayesian inference

In this section, we shed light on the problem of the posterior distribution of parameters from model (9). First, we assume that the functions $a_j(t)$ and $b_j(t)$ in (9) are spline functions with $r_j + 1$ knots such that Assumptions 1 and 2 hold for $j = 1, 2, \dots, m$. Let us assume that the equality $r_j = 0$ for some $j = 1, 2, \dots, m$ means that the functions $a_j(t)$ and $b_j(t)$ are constant. We use the following notation for knots: $\{(t_{i,j}, a_{i,j}), i = 0, 1, \dots, r_j\}$ for $a_j(t)$ and $\{(t_{i,j}, b_{i,j}), i = 0, 1, 2, \dots, r_j\}$ for $b_j(t)$, where j corresponds to frequency λ_j and $j = 1, 2, \dots, m$. Let us denote the following: $\mathbf{a}_j = [a_{0,j} \ \ a_{1,j} \ \ \dots \ \ a_{r_j,j}]$, $\mathbf{b}_j = [b_{0,j} \ \ b_{1,j} \ \ \dots \ \ b_{r_j,j}]$, for $j = 1, 2, \dots, m$ and $\Lambda = [\lambda_1 \ \ \lambda_2 \ \ \dots \ \ \lambda_m]$. The vector of polynomial coefficients for trend function $\mu(t)$ we denote by $\mathbf{p} = [p_0 \ \ p_1 \ \ \dots \ \ p_f]$. We formulate the following additional assumptions.

Assumption 3. *Assume that ϵ_t is Gaussian white noise.*

Assumption 4. *Assume that for any $j = 1, 2, \dots, m$ the functions $a_j(t)$ and $b_j(t)$ are linear splines with equally spaced knots $\{(t_{i,j}, a_{i,j}), i = 0, 1, \dots, r_j\}$ and $\{(t_{i,j}, b_{i,j}), i = 0, 1, \dots, r_j\}$ on the interval $[1, n]$ (if $r_j \geq 2$), i.e., $1 = t_0 < t_1 < < t_2 < \dots < t_{r_j} = n$, $t_{i,j} = \lfloor i(n-1)/r_j + 1 \rfloor$, for $i = 1, 2, \dots, r_j - 1$ (see Section 3.1) or for any $j = 1, 2, \dots, m$ the functions $a_j(t)$ and $b_j(t)$ are first coordinates of Bézier curves (see formula (11) in Section 3.2) with equally spaced knots.*

The above assumptions are quite strong and require weakening. On the other hand, under the above assumptions, we obtain below the closed form of the marginal posterior distribution of the frequency vector $\Lambda = [\lambda_1 \ \ \lambda_2 \ \ \dots \ \ \lambda_m]$. Let us assume that we have a sample path $\mathbf{y} = [y_1 \ y_2 \ \dots \ y_n]'$ from the considered time series. Note that by (10) and (12) model (9) can be equivalently written as partially linear regression model (see Osiewalski (1988))

$$\mathbf{y} = \mathbf{X}(\mathbf{\Lambda})\beta + \epsilon, \tag{13}$$

where the $n \times k$ matrix $\mathbf{X}(\mathbf{\Lambda})$ depends on $\Lambda = [\lambda_1 \ \ \lambda_2 \ \ \dots \ \ \lambda_m]$ and first coordinates of knots and $k = f + 1 + 2\sum_{j=1}^{m}(r_j + 1)$. The vector

$$\beta = [\mathbf{p} \ \mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_m \ \mathbf{b}_1 \ \mathbf{b}_2 \ \dots \ \mathbf{b}_m]'$$

is $k \times 1$ vector of parameters and $\epsilon = [\epsilon_1 \ \epsilon_2 \ \dots \ \epsilon_n]'$, where $\epsilon_t \sim N(0, \tau^{-1})$, for $t = 1, 2, \dots, n$. Note that Osiewalski (1988) considered a more general functional form for $g(\lambda, t)$ then in model (9). It should be emphasized that, in Lenart and Mazur (2016), a similar problem was considered, where the marginal posterior distribution for frequency was obtained under constant amplitude, without a polynomial trend and with the additional autoregressive part. We use similar steps to those presented in Osiewalski (1988) and then repeated in Lenart and Mazur (2016).
Under the notation $\theta = [\beta' \ \tau \ \Lambda]$, the likelihood function has the form

$$p(\mathbf{y}|\theta) = (2\pi)^{\frac{-n}{2}} \tau^{\frac{n}{2}} \exp\left[\frac{-\tau}{2}(\mathbf{y} - \mathbf{X}(\mathbf{\Lambda})\beta)'(\mathbf{y} - \mathbf{X}(\mathbf{\Lambda})\beta)\right].$$

To obtain the marginal posterior distribution of Λ, we use the analogical prior structure as in Lenart and Mazur (2016), i.e., we assume the following structure of prior distribution (see also the related distribution in Osiewalski (1988))

$$p(\theta) = p(\beta,\tau)p(\Lambda) = p(\beta|\tau)p(\tau)p(\Lambda),$$

with

$$p(\beta|\tau) = (2\pi)^{-k/2}(\det(\mathbf{B}))^{1/2}\tau^{k/2}\exp\left\{-\frac{\tau}{2}\beta'\mathbf{B}\beta\right\},$$

$$p(\tau) = f_G(\tau|\frac{n_0}{2},\frac{s_0}{2}) = \frac{(s_0/2)^{n_0/2}}{\Gamma(n_0/2)}\tau^{\frac{n_0}{2}-1}\exp(-\frac{s_0\tau}{2}),$$

$$p(\Lambda) = \prod_{j=1}^{m}\frac{\mathbf{1}\{\lambda_j \in (\lambda_{j,L},\lambda_{j,U})\}}{\lambda_{j,U}-\lambda_{j,L}},$$

where $\mathbf{B}$, n_0, s_0, $(\lambda_{j,L},\lambda_{j,U}) \subset (0,\pi)$, $j = 1,2,\ldots,m$ are prior hyperparameters. That is, given Λ, we assume the natural conjugate, normal-gamma prior for (β,τ). For Λ, we assume uniform prior distribution on $\mathbf{S}_{\mathbf{\Lambda}} = (\lambda_{1,L},\lambda_{1,U}) \times (\lambda_{2,L},\lambda_{2,U})\times$ $\times\ldots\times(\lambda_{m,L},\lambda_{m,U})$. Under this assumption we get

$$\begin{aligned}
p(\theta|\mathbf{y}) = \frac{p(\mathbf{y}|\theta)p(\theta)}{p(\mathbf{y})} &= \\
&= \frac{1}{p(\mathbf{y})}(\det(\mathbf{B}))^{1/2}\frac{(s_0/2)^{n_0/2}}{\Gamma(n_0/2)}(2\pi)^{\frac{-(n+k)}{2}}\tau^{\frac{n+k+n_0}{2}-1}\prod_{j=1}^{m}\frac{\mathbf{1}\{\lambda_j \in (\lambda_{j,L},\lambda_{j,U})\}}{\lambda_{j,U}-\lambda_{j,L}} \\
&\cdot\exp\left\{-\frac{\tau}{2}\left[(\beta-\hat\beta)'\mathbf{X}'\mathbf{X}(\beta-\hat\beta)+\beta'\mathbf{B}\beta+(\mathbf{y}-\mathbf{X}(\mathbf{\Lambda})\hat\beta)'(\mathbf{y}-\mathbf{X}(\mathbf{\Lambda})\hat\beta)+s_0\right]\right\} = \\
&= \frac{1}{p(\mathbf{y})}(\det(\mathbf{B}))^{1/2}\frac{(s_0/2)^{n_0/2}}{\Gamma(n_0/2)}(2\pi)^{\frac{-(n+k)}{2}}\tau^{\frac{n+k+n_0}{2}-1}\prod_{j=1}^{m}\frac{\mathbf{1}\{\lambda_j \in (\lambda_{j,L},\lambda_{j,U})\}}{\lambda_{j,U}-\lambda_{j,L}} \\
&\cdot\exp\left\{-\frac{\tau}{2}\left[(\beta-\mathbf{d})'\mathbf{D}(\beta-\mathbf{d})\right]\right\}\exp\left\{-\frac{\tau}{2}\left[-\mathbf{d}'\mathbf{D}\mathbf{d}+\mathbf{y}'\mathbf{y}+s_0\right]\right\},
\end{aligned}$$

where $\hat\beta = (\mathbf{X}(\mathbf{\Lambda})'\mathbf{X}(\mathbf{\Lambda}))^{-1}\mathbf{X}(\mathbf{\Lambda})'\mathbf{y}$, $\mathbf{D} = \mathbf{X}(\mathbf{\Lambda})'\mathbf{X}(\mathbf{\Lambda})+\mathbf{B}$ and $\mathbf{d} = \mathbf{D}^{-1}\mathbf{X}(\mathbf{\Lambda})'\mathbf{y}$. Note that the conditional posterior of (β,τ) given $\mathbf{\Lambda}$ is obviously the normal-gamma distribution. Integrating over β and τ we get the marginal posterior distribution for $\mathbf{\Lambda}$

$$\begin{aligned}
p(\mathbf{\Lambda}|\mathbf{y}) &= \int_0^\infty\int_{\mathbb{R}^k} p(\theta|\mathbf{y})\mathrm{d}\beta\mathrm{d}\tau = \\
&= \frac{1}{p(\mathbf{y})}(\det(\mathbf{B}))^{1/2}\frac{(s_0/2)^{n_0/2}}{\Gamma(n_0/2)}(2\pi)^{\frac{-(n+k)}{2}}\prod_{j=1}^{m}\frac{\mathbf{1}\{\lambda_j \in (\lambda_{j,L},\lambda_{j,U})\}}{\lambda_{j,U}-\lambda_{j,L}}
\end{aligned}$$

$$\cdot \int_0^\infty \tau^{\frac{n+k+n_0}{2}-1} \exp\left\{-\frac{\tau}{2}\left[-\mathbf{d}'\mathbf{D}\mathbf{d}+\mathbf{y}'\mathbf{y}+s_0\right]\right\} \int_{\mathbb{R}^k} \exp\left\{-\frac{\tau}{2}\left[(\beta-\mathbf{d})'\mathbf{D}(\beta-\mathbf{d})\right]\right\} \mathrm{d}\beta \mathrm{d}\tau =$$

$$= \frac{1}{p(\mathbf{y})}(\det(\mathbf{B}))^{1/2} \frac{(s_0/2)^{n_0/2}}{\Gamma(n_0/2)} (2\pi)^{\frac{-(n+k)}{2}} \prod_{j=1}^m \frac{\mathbf{1}\{\lambda_j \in (\lambda_{j,L}, \lambda_{j,U})\}}{\lambda_{j,U}-\lambda_{j,L}}$$

$$\cdot \int_0^\infty \tau^{\frac{n+k+n_0}{2}-1} \exp\left\{-\frac{\tau}{2}\left[-\mathbf{d}'\mathbf{D}\mathbf{d}+\mathbf{y}'\mathbf{y}+s_0\right]\right\} (2\pi)^{\frac{k}{2}} \tau^{-\frac{k}{2}} \det(\mathbf{D})^{-\frac{1}{2}} \mathrm{d}\tau =$$

$$= \frac{1}{p(\mathbf{y})}(\det(\mathbf{B}))^{1/2} \frac{(s_0/2)^{n_0/2}}{\Gamma(n_0/2)} (2\pi)^{-\frac{n}{2}} \prod_{j=1}^m \frac{\mathbf{1}\{\lambda_j \in (\lambda_{j,L}, \lambda_{j,U})\}}{\lambda_{j,U}-\lambda_{j,L}}$$

$$\cdot \left[\frac{-\mathbf{d}'\mathbf{D}\mathbf{d}+\mathbf{y}'\mathbf{y}+s_0}{2}\right]^{-\frac{n+n_0}{2}} \Gamma\left(\frac{n+n_0}{2}\right) \det(\mathbf{D})^{-\frac{1}{2}} =$$

$$= \frac{1}{p(\mathbf{y})}(\det(\mathbf{B}))^{1/2} \frac{s_0^{n_0/2}\pi^{-\frac{n}{2}}}{\Gamma(n_0/2)} \Gamma\left(\frac{n+n_0}{2}\right) \prod_{j=1}^m \frac{\mathbf{1}\{\lambda_j \in (\lambda_{j,L}, \lambda_{j,U})\}}{\lambda_{j,U}-\lambda_{j,L}}$$

$$\cdot (\det(\mathbf{X}(\mathbf{\Lambda})'\mathbf{X}(\mathbf{\Lambda})+\mathbf{B}))^{-\frac{1}{2}} \cdot \left(\mathbf{y}'\left[\mathbf{I}-\mathbf{X}(\mathbf{\Lambda})(\mathbf{X}(\mathbf{\Lambda})'\mathbf{X}(\mathbf{\Lambda})+\mathbf{B})^{-1}\mathbf{X}(\mathbf{\Lambda})'\right]\mathbf{y}+s_0\right)^{-\frac{n+n_0}{2}}. \tag{14}$$

Hence,

$$p(\mathbf{y}) = \frac{s_0^{n_0/2}\pi^{-\frac{n}{2}}}{\Gamma(n_0/2)} \Gamma\left(\frac{n+n_0}{2}\right) (\det(\mathbf{B}))^{1/2} \int_{\mathbf{S_\Lambda}} \prod_{j=1}^m \frac{\mathbf{1}\{\lambda_j \in (\lambda_{j,L}, \lambda_{j,U})\}}{\lambda_{j,U}-\lambda_{j,L}}$$

$$\cdot (\det(\mathbf{X}(\mathbf{\Lambda})'\mathbf{X}(\mathbf{\Lambda})+\mathbf{B}))^{-\frac{1}{2}} \left(\mathbf{y}'\left[\mathbf{I}-\mathbf{X}(\mathbf{\Lambda})(\mathbf{X}(\mathbf{\Lambda})'\mathbf{X}(\mathbf{\Lambda})+\mathbf{B})^{-1}\mathbf{X}(\mathbf{\Lambda})'\right]\mathbf{y}+s_0\right)^{-\frac{n+n_0}{2}} \mathrm{d}\mathbf{\Lambda}. \tag{15}$$

In this paper, we are not interested in the construction of the MCMC sampler for posterior inference. This is because the assumptions are too strong and should first be weakened. The most restrictive assumption which can have a significant impact on the results obtained is Assumption 4. It is obvious that this assumption should be weakened. It is widely known that fitting splines to data can be improved significantly if the knots can be adjusted. The Bayesian estimation of free-knot splines was considered many times in the literature (see Dimatteo, Genovese, and Kass (2001), Lindstrom (2002), Wang (2008)). However, in our approach the problem is more difficult since $g(\lambda, t)$ is a mix of two spline functions and trigonometric functions. Hence, the fully Bayesian approach to estimating the function $g(\lambda, t)$ is not trivial, especially when we use few components $g(\lambda, t)$ in one model. Therefore, the construction of the MCMC sampler should be considered in the future as a separate problem.

From the practical point of view, the posterior distribution for $\mathbf{X}(\mathbf{\Lambda})\beta|\mathbf{y}$ can be interpreted as cyclical fluctuations in probabilistic terms. Since we are not interested in the construction of the MCMC sampler, we show only the possibility to numerically evaluate the first and second moment of the posterior distribution of $\mathbf{X}(\mathbf{\Lambda})\beta|\mathbf{y}$. Note that the conditional posterior pdf for β given $\mathbf{\Lambda}, \mathbf{y}$ is a multivariate Student t distribution with location vector $\overline{\mu} = \mathbf{d}$, shape matrix $\overline{\mathbf{\Sigma}} = (-\mathbf{d}'\mathbf{D}\mathbf{d} + \mathbf{y}'\mathbf{y}+ +s_0)\mathbf{D}^{-1}/(n_0+n)$ and $\overline{\nu} = n_0+n$ degrees of freedom (see for example Zellner (1971), page 75-76). Hence,

$$\begin{aligned} E(\mathbf{X}(\mathbf{\Lambda})\beta|\mathbf{y}) &= E[E(\mathbf{X}(\mathbf{\Lambda})\beta|\mathbf{\Lambda},\mathbf{y})|\mathbf{y}] = E[\mathbf{X}(\mathbf{\Lambda})E(\beta|\mathbf{\Lambda},\mathbf{y})|\mathbf{y}] = \\ &= E[\mathbf{X}(\mathbf{\Lambda})(\mathbf{X}(\mathbf{\Lambda})'\mathbf{X}(\mathbf{\Lambda})+\mathbf{B})^{-1}\mathbf{X}(\mathbf{\Lambda})'\mathbf{y}|\mathbf{y}] \end{aligned} \tag{16}$$

and the above expectation can be calculated using numerical integration based on posterior distribution (14). Similarly, the second order moments of $\mathbf{X}(\mathbf{\Lambda})\beta|\mathbf{y}$ can be calculated using numerical integration based on relation

$$\begin{aligned} E(\mathbf{X}(\mathbf{\Lambda})\beta\beta'\mathbf{X}(\mathbf{\Lambda})'|\mathbf{y}) &= E[\mathbf{X}(\mathbf{\Lambda})E(\beta\beta'|\mathbf{\Lambda},\mathbf{y})\mathbf{X}(\mathbf{\Lambda})'|\mathbf{y}] = \\ &= E\left[\mathbf{X}(\mathbf{\Lambda})\left(\mathbf{d}\mathbf{d}' + \frac{\overline{\nu}}{\overline{\nu}-2}\overline{\mathbf{\Sigma}}\right)\mathbf{X}(\mathbf{\Lambda})'|\mathbf{y}\right] \end{aligned} \tag{17}$$

and posterior distribution (14).

5 Real data example

In this section, we show some results based on the marginal posterior distribution (14) and the expectation (16) for $k = 1$ (i.e., we consider only one frequency). We consider the following hyperparameters $\mathbf{B} = \mathbf{I}$, $s_0 = 1.05$, $n_0 = 2.1$, $\lambda_{1,L} = \frac{2\pi}{1.5\times 12}$ and $\lambda_{1,U} = \frac{2\pi}{10\times 12}$. We choose such $\lambda_{1,L}$ and $\lambda_{1,U}$ because, for the analysis of Kitchin cycles (i.e., from 3 to 5 years), the range of fluctuations from 1.5 to 10 years seems to be sufficient, assuming $k = 1$. Note that a wider window for fluctuations (for example, up to 11 or 12 years or longer) would probably require an additional analysis taking into account more frequencies (i.e., $k > 1$). It is possible, but such analysis devoid of the weakening of Assumption 4 is not the primary aim of this article. We consider constant amplitude together with $r \in \{1, 2, \ldots, 9\}$. We use both linear splines and Bézier curve. We consider the order of polynomial: $f = 0, 1, 2$.

5.1 Bayesian model comparision

We calculate the marginal data density $p(\mathbf{y})$ using (15) and numerical integration. In Table 2 we present the marginal data density comparison for different values of polynomial order, $f = 0, 1, 2$ in the case of linear splines. In Table 3 we present the same marginal data density comparison in the case of Bézier curves. Note that the case of Bézier curves reduce to constant amplitude for $r = 0$, and to a linear spline

Table 2: Results for linear splines and $f = 0, 1, 2$. The difference between maximum value in the row and $L = \log_{10} p(y)$. The underlined value (zero) means that the model has the maximum $\log_{10} p(y)$ (last column)

	$f=0$										$f=1$										$f=2$										max
	r=0	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=0	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=0	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	L
EU	4.4	5.3	1.	3.9	0.2	2.9	$\underline{0.}$	2.	0.6	1.4	5.	5.9	1.6	4.5	0.8	3.3	0.5	2.3	1.	1.7	5.	5.9	1.8	4.6	0.9	3.3	0.7	2.4	1.1	1.8	95.
EA	5.4	6.3	1.4	4.6	0.3	3.3	$\underline{0.}$	2.3	0.7	1.5	6.	6.9	2.	5.2	0.9	3.8	0.5	2.6	1.1	1.8	6.1	7.	2.2	5.3	1.	3.8	0.7	2.7	1.3	1.9	94.45
BE	2.3	3.1	0.7	2.7	0.5	1.9	0.2	1.6	0.8	0.4	2.4	3.1	0.7	2.8	0.4	1.7	$\underline{0.}$	1.6	0.9	0.7	2.6	3.3	0.9	3.	0.6	1.9	0.2	1.8	1.	0.8	90.27
BG	12.5	10.6	8.5	9.3	3.8	3.4	0.3	2.1	$\underline{0.}$	1.4	11.1	10.2	6.5	9.3	1.4	3.3	0.	2.1	0.1	1.5	11.1	10.3	6.5	9.5	1.4	3.5	0.1	2.2	0.2	1.6	87.33
CZ	6.6	7.3	0.9	4.4	0.8	1.4	0.2	1.3	$\underline{0.}$	0.7	6.7	7.4	0.8	4.4	0.2	1.7	0.2	1.5	0.1	0.9	6.6	7.4	0.9	4.5	0.4	1.8	0.3	1.6	0.3	1.	89.36
DK	2.4	3.2	0.9	2.6	0.8	2.3	$\underline{0.}$	1.5	0.1	1.1	3.	3.8	1.4	2.8	1.3	2.5	0.4	1.8	0.4	1.4	2.9	3.7	1.5	2.9	1.3	2.6	0.6	1.9	0.5	1.4	89.15
DE	8.6	9.4	2.7	6.7	0.4	3.9	$\underline{0.}$	2.7	1.2	1.8	9.2	10.1	3.2	7.2	1.	4.4	0.5	3.1	1.7	2.1	9.4	10.3	3.4	7.4	1.2	4.5	0.7	3.2	1.8	2.2	92.79
EE	27.6	28.1	15.3	21.2	6.9	10.2	2.5	3.4	$\underline{0.}$	1.8	27.	27.5	13.5	20.1	5.9	10.1	2.2	2.9	0.2	2.1	27.2	27.7	13.7	20.3	6.1	10.1	2.2	3.1	0.4	2.3	82.55
IE	11.8	7.4	6.7	7.7	2.4	1.6	$\underline{0.}$	0.4	0.4	1.2	11.6	6.6	6.6	7.2	2.3	1.8	0.3	0.6	0.7	1.4	10.9	4.4	4.2	5.	2.2	1.9	0.4	0.7	0.8	1.5	54.59
EL	1.9	1.	$\underline{0.}$	0.9	0.5	1.	1.	1.6	1.4	2.	2.5	1.6	0.5	1.2	0.7	1.2	1.2	1.8	1.6	2.2	2.1	1.6	0.6	1.3	0.9	1.3	1.3	1.9	1.7	2.3	91.85
ES	7.	6.9	4.6	6.1	1.5	2.4	$\underline{0.}$	1.4	0.1	1.	7.6	7.4	5.	6.1	1.4	2.6	0.3	1.7	0.3	1.3	6.9	7.4	4.7	5.9	1.5	2.7	0.4	1.8	0.5	1.3	92.95
FR	2.4	3.3	0.3	2.9	0.1	2.1	$\underline{0.}$	1.4	0.5	1.3	2.9	3.8	0.8	3.4	0.5	2.4	0.4	1.7	0.8	1.5	3.	3.8	0.9	3.5	0.6	2.3	0.6	1.8	0.9	1.6	94.73
HR	3.6	3.	1.	0.7	$\underline{0.}$	0.7	0.	0.8	0.4	0.9	2.5	1.4	0.1	0.8	0.1	0.8	0.2	1.	0.6	1.1	2.2	1.6	0.1	0.9	0.1	0.9	0.3	1.1	0.7	1.2	91.5
IT	7.5	8.4	2.	5.	0.9	4.1	$\underline{0.}$	2.6	0.3	1.5	8.1	9.	2.6	5.6	1.5	4.6	0.5	2.9	0.8	1.8	7.9	8.8	2.6	5.6	1.4	4.5	0.6	3.	0.9	1.8	91.2
CY	8.5	2.3	1.6	1.9	0.	1.1	0.3	0.7	$\underline{0.}$	1.3	7.6	1.4	1.3	2.1	0.3	1.3	0.5	0.9	0.2	1.4	6.4	1.5	1.4	2.2	0.4	1.4	0.6	1.	0.3	1.5	88.6
LV	12.2	12.6	6.	8.7	3.4	5.2	0.3	2.9	$\underline{0.}$	1.4	11.8	11.8	6.	9.	3.2	5.4	0.2	3.1	0.1	1.6	11.6	11.9	6.	9.	3.3	5.5	0.4	3.1	0.3	1.7	85.53
LT	5.	5.6	4.	5.7	3.6	4.4	0.5	3.1	0.3	0.9	3.7	4.4	2.8	4.8	2.5	4.2	$\underline{0.}$	2.9	0.2	1.1	3.5	4.3	2.7	4.7	2.4	4.1	0.2	3.	0.4	1.2	69.25
LU	6.6	7.3	4.	6.3	2.6	4.1	1.1	3.2	0.3	1.3	6.4	7.2	3.7	6.	2.	4.3	0.9	3.4	$\underline{0.}$	1.4	6.2	6.9	3.6	5.9	2.	4.4	1.	3.5	0.1	1.6	80.58
HU	9.3	10.1	4.4	8.3	2.5	5.	0.6	3.	0.2	1.5	9.5	10.4	4.5	8.5	2.3	5.2	0.6	3.2	$\underline{0.}$	1.6	9.5	10.3	4.6	8.5	2.3	5.3	0.7	3.3	0.1	1.8	85.39
MT	8.1	7.4	4.9	5.4	0.5	3.3	$\underline{0.}$	1.7	0.5	1.2	8.6	8.	5.4	6.	1.1	3.8	0.4	2.	1.	1.6	8.8	8.1	5.6	6.2	1.3	4.	0.6	2.2	1.1	1.7	80.17
NL	0.8	1.5	$\underline{0.}$	2.	0.1	1.3	0.5	1.3	1.1	0.7	1.3	2.	0.4	2.4	0.4	1.6	0.8	1.6	1.4	1.2	1.5	2.2	0.6	2.6	0.6	1.9	1.	1.8	1.6	1.3	89.58
AT	4.3	5.1	0.7	3.6	0.2	2.	$\underline{0.}$	1.4	0.4	0.7	4.7	5.3	0.9	3.7	0.4	2.3	0.1	1.6	0.6	1.	4.9	5.5	1.1	3.9	0.6	2.4	0.3	1.7	0.7	1.1	93.18
PL	2.8	2.8	1.1	2.4	2.	0.9	0.3	1.1	1.3	0.3	3.1	2.9	0.5	1.6	1.3	1.2	$\underline{0.}$	1.2	1.3	0.5	3.2	3.	0.6	1.8	1.3	1.4	0.1	1.4	1.3	0.7	90.41
PT	2.1	2.7	1.4	2.3	0.5	1.5	$\underline{0.}$	0.9	0.3	0.7	2.5	3.	1.7	2.2	0.9	1.7	0.2	1.2	0.5	1.	1.8	2.5	1.4	2.1	0.9	1.8	0.3	1.3	0.6	1.1	88.67
RO	2.7	3.3	0.9	0.6	$\underline{0.}$	0.7	0.5	0.9	0.8	1.4	2.7	3.4	0.7	0.5	0.2	0.7	0.6	1.1	1.	1.3	2.9	3.6	0.9	0.7	0.4	0.9	0.7	1.3	1.1	1.4	88.67
SI	9.4	10.	4.4	7.8	2.8	5.	0.9	3.5	$\underline{0.}$	1.9	9.9	10.6	4.9	8.3	3.2	5.3	1.3	3.8	0.4	2.2	9.7	10.5	4.9	8.2	3.1	5.3	1.4	3.8	0.5	2.2	88.25
SK	9.5	9.1	6.4	6.7	4.6	6.5	2.1	3.9	2.8	$\underline{0.}$	9.6	9.	6.3	6.4	4.3	6.5	2.	4.	2.6	0.2	9.8	9.2	6.4	6.6	4.5	6.6	2.1	4.2	2.4	0.3	76.76
FI	8.3	9.	4.4	6.8	3.8	5.	0.4	4.1	1.2	$\underline{0.}$	8.6	9.2	4.3	6.4	3.5	5.1	$\underline{0.}$	4.2	1.2	0.4	8.8	9.4	4.5	6.6	3.7	5.3	0.2	4.3	1.3	0.5	87.07
SE	9.2	9.8	3.8	7.3	1.6	4.7	0.3	2.7	$\underline{0.}$	1.3	9.7	10.3	4.2	7.6	2.2	5.1	0.6	2.9	0.3	1.5	9.7	10.3	4.2	7.6	2.1	5.	0.7	3.	0.4	1.6	88.76
UK	$\underline{0.}$	0.8	0.3	1.6	0.6	1.6	1.	1.7	1.3	1.9	0.3	1.2	0.7	1.9	1.1	1.9	1.3	1.9	1.6	2.1	0.3	1.2	0.7	1.9	1.1	2.	1.4	2.	1.7	2.2	97.38
NO	$\underline{0.}$	0.5	1.2	1.3	2.	2.1	2.3	2.5	2.5	2.7	0.5	1.	1.7	1.8	2.4	2.5	2.7	2.7	2.9	3.	0.5	1.	1.7	1.9	2.5	2.6	2.7	2.8	3.	3.1	88.56
MK	$\underline{0.}$	0.5	0.2	1.3	0.3	1.6	0.8	1.	0.7	0.7	0.5	1.	0.7	1.8	0.8	1.9	1.1	1.3	0.9	1.	0.8	1.2	0.9	1.9	1.	2.	1.3	1.4	1.1	1.1	62.03
RS	3.4	3.4	3.1	3.9	1.4	2.8	$\underline{0.}$	1.6	0.4	1.	3.8	3.7	3.4	4.3	1.9	3.3	0.5	1.9	0.7	1.3	3.6	3.5	3.3	4.2	1.8	3.2	0.6	2.	0.8	1.4	80.43
TR	8.8	7.5	3.1	5.1	1.	3.6	0.6	1.5	1.3	0.9	9.3	7.8	3.2	5.4	0.7	3.8	$\underline{0.}$	1.7	1.	1.1	9.5	8.	3.4	5.6	0.9	4.	0.2	1.9	1.	1.2	82.55

Table 3: Results for Bézier curves and $f = 0, 1, 2$. The difference between maximum value in the row and $L = \log_{10} p(y)$. The underlined value (zero) means that the model has the maximum $\log_{10} p(y)$ (last column)

	$f = 0$										$f = 1$										$f = 2$										max
	r=0	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=0	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=0	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	L
EU	2.2	3.1	1.2	1.	0.8	0.6	0.4	0.2	0.1	$\underline{0.}$	2.9	3.7	1.7	1.6	1.3	1.1	0.9	0.8	0.7	0.5	2.9	3.8	1.9	1.7	1.5	1.3	1.1	0.9	0.8	0.7	92.89
EA	2.7	3.6	1.3	1.1	0.9	0.7	0.4	0.3	0.1	$\underline{0.}$	3.3	4.2	1.9	1.7	1.5	1.2	1.	0.8	0.7	0.5	3.4	4.3	2.1	1.9	1.6	1.4	1.1	1.	0.8	0.7	91.77
BE	1.3	2.	1.	1.	0.8	0.6	0.5	0.3	0.2	0.2	1.4	2.1	1.1	1.	0.9	0.6	0.4	0.2	0.1	$\underline{0.}$	1.6	2.3	1.3	1.3	1.1	0.8	0.6	0.4	0.3	0.2	89.26
BG	5.7	3.7	3.	2.8	2.6	2.4	2.2	1.9	1.6	1.2	4.2	3.3	2.3	2.1	1.9	1.6	1.2	0.7	0.3	$\underline{0.}$	4.2	3.4	2.4	2.2	2.	1.6	1.2	0.8	0.4	0.1	80.44
CZ	3.5	4.3	1.2	0.8	0.6	0.5	0.4	0.3	0.2	0.1	3.6	4.3	1.3	0.8	0.6	0.4	0.3	0.2	0.1	$\underline{0.}$	3.6	4.3	1.4	1.	0.7	0.5	0.4	0.2	0.2	0.1	86.29
DK	0.9	1.7	0.7	0.5	0.4	0.3	0.2	0.1	0.1	$\underline{0.}$	1.5	2.3	1.2	1.	0.8	0.7	0.6	0.5	0.4	0.4	1.4	2.2	1.2	1.	0.9	0.8	0.7	0.6	0.5	0.5	87.66
DE	4.2	5.1	2.2	1.8	1.4	1.	0.7	0.4	0.2	$\underline{0.}$	4.8	5.7	2.8	2.4	1.9	1.5	1.2	0.9	0.7	0.5	5.	5.9	3.	2.5	2.1	1.7	1.4	1.1	0.9	0.7	88.4
EE	11.9	12.4	6.9	6.2	5.	3.5	2.3	1.4	0.7	0.1	11.3	11.9	6.	5.2	4.	2.8	1.8	1.1	0.5	$\underline{0.}$	11.5	12.1	6.2	5.3	4.1	3.	2.	1.2	0.6	0.2	66.87
IE	9.2	4.8	4.9	4.3	3.6	2.8	2.	1.3	0.9	0.5	9.	4.	4.1	3.5	3.	2.4	1.8	1.2	0.8	0.5	8.3	1.8	1.6	1.3	1.2	1.	0.7	0.4	0.2	$\underline{0.}$	52.03
EL	1.8	0.8	0.4	0.3	0.3	0.2	0.1	0.1	$\underline{0.}$	0.	2.3	1.4	0.8	0.7	0.6	0.5	0.4	0.4	0.3	0.3	1.9	1.5	1.	0.8	0.7	0.6	0.5	0.5	0.4	0.4	91.67
ES	2.8	2.7	1.9	1.7	1.5	1.3	0.9	0.6	0.3	$\underline{0.}$	3.4	3.3	2.3	2.1	2.	1.7	1.3	0.9	0.6	0.3	2.8	3.2	2.	1.9	1.7	1.5	1.1	0.8	0.5	0.3	88.78
FR	1.3	2.2	0.9	0.8	0.7	0.5	0.4	0.2	0.1	$\underline{0.}$	1.8	2.7	1.4	1.4	1.2	1.	0.9	0.7	0.6	0.5	1.9	2.7	1.5	1.4	1.3	1.1	0.9	0.8	0.7	0.6	93.64
HR	3.4	2.9	1.2	0.7	0.5	0.3	0.2	0.1	0.1	0.	2.4	1.3	0.6	0.4	0.3	0.2	0.1	0.1	0.	$\underline{0.}$	2.1	1.4	0.7	0.6	0.4	0.3	0.3	0.2	0.2	0.1	91.39
IT	4.1	5.	1.5	1.1	0.8	0.6	0.5	0.3	0.1	$\underline{0.}$	4.7	5.6	2.	1.6	1.4	1.2	1.	0.8	0.7	0.5	4.5	5.4	2.1	1.7	1.5	1.2	1.	0.8	0.7	0.6	87.79
CY	7.1	1.	0.1	0.1	0.1	0.1	0.2	0.2	0.3	0.3	6.3	$\underline{0.}$	0.	0.1	0.2	0.2	0.3	0.3	0.4	0.4	5.	0.2	0.2	0.2	0.3	0.4	0.4	0.5	0.5	0.5	87.23
LV	5.8	6.3	3.	2.5	2.1	1.5	0.9	0.5	0.2	$\underline{0.}$	5.4	5.5	3.2	2.8	2.3	1.7	1.2	0.8	0.5	0.3	5.2	5.5	3.1	2.7	2.3	1.7	1.2	0.8	0.5	0.3	79.17
LT	1.5	2.1	1.6	1.4	1.3	1.3	1.2	1.1	1.	1.	0.2	0.9	0.5	0.5	0.5	0.4	0.3	0.3	0.2	0.1	$\underline{0.}$	0.8	0.4	0.4	0.4	0.3	0.2	0.2	0.1	0.	65.74
LU	2.2	2.9	1.2	1.1	1.	0.9	0.8	0.6	0.4	0.2	2.	2.8	1.	0.8	0.7	0.6	0.5	0.3	0.2	0.1	1.8	2.5	0.9	0.7	0.6	0.5	0.4	0.2	0.1	$\underline{0.}$	76.18
HU	2.8	3.6	1.2	1.	0.8	0.6	0.5	0.3	0.2	$\underline{0.}$	3.	3.9	1.5	1.2	0.9	0.7	0.6	0.4	0.2	0.	3.	3.8	1.5	1.3	1.	0.8	0.6	0.4	0.2	0.1	78.89
MT	3.9	3.2	1.7	1.2	1.	0.8	0.6	0.4	0.2	$\underline{0.}$	4.5	3.8	2.3	1.7	1.5	1.3	1.1	0.9	0.7	0.5	4.6	3.9	2.5	1.9	1.7	1.5	1.3	1.1	0.9	0.7	75.98
NL	0.4	1.1	0.5	0.5	0.5	0.4	0.3	0.2	0.1	$\underline{0.}$	0.9	1.6	0.8	0.8	0.8	0.7	0.6	0.5	0.4	0.3	1.1	1.8	1.	1.	1.	0.9	0.8	0.7	0.6	0.5	89.17
AT	2.4	3.1	1.3	1.1	0.8	0.6	0.4	0.3	0.1	$\underline{0.}$	2.7	3.4	1.5	1.1	0.9	0.7	0.5	0.4	0.2	0.1	2.9	3.6	1.7	1.3	1.1	0.9	0.7	0.5	0.4	0.3	91.21
PL	1.7	1.7	1.4	1.3	1.1	0.9	0.8	0.7	0.6	0.6	1.9	1.7	1.1	0.7	0.5	0.3	0.2	0.1	$\underline{0.}$	0.	2.1	1.9	1.3	0.8	0.6	0.4	0.3	0.2	0.2	0.1	89.27
PT	1.2	1.8	1.	0.9	0.7	0.5	0.4	0.2	0.1	$\underline{0.}$	1.6	2.2	1.4	1.3	1.1	0.9	0.8	0.6	0.5	0.4	0.9	1.6	1.1	1.	0.9	0.8	0.6	0.5	0.4	0.3	87.79
RO	2.5	3.1	2.	1.1	0.7	0.4	0.3	0.1	0.1	0.	2.5	3.1	1.8	1.	0.5	0.3	0.2	0.1	0.	$\underline{0.}$	2.7	3.4	2.	1.2	0.7	0.5	0.4	0.3	0.2	0.2	88.46
SI	3.7	4.3	1.7	1.3	1.	0.8	0.6	0.4	0.2	$\underline{0.}$	4.2	4.9	2.3	1.9	1.6	1.4	1.1	0.9	0.7	0.5	4.	4.8	2.1	1.8	1.5	1.3	1.	0.8	0.6	0.4	82.55
SK	3.	2.6	1.3	0.7	0.4	0.3	0.2	0.2	0.2	0.1	3.1	2.5	1.2	0.7	0.4	0.2	0.1	0.1	$\underline{0.}$	0.	3.3	2.7	1.3	0.8	0.6	0.4	0.3	0.2	0.2	0.2	70.28
FI	3.1	3.8	1.5	1.3	1.1	0.9	0.8	0.6	0.5	0.4	3.4	4.	1.3	1.	0.8	0.6	0.4	0.2	0.1	$\underline{0.}$	3.6	4.2	1.6	1.2	1.	0.8	0.6	0.4	0.3	0.2	81.85
SE	4.3	4.9	1.9	1.5	1.3	1.	0.8	0.5	0.2	$\underline{0.}$	4.9	5.4	2.3	1.8	1.6	1.3	1.1	0.8	0.5	0.3	4.8	5.5	2.3	1.9	1.6	1.4	1.1	0.9	0.6	0.4	83.91
UK	$\underline{0.}$	0.8	0.6	0.6	0.5	0.5	0.4	0.4	0.3	0.3	0.3	1.2	0.9	1.	1.	0.9	0.9	0.8	0.8	0.7	0.3	1.2	0.9	1.	1.	0.9	0.9	0.8	0.8	0.8	97.38
NO	$\underline{0.}$	0.5	0.7	0.7	0.7	0.8	0.7	0.7	0.7	0.7	0.5	1.	1.2	1.3	1.3	1.3	1.3	1.2	1.2	1.2	0.5	1.	1.3	1.3	1.3	1.3	1.3	1.3	1.2	1.2	88.56
MK	$\underline{0.}$	0.5	0.5	0.5	0.5	0.4	0.4	0.3	0.3	0.2	0.5	1.	1.	0.9	0.9	0.9	0.8	0.7	0.7	0.6	0.8	1.2	1.2	1.1	1.1	1.	1.	0.9	0.9	0.8	62.03
SR	1.4	1.4	1.4	1.	0.7	0.6	0.4	0.2	0.1	$\underline{0.}$	1.8	1.8	1.7	1.4	1.2	1.	0.9	0.8	0.6	0.5	1.7	1.5	1.6	1.3	1.1	1.	0.8	0.7	0.6	0.5	78.48
TR	4.6	3.3	2.1	1.3	0.9	0.7	0.5	0.4	0.3	0.2	5.1	3.6	2.2	1.1	0.7	0.5	0.3	0.2	0.1	$\underline{0.}$	5.3	3.8	2.4	1.3	0.9	0.7	0.5	0.4	0.3	0.2	78.36

with two knots for $r = 1$. Therefore the results for $r = 0$ and $r = 1$ are the same for linear splines and for the Bézier curve.
The zero degree polynomial (constant value) is most often supported by the data for both linear splines and Bézier curves. The first degree polynomial is supported in only a few cases, while the second degree polynomial is not supported for linear splines and only three times in the case of Bézier curves (for Ireland, Lithuania and Luxembourg).
In Table 4 we present selected characteristics of the model with the maximum value of $\log_{10} p(y)$ from the models considered in Tables 2 and 3. Linear spline is supported by all data, i.e., for the European Union, the Euro area and all considered countries. It should be emphasized that in most cases, the data does not support any trend. The data support a linear trend in only seven cases. Moreover, most of the data support models in which the number of knots is odd. An odd number of knots equal to seven or nine is mostly supported by the data (22 times). The odd number of knots is probably related to a strong change in the amplitude of cyclic fluctuations for most data during the crisis in 2009. In the case of an odd number of knots, the middle knot falls in mid-2009, which is supported by the data. Only in the case of the United Kingdom, Norway and Macedonia does the data support a constant amplitude of fluctuations ($r = 0$). In the next two subsections, we will interpret the posterior expectation $E(\mathbf{X}(\mathbf{\Lambda})\beta|\mathbf{y})$ and the posterior distributions for frequencies Λ for models with a maximum $p(y)$ value from Table 4.

5.2 The posterior expectation of cyclical fluctuations

In Figures 2-7 we present the industrial production together with the posterior expectation $E(\mathbf{X}(\mathbf{\Lambda})\beta|\mathbf{y})$. We present the results for the models with the higher marginal data density from Table 4. Additionally, in order to compare obtained results we presents the cycle extracted by the standard bandpass filtration procedure. We use the Christiano-Fitzgerald (CF) filter with a range of 1.5 year to 10 years (see Christiano and Fitzgerald (1999))).
It seems likely that the proposed approach based on a time-varying amplitude is dynamic enough to describe the dynamics of the business cycle. In most cases, the posterior expected value $E(\mathbf{X}(\mathbf{\Lambda})\beta|\mathbf{y})$ and the cycle extracted by the CF filter have very similar dynamics. Moreover, in most cases, the results indicate similar turning points of the cycle and amplitude. Let us emphasize that, in three cases (the United Kingdom, Norway and Macedonia), the model with the maximum $p(y)$ value assumes a constant amplitude of fluctuations (see Table 4). This is clearly visible in the dynamics of the expected value $E(\mathbf{X}(\mathbf{\Lambda})\beta|\mathbf{y})$ for these countries in Figures 6-7.

Table 4: Optimal model with the maximum $\log_{10} p(y)$ (last column), $r+1=1$ means constant amplitude

Country or region	Linear splines	Bézier curves	Optimal polynomial order $0 \leq f \leq 2$	Optimal knots number $r+1$ (where $1 \leq r+1 \leq 10$)	max $\log_{10} p(y)$
European Union	+	-	0	7	97.18
Euro area	+	-	0	7	96.63
Belgium	+	-	1	7	92.46
Bulgaria	+	-	0	9	89.51
Czech Republic	+	-	0	9	91.54
Denmark	+	-	0	7	91.34
Germany	+	-	0	7	94.97
Estonia	+	-	0	9	84.73
Ireland	+	-	0	7	56.78
Greece	+	-	0	3	94.03
Spain	+	-	0	7	95.13
France	+	-	0	7	96.91
Croatia	+	-	0	5	93.68
Italy	+	-	0	7	93.39
Cyprus	+	-	0	9	90.78
Latvia	+	-	0	9	87.72
Lithuania	+	-	1	7	71.43
Luxembourg	+	-	1	9	82.76
Hungary	+	-	1	9	87.57
Malta	+	-	0	7	82.36
Netherlands	+	-	0	3	91.76
Austria	+	-	0	7	95.36
Poland	+	-	1	7	92.6
Portugal	+	-	0	7	90.86
Romania	+	-	0	5	90.86
Slovenia	+	-	0	9	90.43
Slovakia	+	-	0	10	78.94
Finland	+	-	1	7	89.25
Sweden	+	-	0	9	90.95
United Kingdom	+	+	0	1	99.56
Norway	+	+	0	1	90.74
Macedonia	+	+	0	1	64.21
Serbia	+	-	0	7	82.61
Turkey	+	-	1	7	84.74

Figure 2: Growth cycle for industrial production from January 2001 to December 2017 - gray line; posterior expectation of cyclical fluctuations with trend - black line; cycle extracted by CF filter - dashed line

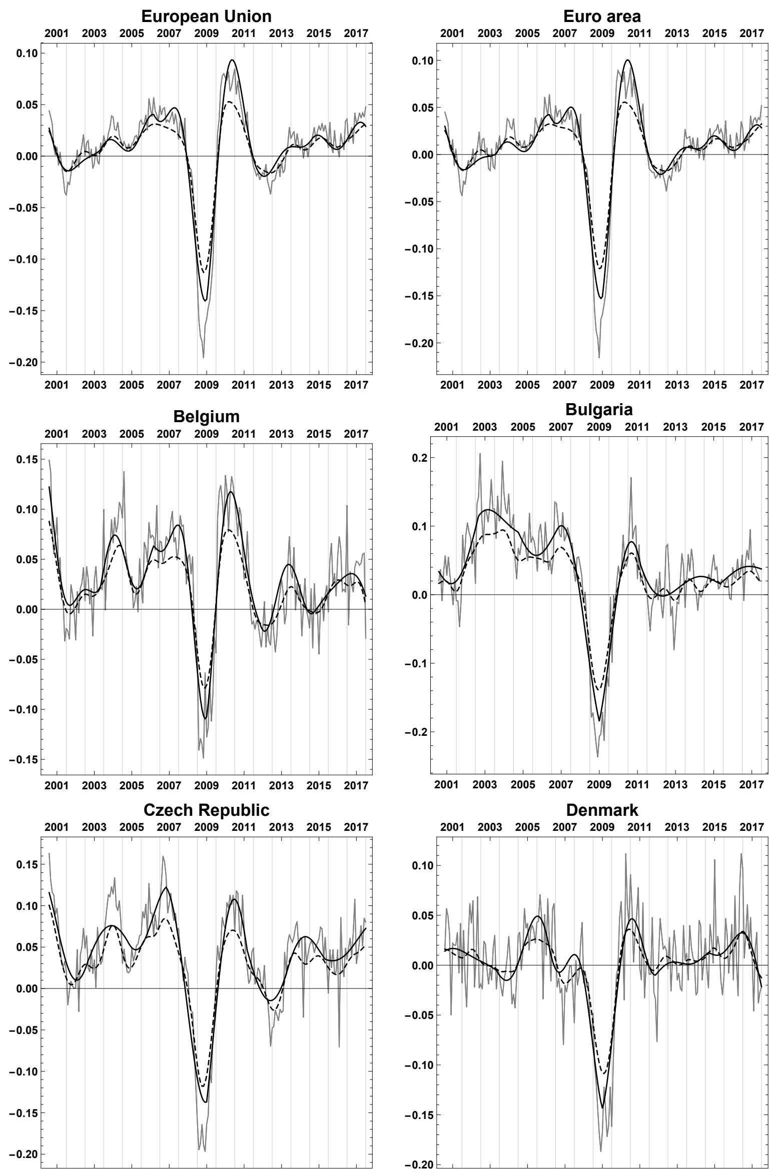


Figure 3: Growth cycle for industrial production from January 2001 to December 2017 - gray line; posterior expectation of cyclical fluctuations with trend - black line; cycle extracted by CF filter - dashed line

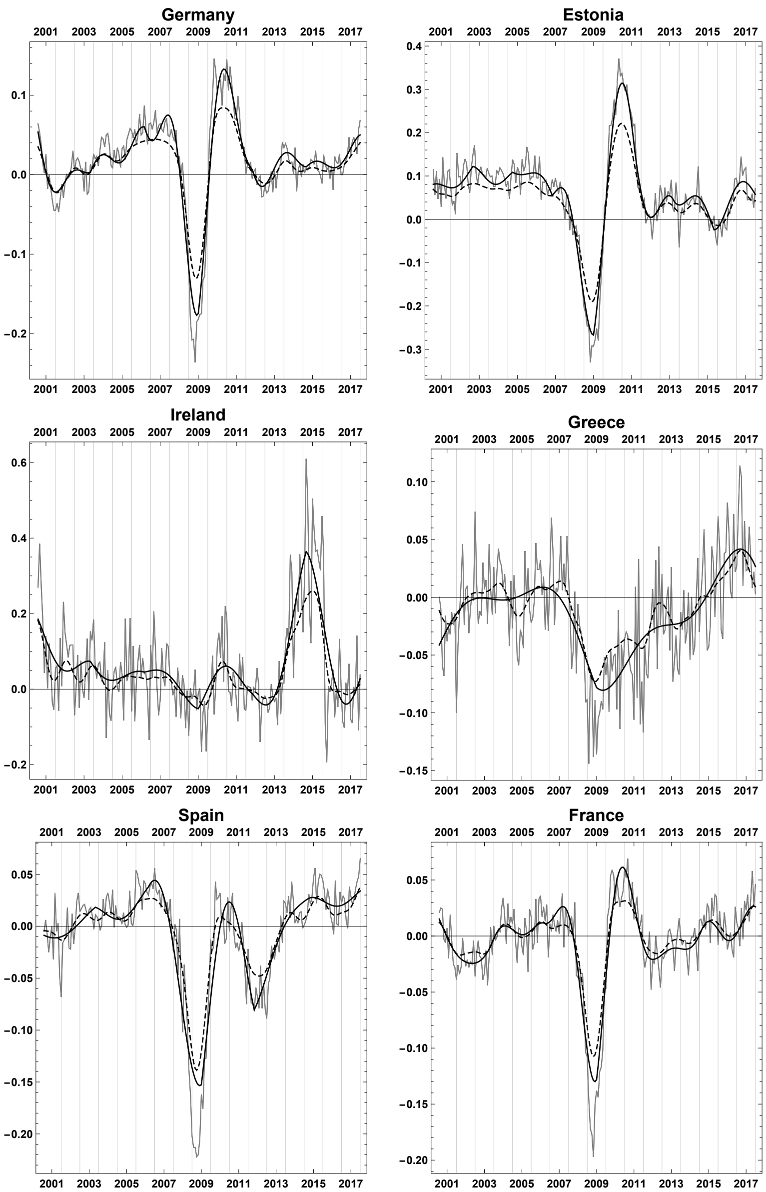


Figure 4: Growth cycle for industrial production from January 2001 to December 2017 - gray line; posterior expectation of cyclical fluctuations with trend - black line; cycle extracted by CF filter - dashed line

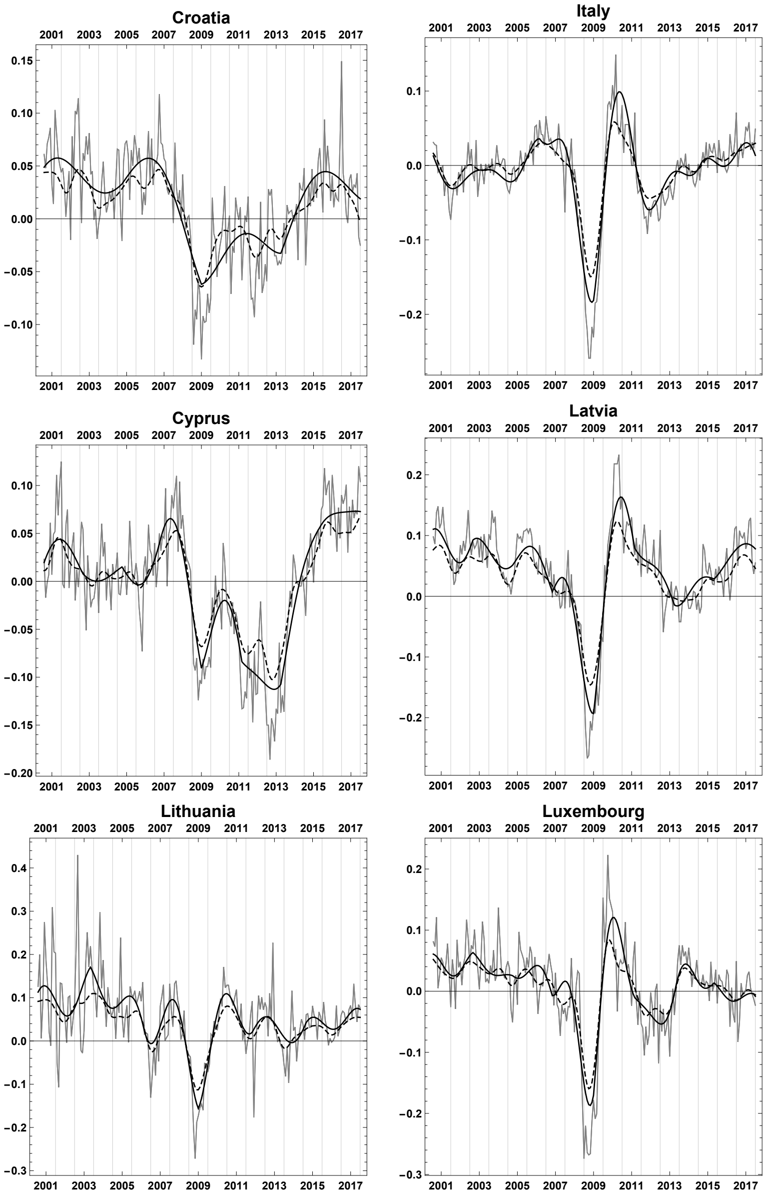


Figure 5: Growth cycle for industrial production from January 2001 to December 2017 - gray line; posterior expectation of cyclical fluctuations with trend - black line; cycle extracted by CF filter - dashed line

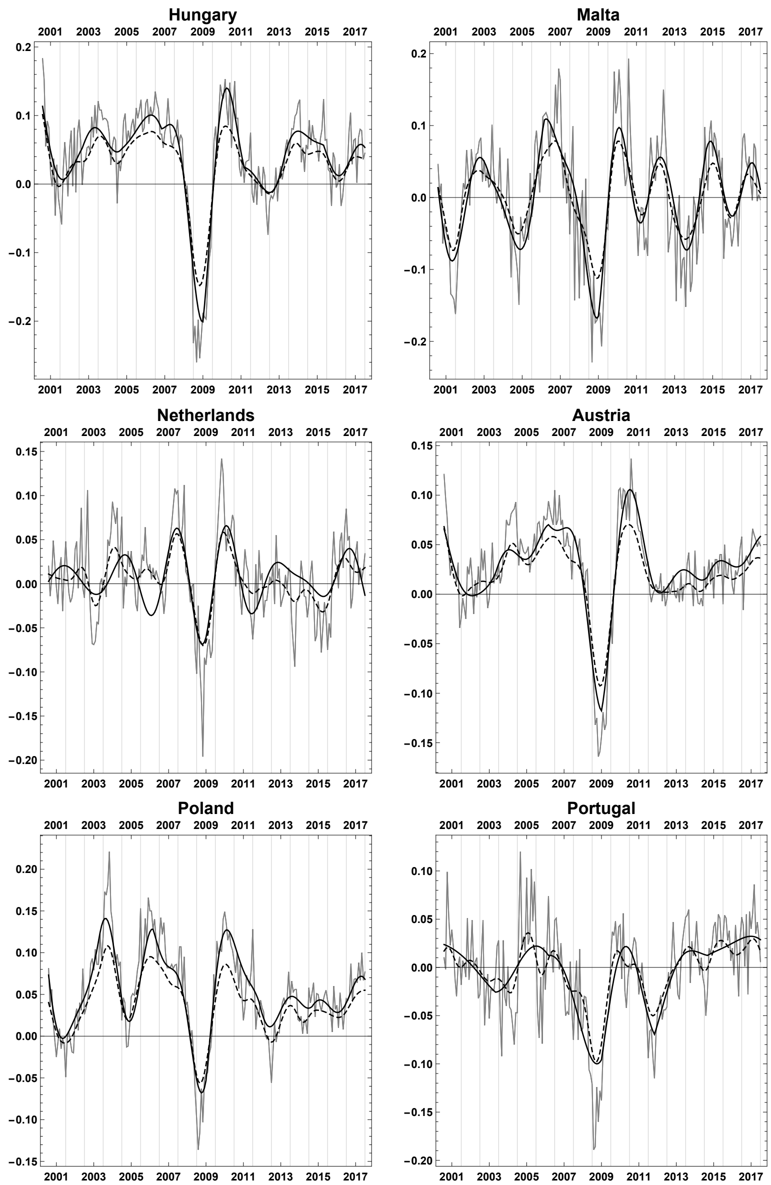


Figure 6: Growth cycle for industrial production from January 2001 to December 2017 - gray line; posterior expectation of cyclical fluctuations with trend - black line; cycle extracted by CF filter - dashed line

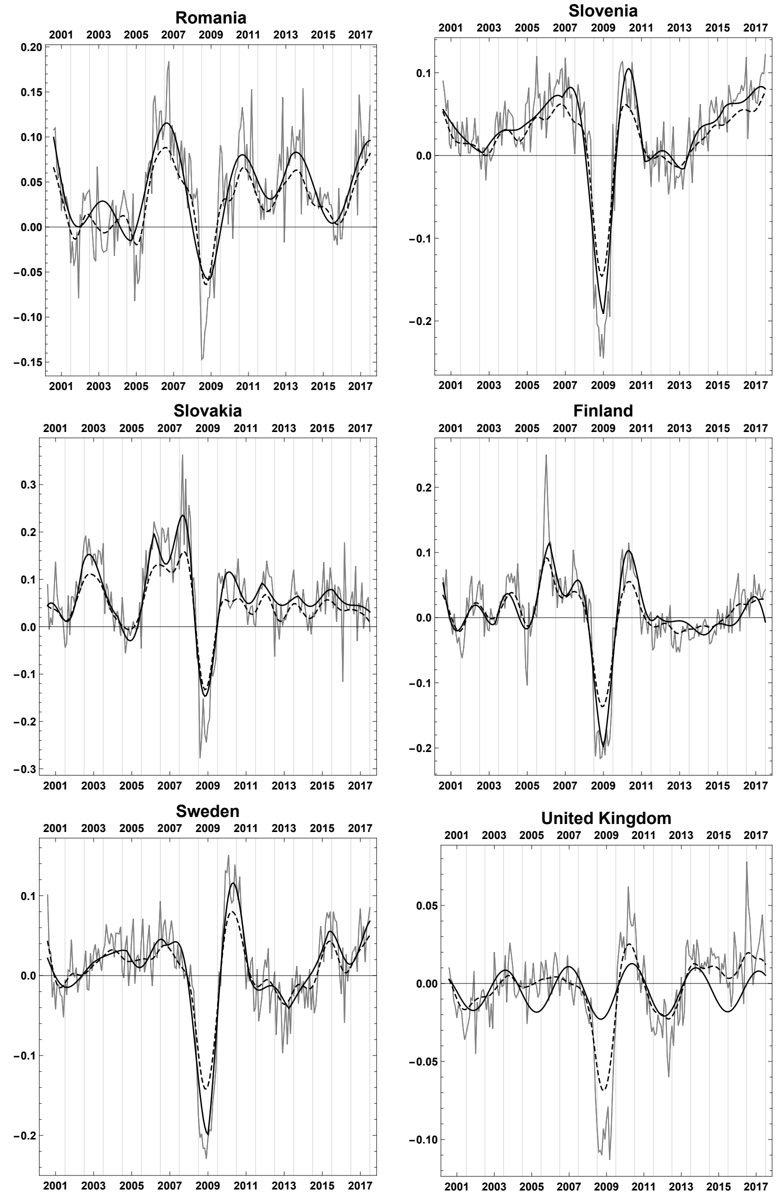
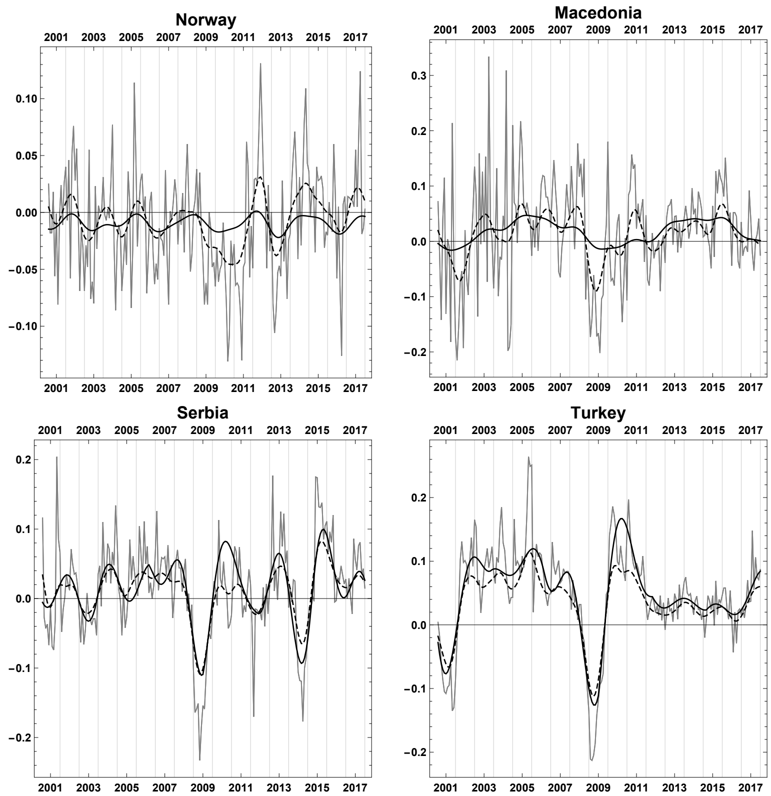


Figure 7: Growth cycle for industrial production from January 2001 to December 2017 - gray line; posterior expectation of cyclical fluctuations with trend - black line; cycle extracted by CF filter - dashed line



5.3 The posterior for frequency and the Kitchin cycle analysis

In Figures 9-10 we present the marginal posterior distribution for frequency for the best model from Table 4. The gray field indicates the standard periodogram for the data used. Note that in many cases the concentration of the mass for posterior distribution for frequency is quite different from the mass concentration on the periodogram. This is probably due to two completely different approaches. The parametric approach proposed in this paper assumes a time-varying amplitude, whereas on the periodogram, we can only identify a constant amplitude.

In a later part of this section we will measure the share of identified fluctuations $\mathbf{y}_E = E(\mathbf{X}(\mathbf{\Lambda})\beta|\mathbf{y})$ in relation to noise ϵ_t variance. For this purpose, we will use the very well known in applications idea of signal-to-noise ratio (see for example Loizou (2013)). In the considered model the signal-to-noise ratio (SNR for short) for the observed process on set $t \in \{1, 2, \ldots, n\}$ can be defined as

$$SNR = \frac{\frac{1}{n}\sum_{t=1}^{n}(g(\lambda, t) + \mu(t))^2}{\mathrm{var}(\epsilon_t)}.$$

Higher SNR values indicate a higher signal share with respect to the noise share. To obtain an empirical SNR, we propose replacing $g(\lambda, t) + \mu(t)$ with the posterior expectation $E(g(\lambda, t) + \mu(t)|\mathbf{y})$. In place of $\mathrm{var}(\epsilon_t)$, we propose basing the empirical variance on $\mathbf{y} - \mathbf{y}_E$. More precisely, we propose using an empirical measure of the form

$$S\hat{N}R = \frac{\mathbf{y}_E'\mathbf{y}_E}{(\mathbf{y} - \mathbf{y}_E)'(\mathbf{y} - \mathbf{y}_E)}.$$

Some portion of the posterior frequency distributions data supports typical frequencies for the Kitchin cycle, i.e., from 3 to 5 years. To better analyze this statement, we present the probability mass related to the Kitchin cycle (in the range of 3 to 5 years) together with the $S\hat{N}R$ statistics in Figure 8. Estonia, Germany, the European Union and the Euro area have the highest $S\hat{N}R$ statistics, which means that the share of noise in fluctuations is the lowest. In analyzing the map of $S\hat{N}R$ values (see Figure 11), one can see that a large portion of Balkan countries are characterized by a high share of noise in fluctuations in industrial production. In turn, a low share of noise is observed in the majority of Central European countries and Southern Europe (excluding Portugal), among others. This is probably the result of the structure of industrial production in these countries. The probability mass related to the Kitchen cycle is higher than 0.9 for 4 countries (Lithuania, Estonia, Turkey and Hungary). For 7 countries (Cyprus, Spain, Bulgaria, Macedonia, the Netherlands, Croatia, Greece), this probability is less then 0.2. However, for almost all of these 7 countries (except the Netherlands), the probability mass for frequency is focused on values greater than 5 years, which may suggest the occurrence of longer (than 5 year) cycles.

Let us note that Estonia and Germany have the highest $S\hat{N}R$ statistics with simultaneous high concentrations of posterior probability mass related to Kitchen cycles. The European Union and Euro area are in a similar situation. This means that the Kitchen cycle is predominant and the share of identified cyclical fluctuations in relation to noise variance is low for total industrial production for the European Union and the Euro area. This is especially evident for Estonia and Germany. However, in many countries the share of noise is much larger and the identified cycle lengths are not concentrated in the range of 3-5 years. This indicates the diversity of results between considered European countries.

It should be emphasized that the above empirical results should be interpreted with special caution. Note that long-term fluctuations are modelled in a very simple way in the proposed model (i.e., using a polynomial).

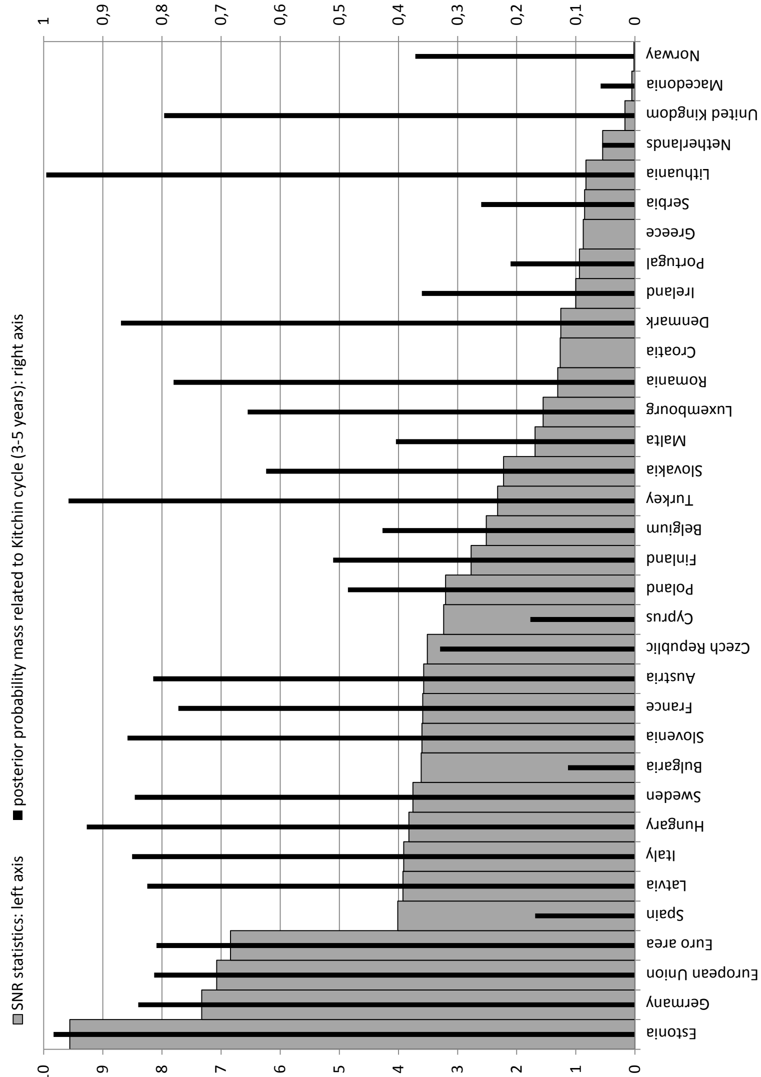


Figure 8: The $\hat{SNR}$ statistics with the posterior probability mass related to the Kitchin cycle (from 3 to 5 years)

Figure 9: The shape of empirical periodograms (gray field) and the shape of marginal posterior distributions (14) from Table 4 with maximum marginal data density value (black line). Instead of frequency, the cycle length (in years) is indicated on the horizontal axis

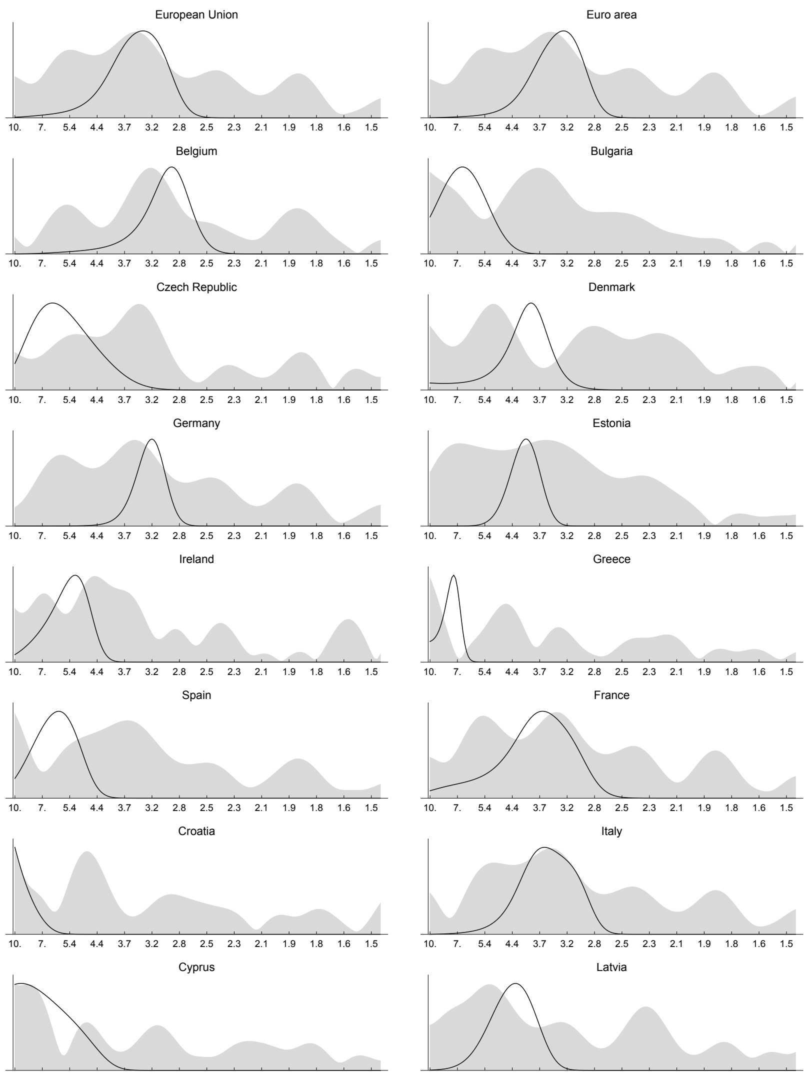


Figure 10: The shape of empirical periodograms (gray field) and the shape of marginal posterior distributions (14) from Table 4 with maximum marginal data density value (black line). Instead of frequency, the cycle length (in years) is indicated on the horizontal axis

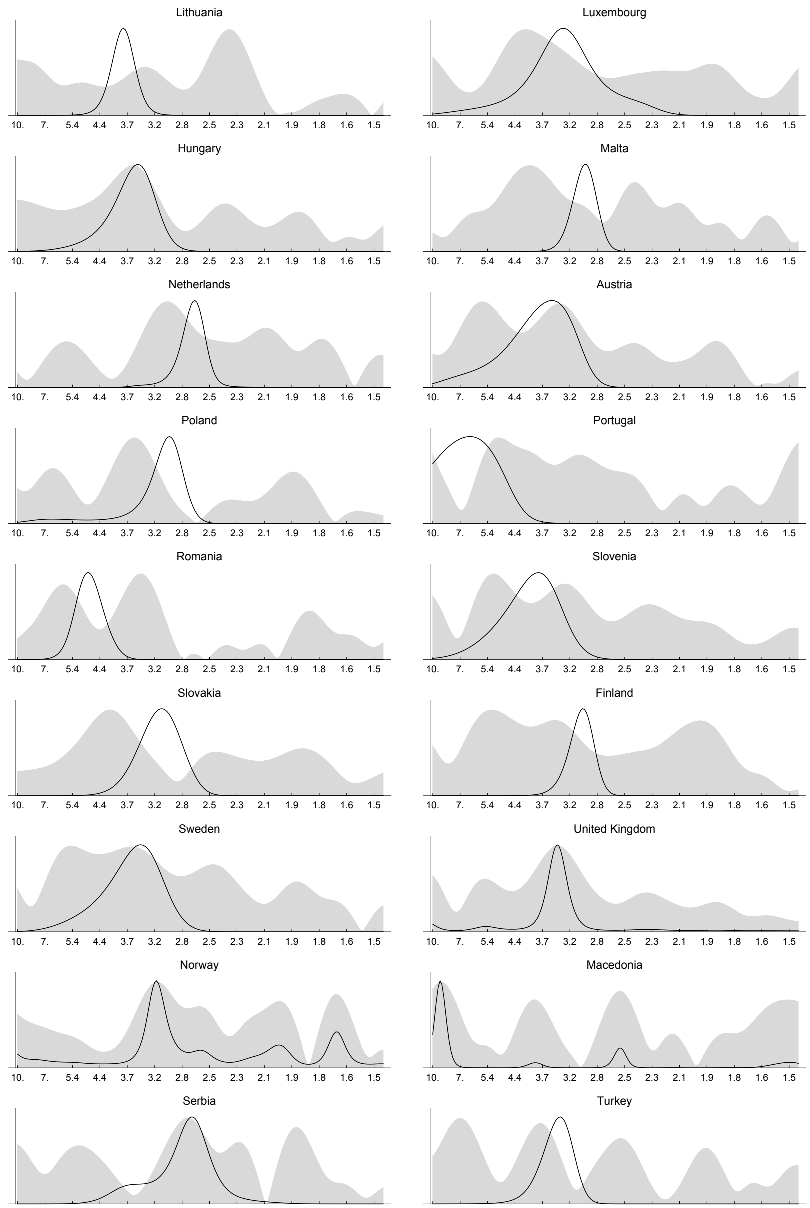
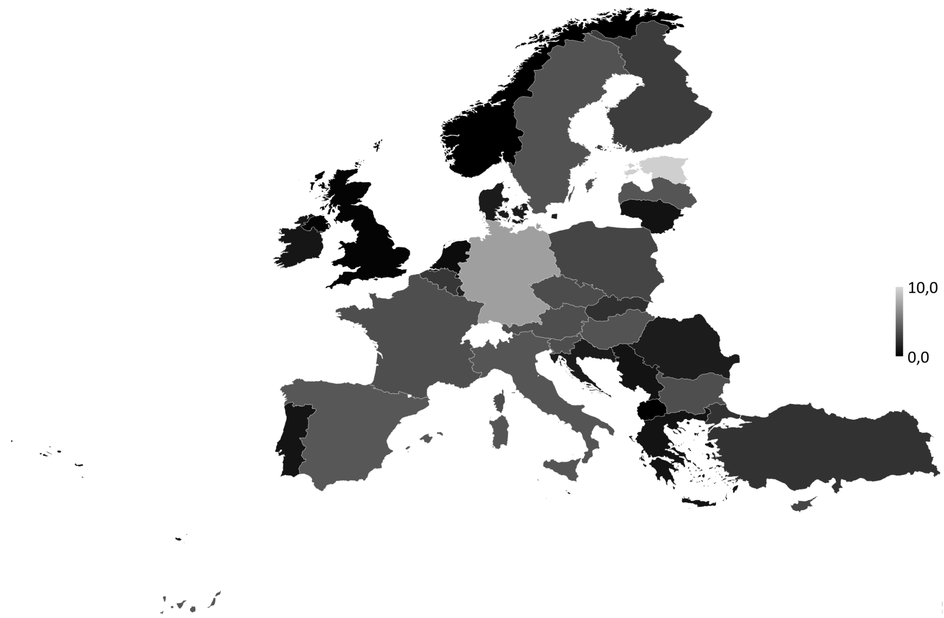


Figure 11: $S\hat{N}R$ map



6 Conclusions

This paper proposes a probabilistic approach to business cycle analysis. This approach combines the new concept of the deterministic cycle with time-varying amplitude and the Bayesian approach. The closed form of the marginal posterior distribution for frequencies (related to cycle length) in the case of time-varying amplitude is shown. This gives an opportunity to expand the statistical inference proposed in Lenart and Mazur (2016). The method is illustrated by the example of a growth cycle of industrial production in European countries. Some initial results were obtained under strong assumptions. These results are promising and point to sufficient dynamics for the proposed deterministic cycle (with time-varying amplitude) to describe a growth cycle that is variable in time. The most important assumption requiring weakening is the assumption of equally spaced knots of spline functions determining the amplitude of fluctuations.

The proposed model is very simple. Therefore, the empirical results are more illustrative than comprehensive. Let us note that long-term fluctuations are modelled only through polynomials in the proposed model. This assumption should be weakened and more advanced methods of eliminating the trend from the data should definitely be considered. One proposal is to consider classical non-parametric methods for trend and cycle filtration (for example, using bandpass filters) and then detrended data analysis. This is the object of the author's interest.

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