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Some q-rung orthopair linguistic Heronian mean operators with their application to multi-attribute group decision making

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The recently proposed q-rung orthopair fuzzy set (q-ROFS) characterized by a membership degree and a non-membership degree is powerful tool for handling uncertainty and vagueness. This paper proposes the concept of q-rung orthopair linguistic set (q-ROLS) by combining the linguistic term sets with q-ROFSs. Thereafter, we investigate multi-attribute group decision making (MAGDM) with q-rung orthopair linguistic information. To aggregate q-rung orthopair linguistic numbers (q-ROLNs), we extend the Heronian mean (HM) to q-ROLSs and propose a family of q-rung orthopair linguistic Heronian mean operators, such as the q-rung orthopair linguistic Heronian mean (q-ROLHM) operator, the q-rung orthopair linguistic weighted Heronian mean (q-ROLWHM) operator, the q-rung orthopair linguistic geometric Heronian mean (q-ROLGHM) operator and the q-rung orthopair linguistic weighted geometric Heronian mean (q-ROLWGHM) operator. Some desirable properties and special cases of the proposed operators are discussed. Further, we develop a novel approach to MAGDM within q-rung orthopair linguistic context based on the proposed operators. A numerical instance is provided to demonstrate the effectiveness and superiorities of the proposed method.

Key words: q-rung orthopair fuzzy set, q-rung orthopair linguistic set, Heronian mean, q-rung orthopair linguistic Heronian mean, multi-attribute group decision making

1. Introduction

Multi-attribute decision making is an activity that aims to select the best alternative from a set of candidates with respect to a set of attributes. Due to the increase of complexity in decision making, we have to face the difficulties of representing the attribute values in different complicated and fuzzy environments. Zadeh's fuzzy set (FS) theory [1] is a powerful tool to describe and depict fuzziness and uncertainty. Thereafter, Atanassov [2] pro-

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posed the concept of intuitionistic fuzzy set (IFS), which has a membership degree and a non-membership degree. IFS can be viewed as an extension of the classic FS theory and it can cope with uncertainty more comprehensively. Since the introduction of IFS, it has drawn much scholars' attention and quite a few works have been reported. For instance, Xu [3] proposed a family of intuitionistic fuzzy ordered weighted average operators by extending the ordered weighted average operator to IFSs. Jiang et al. [4] developed the entropy-based intuitionistic fuzzy power operator and applied it to MAGDM. Ren et al. [5] proposed a thermodynamic method for multiple criteria decision making with intuitionistic fuzzy numbers (IFNs). Zhang [6] proposed a family of intuitionistic fuzzy Einstein hybrid weighted operators based on Einstein operations for IFNs. In addition, IFSs have been widely applied to medical decision making [7, 8], cluster analysis [9, 10] and pattern recognition [11, 12].

From above analysis, we can find that IFSs are an effective tool in decision making. However, there are quite a few situations that IFSs cannot effectively deal with. For instance, if the membership degree and the non-membership degree provided by a decision maker are 0.6 and 0.7 respectively, then it is not valid for IFNs. In other words, the ordered pair (0.6, 0.7) cannot be denoted by an IFN. In order to address these kinds of circumstances, Yager [13] put forward the concept of Pythagorean fuzzy set (PFS), which also has a membership degree and a non-membership degree. The prominent feature of PFS is that the sum of its membership and non-membership degrees is allowed to be greater than one, with their square sum is less than or equal to one. Therefore, PFS is more general than IFS and all intuitionistic membership degrees are part of Pythagorean fuzzy membership degrees. Since its appearance, it has received more and more attention. Garg [14, 15] and Rahman et al. [16] proposed a family of Pythagorean fuzzy Einstein operators based on Einstein t-norm and t-conorm respectively. To consider the relationship between Pythagorean fuzzy numbers (PFNs), Wei and Lu [17] proposed some Pythagorean fuzzy power aggregation operators by extending Yager's power average operator [18] to PFSs. To process the interactions between membership and non-membership degrees of PFSs, Wei [19] developed a series of Pythagorean fuzzy interaction operators. To capture the interrelationship between aggregated PFNs, Liang et al. [20] and Zhang et al. [21] proposed some Pythagorean fuzzy Bonferroni mean operators. Wei and Lu [22] developed a series of Pythagorean fuzzy Maclaurin symmetric mean operators. Considering there are situations in which decision makers may be hesitant when determining the membership degrees between a set of possible values, Liang et al. [23], Khan et al. [24] and Lu et al. [25] proposed the concept of hesitant Pythagorean fuzzy set (HPFS) respectively. Concretely, Liang et al. [23] employed the technique for order preference by similarity to ideal solution (TOPSIS) method to MAGDM based on HPFSs. Khan et al. [24] proposed a family of hesitant Pythagorean fuzzy aggregation operators based on algebraic t-norm and t-conorm. Lu et al. [25] proposed some novel hesitant Pythagorean fuzzy aggregation operators based on Hamacher t-norm and t-conorm. Recently, Wei and Lu [26] proposed the concept of dual hesitant Pythagorean fuzzy sets as well as their aggregation operators based on Hamacher t-norm and t-conorm.

More recently, Yager [27] introduced a new extension of FS, called q-ROFS. The prominent feature of the q-ROFS is that the sum and square sum of membership and non-membership degrees are allowed to be greater than one with their sum of qth power of the membership degree and the qth power of the degree of non-membership being equal to or less than 1. Thus, q-ROFS is more general and powerful than IFS and PFS. To effectively aggregate q-rung orthopair fuzzy information, Liu and Wang [28] proposed a family of q-rung orthopair fuzzy weighted aggregation operators. To capture the interrelationship between q-rung orthopair fuzzy numbers (q-ROFNs), Liu PD and Liu JL [29] put forward a family of q-rung orthopair fuzzy Bonferroni mean operators.

There are some situations in which decision makers prefer to make qualitative decisions instead of quantitative decisions due to a lack of time and expertise. Zadeh's [30] linguistic variables are effective tools to make qualitative decisions. However, the traditional linguistic variables can only reflect decision makers' qualitative preferences and the membership and non-membership degrees of an element to a particular concept are ignored. Therefore, motivated by the concept of IFS, Wang and Li [31] proposed the concept of intuitionistic linguistic set (ILS) by combining linguistic term set with IFS. Du et al. [32] proposed the interval-valued Pythagorean fuzzy linguistic set as well as some aggregation operators. In this paper, we first give the definition of q-ROLS as well as their operations. To effectively aggregate q-rung orthopair linguistic information, we investigate HM under q-ROLSs and propose some q-rung orthopair linguistic Heronian mean operators. Moreover, we apply the proposed operators to solve MAGDM.

In order to do this, the remainder of the paper is organized as follows. Sections 2 briefly recalls some notions. Section 3 develops some q-rung orthopair linguistic Heronian means. In addition, we also investigate some properties of the proposed operators. Section 4 presents a novel approach to q-rung orthopair linguistic MAGDM and a numerical experiment is conducted in Section 5. Conclusions are given in Section 6.

2. Preliminaries

In this section, we briefly review concepts about q-ROFS, linguistic term set and HM.

2.1. The q-rung orthopair fuzzy set

Definition 1 [27] *Let X be an ordinary fixed set, a q-ROFS A defined on X is given by*

$$A = \{\langle x, u_A(x), v_A(x)\rangle \,|x \in X\}, \tag{1}$$

where $u_A(x)$ and $v_A(x)$ represent the membership degree and non-membership degree respectively, satisfying $u_A(x) \in [0,1]$, $v_A(x) \in [0,1]$ and $0 \leqslant u_A(x)^q + v_A(x)^q \leqslant 1$, $(q \geqslant 1)$. The indeterminacy degree of A is defined as $\pi_A(x) = \left(u_A(x)^q + v_A(x)^q - u_A(x)^q v_A(x)^q\right)^{1/q}$. For convenience, $(u_A(x), v_A(x))$ is called a q-ROFN by Liu and Wang [28], which can be denoted as $A = (u_A, v_A)$.

Liu and Wang [28] also proposed some operations for q-ROFNs.

Definition 2 [28] *Let $\widetilde{a}_1 = (u_1, v_1)$ and $\widetilde{a}_2 = (u_2, v_2)$ be two q-ROFNs, λ be a positive real number, then*

(1) $\widetilde{a}_1 \oplus \widetilde{a}_2 = \left(\left(u_1^q + u_2^q - u_1^q u_2^q\right)^{1/q}, v_1 v_2\right)$;

(2) $\widetilde{a}_1 \otimes \widetilde{a}_2 = \left(u_1 u_2, \left(v_1^q + v_2^q - v_1^q v_2^q\right)^{1/q}\right)$;

(3) $\lambda \widetilde{a}_1 = \left(\left(1 - \left(1 - u_1^q\right)^{\lambda}\right)^{1/q}, v_1^{\lambda}\right)$;

(4) $\widetilde{a}_1^{\lambda} = \left(u_1^{\lambda}, \left(1 - \left(1 - v_1^q\right)^{\lambda}\right)^{1/q}\right)$.

To compare two q-ROFNs, Liu and Wang [28] proposed a comparison method for q-ROFNs.

Definition 3 [28] *Let $\widetilde{a} = (u_a, v_a)$ be a q-ROFN, then the score function of $\widetilde{a}$ is defined as $S(\widetilde{a}) = \mu_a^q - \nu_a^q$, the accuracy function of $\widetilde{a}$ is defined as $H(\widetilde{a}) = \mu_a^q + \nu_a^q$. For any two q-ROFNs, $\widetilde{a}_1 = (u_1, v_1)$ and $\widetilde{a}_2 = (u_2, v_2)$. Then*

(1) *If $S(\widetilde{a}_1) > S(\widetilde{a}_2)$, then $\widetilde{a}_1 > \widetilde{a}_2$;*

(2) *If $S(\widetilde{a}_1) > S(\widetilde{a}_2)$, then*

If $H(\widetilde{a}_1) > H(\widetilde{a}_2)$, then $\widetilde{a}_1 > \widetilde{a}_2$;

If $H(\widetilde{a}_1) = H(\widetilde{a}_2)$, then $\widetilde{a}_1 = \widetilde{a}_2$.

2.2. Linguistic term set and q-rung orthopair linguistic set

Let $S = \{s_i \,| i = 1, 2, \ldots, t\}$ be a linguistic term set with odd cardinality and t is the cardinality of S. The label s_i represents a possible value for a linguistic variable. For instance, a possible linguistic term set can be defined as follows:

$$\begin{aligned} S &= (s_1, s_2, s_3, s_4, s_5, s_6, s_7) \\ &= \{\text{very poor, poor, slightly poor, fair, slightly good, good, very good}\}. \end{aligned}$$

Motivated by the concept of intuitionistic linguistic set (ILS) [31], we propose the concept of q-ROFLS by combining the linguistic term set with q-ROFS.

Definition 4 *Let X be an ordinary fixed set and $\overline{S}$ be a continuous linguistic term set of $S = \{s_i \,|\, i = 1,2,\ldots,t\}$, then a q-rung orthopair linguistic set (q-ROLS) A on X can be given as follows*

$$A = \left\{ \left\langle x, s_{\theta(x)}, (u_A(x), v_A(x)) \right\rangle | x \in X \right\}, \tag{2}$$

where $s_{\theta(x)} \in \overline{S}$, $u_A(x)\colon X \to [0,1]$ and $v_A(x)\colon X \to [0,1]$, satisfying $0 \leqslant (u_A(x))^q + (v_A(x))^q \leqslant 1$, $(q \geqslant 1)$, then $\left\langle s_{\theta(x)}, (u_A(x), v_A(x)) \right\rangle$ is called a q-rung orthopair linguistic number (q-ROLN), which can be simply denoted by $\alpha = \langle s_\theta, (u,v) \rangle$.

Based on the operations for q-ROFNs, we provide some operations for q-ROLNs.

Definition 5 *Let $\alpha_1 = \left\langle s_{\theta_1}, (u_1, v_1) \right\rangle$ and $\alpha_2 = \left\langle s_{\theta_2}, (u_2, v_2) \right\rangle$ be any two q-ROLNs, and λ be a positive real number, then*

(1) $\alpha_1 \oplus \alpha_2 = \left\langle s_{\theta_1+\theta_2}, \left(\left(u_1^q + u_2^q - u_1^q u_2^q \right)^{1/q}, v_1 v_2 \right) \right\rangle$;

(2) $\alpha_1 \otimes \alpha_2 = \left\langle s_{\theta_1\times\theta_2}, \left(u_1 u_2, \left(v_1^q + v_2^q - v_1^q v_2^q \right)^{1/q} \right) \right\rangle$;

(3) $\lambda\alpha_1 = \left\langle s_{\lambda\times\theta_1}, \left(\left(1 - \left(1 - u_1^q \right)^\lambda \right)^{1/q}, v_1^\lambda \right) \right\rangle$;

(4) $\alpha_1^\lambda = \left\langle s_{\theta_1^\lambda}, \left(u_1^\lambda, \left(1 - \left(1 - v_1^q \right)^\lambda \right)^{1/q} \right) \right\rangle$.

To compare two q-ROLNs, we firstly propose the concepts of score function and accuracy function of a q-ROLN. Then based on the two concepts, we propose a comparison rule for q-ROLNs.

Definition 6 *Let $\alpha = \langle s_\theta, (u,v) \rangle$ be a q-ROLN, the score function of α is given by*

$$S(\alpha) = (u^q + 1 - v^q) \times \theta. \tag{3}$$

Definition 7 *Let $\alpha = \langle s_\theta, (u,v) \rangle$ be a q-ROLN, then the accuracy function of α is defined as*

$$H(\alpha) = (u^q + v^q) \times \theta. \tag{4}$$

Then we provide a comparison law for q-ROLNs.

Definition 8 *Let $\alpha_1 = \left\langle s_{\theta_1}, (u_1, v_1) \right\rangle$ and $\alpha_2 = \left\langle s_{\theta_2}, (u_2, v_2) \right\rangle$ be any two q-ROLNs, $S(\alpha_1)$ and $S(\alpha_2)$ be the score functions of α_1 and α_2 respectively, $H(\alpha_1)$ and $H(\alpha_2)$ be the accuracy functions of α_1 and α_2 respectively, then*

(1) *If* $S(\alpha_1) > S(\alpha_2)$, *then* $\alpha_1 > \alpha_2$;

(2) *If* $S(\alpha_1) = S(\alpha_2)$, *then*

If $H(\alpha_1) > H(\alpha_2)$, *then* $\alpha_1 > \alpha_2$;

If $H(\alpha_1) = H(\alpha_2)$, *then* $\alpha_1 = \alpha_2$.

2.3. Heronian mean

Definition 9 [33, 34] *Let* a_i $(i = 1, 2, \ldots, n)$ *be a collection of crisp numbers, and* $s,\ t > 0$, *then the Heronian mean (HM) is defined as follows:*

$$HM^{s,t}(a_1, a_2, \ldots, a_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} a_i^s a_j^t\right)^{1/(s+t)}. \tag{5}$$

Definition 10 [35] *Let* a_i $(i = 1, 2, \ldots, n)$ *be a collection of crisp numbers, and* $s,\ t > 0$, *then the geometric Heronian mean (GHM) is defined as follows:*

$$GHM^{s,t}(a_1, a_2, \ldots, a_n) = \frac{1}{s+t} \prod_{i=1}^{n} \prod_{j=i}^{n} (sa_i + ta_j)^{\frac{1}{n(n+2)}}. \tag{6}$$

3. The q-rung orthopair linguistic Heronian mean operators

In this section, we extend HM to q-ROLSs and proposed a series of q-rung orthopair linguistic Heronian mean operators.

3.1. The q-rung orthopair linguistic Heronian mean (q-ROLHM) operator

Definition 11 *Let* α_i $(i = 1, 2, \ldots, n)$ *be a collection of q-ROLNs, and* $s,\ t > 0$. *If*

$$q\text{-}ROLHM^{s,t}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} \alpha_i^s \alpha_j^t\right)^{1/(s+t)}, \tag{7}$$

then q-ROLHM^{s,t} *is called the q-rung orthopair linguistic Heronian mean (q-ROLHM) operator.*

According to the operations for q-ROLNs, the following theorem can be obtained.

Theorem 1 *Let* α_i $(i = 1, 2, \ldots, n)$ *be a collection of q-ROLNs, then the aggregated value by using q-ROLHM is also a q-ROLN and*

$$q\text{-}ROLHM^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left\langle s_{\left(\frac{2}{n(n+1)}\sum\limits_{i=1}^{n}\sum\limits_{j=i}^{n}\theta_i^s\theta_j^t\right)^{1/(s+t)}},\right.$$
$$\left(\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-u_i^{sq}u_j^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{1/q(s+t)},\right.$$
$$\left.\left.\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-v_i^q\right)^s\left(1-v_j^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{1/(s+t)}\right)^{1/q}\right)\right\rangle. \tag{8}$$

Proof. According to the operations for q-ROLNs, we can obtain the followings

$$\alpha_i^s=\left\langle s_{\theta_i^s},\left(u_i^s,\left(1-\left(1-v_i^q\right)^s\right)^{1/q}\right)\right\rangle,$$
$$\alpha_j^t=\left\langle s_{\theta_j^t},\left(u_j^t,\left(1-\left(1-v_j^q\right)^t\right)^{1/q}\right)\right\rangle.$$

Therefore,

$$\alpha_i^s\alpha_j^t=\left\langle s_{\theta_i^s\theta_j^t},\left(u_i^su_j^t,\left(1-\left(1-v_i^q\right)^s\left(1-v_j^q\right)^t\right)\right)\right\rangle.$$

Further,

$$\sum_{j=i}^{n}\alpha_i^s\alpha_j^t=\left\langle s_{\sum\limits_{j=i}^{n}\theta_i^s\theta_j^t},\right.$$
$$\left.\left(\left(1-\prod_{j=i}^{n}\left(1-u_i^{sq}u_j^{tq}\right)\right)^{1/q},\prod_{j=i}^{n}\left(1-\left(1-v_i^q\right)^s\left(1-v_j^q\right)^t\right)\right)\right\rangle.$$

In addition,

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_i^s\alpha_j^t=\left\langle s_{\sum\limits_{i=1}^{n}\sum\limits_{j=i}^{n}\theta_i^s\theta_j^t},\right.$$
$$\left.\left(\left(1-\prod_{i=1}^{n}\left(\prod_{j=i}^{n}\left(1-u_i^{sq}u_j^{tq}\right)\right)\right)^{1/q},\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-v_i^q\right)^s\left(1-v_j^q\right)^t\right)\right)\right\rangle.$$

Thus,

$$\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_i^s\alpha_j^t=\Bigg\langle s_{\frac{2}{n(n+1)}\sum\limits_{i=1}^{n}\sum\limits_{j=i}^{n}\theta_i^s\theta_j^t},\\ \left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-u_i^{sq}u_j^{tq}\right)\right)^{\frac{2}{n(n+1)}}\right)^{1/q},\\ \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-v_i^q\right)^s\left(1-v_j^q\right)^t\right)\right)^{\frac{2}{n(n+1)}}\right)\Bigg\rangle.$$

So,

$$q\text{-ROLHM}^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_i^s\alpha_j^t\right)^{1/(s+t)}\\ =\Bigg\langle s_{\left(\frac{2}{n(n+1)}\sum\limits_{i=1}^{n}\sum\limits_{j=i}^{n}\theta_i^s\theta_j^t\right)^{1/(s+t)}},\left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-u_i^{sq}u_j^{tq}\right)\right)^{\frac{2}{n(n+1)}}\right)^{1/q(s+t)},\\ \left(1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-v_i^q\right)^s\left(1-v_j^q\right)^t\right)\right)^{\frac{2q}{n(n+1)}}\right)^{1/(s+t)}\right)^{1/q}\right)\Bigg\rangle. \qquad \square$$

In addition, the q-ROLHM operator has the following properties.

Theorem 2 (Monotonicity) *Let α_i and β_i $(i=1,2,\ldots,n)$ be two collections of q-ROLNs, if $\alpha_i\leqslant\beta_i$ for all $i=1,2,\ldots,n$, then*

$$q\text{-}ROLHM^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n)\leqslant q\text{-}ROLHM^{s,t}(\beta_1,\beta_2,\ldots,\beta_n). \tag{9}$$

Proof. As $\alpha_i=\alpha$ for all i, we can obtain

Since $\alpha_i\leqslant\beta_i$ and $\alpha_j\leqslant\beta_j$ for $i=1,2,\ldots,n$ and $j=i,i+1,\ldots,n$, we have $\alpha_i^s\alpha_j^t\leqslant\beta_i^s\beta_j^t$.

Then

$$\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_i^s\alpha_j^t\leqslant\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\beta_i^s\beta_j^t.$$

So,

$$\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_i^s\alpha_j^t\right)^{1/(s+t)} \leqslant \left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\beta_i^s\beta_j^t\right)^{1/(s+t)},$$

i.e.

$$q\text{-ROLHM}^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n) \leqslant q\text{-ROLHM}^{s,t}(\beta_1,\beta_2,\ldots,\beta_n). \qquad \square$$

Theorem 3 (Idempotency) *Let α_i $(i = 1,2,\ldots,n)$ be a collection of q-ROLNs, if $\alpha_i = \alpha$, for all $i = 1,2,\ldots,n$, then*

$$q\text{-}ROLHM^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \alpha. \tag{10}$$

Proof. Since $\alpha_i = \alpha$, for all i, we have

$$q\text{-ROLHM}^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_i^s\alpha_j^t\right)^{1/(s+t)} = \left(\alpha^{s+t}\right)^{1/(s+t)} = \alpha. \qquad \square$$

Theorem 4 (Boundedness) *The q-ROLHM operator lies between the* max *and* min *operators*

$$\min(\alpha_1,\alpha_2,\ldots,\alpha_n) \leqslant q\text{-}ROLHM^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n) \leqslant \max(\alpha_1,\alpha_2,\ldots,\alpha_n). \tag{11}$$

Proof. Let $a = \min(\alpha_1,\alpha_2,\ldots,\alpha_n)$, $b = \max(\alpha_1,\alpha_2,\ldots,\alpha_n)$ according to Theorem 2, we have

$$q\text{-ROLHM}^{s,t}(a,a,\ldots,a) \leqslant q\text{-ROLHM}^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n) \leqslant q\text{-ROLHM}^{s,t}(b,b,\ldots,b).$$

Further,

$$q\text{-ROLHM}^{s,t}(a,a,\ldots,a) = a \quad \text{and} \quad q\text{-ROLHM}^{s,t}(b,b,\ldots,b) = b.$$

So,

$$a \leqslant q\text{-ROLHM}^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n) \leqslant b,$$

i.e.

$$\min(\alpha_1,\alpha_2,\ldots,\alpha_n) \leqslant q\text{-ROLHM}^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n) \leqslant \max(\alpha_1,\alpha_2,\ldots,\alpha_n). \qquad \square$$

The parameters s and t play a very important role in the aggregated results. In the followings, we discuss some special cases of the q-ROLHM operator with respect to the parameters s and t.

Case 1 When $t \to 0$, then the q-ROLHM operator reduces to the followings,

$$q\text{-ROLHM}^{s,0}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \lim_{t\to 0}\left\langle s_{\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\theta_i^s\theta_j^t\right)^{1/(s+t)}}, \left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-u_i^{sq}u_j^{tq}\right)\right)^{\frac{2}{n(n+1)}}\right)^{1/q(s+t)}, \left(1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-v_i^q\right)^s\left(1-v_j^q\right)^t\right)\right)^{\frac{2q}{n(n+1)}}\right)^{1/(s+t)}\right)^{1/q}\right)\right\rangle$$

$$= \left\langle s_{\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}(n+1-i)\theta_i^s\right)^{1/s}}, \left(\left(1-\left(\prod_{i=1}^{n}\left(1-u_i^{sq}\right)^{n+1-i}\right)^{\frac{2}{n(n+1)}}\right)^{1/qs}, \left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-v_i^q\right)^s\right)^{n+1-i}\right)^{\frac{2q}{n(n+1)}}\right)^{1/s}\right)^{1/q}\right)\right\rangle, \tag{12}$$

which is a q-rung orthopair linguistic generalized linear descending weighted mean operator. Evidently, it is equivalent to weight the information $(\alpha_1^s,\alpha_2^s,\ldots,\alpha_n^s)$ with $(n,n-1,\ldots,1)$.

Case 2 When $s \to 0$, then the q-ROLHM operator reduces to the followings,

$$q\text{-ROLHM}^{0,t}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \lim_{s\to 0}\left\langle s_{\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\theta_i^s\theta_j^t\right)^{1/(s+t)}}, \left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-u_i^{sq}u_j^{tq}\right)\right)^{\frac{2}{n(n+1)}}\right)^{1/q(s+t)}, \left(1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-v_i^q\right)^s\left(1-v_j^q\right)^t\right)\right)^{\frac{2q}{n(n+1)}}\right)^{1/(s+t)}\right)^{1/q}\right)\right\rangle$$

$$= \left\langle s_{\left(\frac{2}{n(n+1)}\sum\limits_{i=1}^{n} i\theta_i^t\right)^{1/t}}, \left(\left(1-\left(\prod_{i=1}^{n}\left(1-u_i^{tq}\right)^{i}\right)^{\frac{2}{n(n+1)}}\right)^{1/s},\right.\right.$$
$$\left.\left.\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-v_i^q\right)^t\right)^{i}\right)^{\frac{2q}{n(n+1)}}\right)^{1/t}\right)^{1/q}\right)\right\rangle, \tag{13}$$

which is a q-rung orthopair linguistic generalized linear ascending weighted mean operator. Obviously, it is equivalent to weight the information $(\alpha_1^t, \alpha_2^t, \ldots, \alpha_n^t)$ with $(1,2,\ldots,n)$, i.e., when $t=0$ or $s=0$, the q-ROLHM operator has the linear weighted function for input data.

Case 3 When $s=t=1$, then the q-ROLHM operator reduces to the followings,

$$q\text{-ROLHM}^{1,1}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left\langle s_{\left(\frac{2}{n(n+1)}\sum\limits_{i=1}^{n}\sum\limits_{j=i}^{n}\theta_i\theta_j\right)^{1(s+t)}},\right.$$
$$\left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(u_iu_j)^q\right)\right)^{\frac{2}{n(n+1)}}\right)^{1/2q},\right.$$
$$\left.\left.\left(1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-v_i^q\right)\left(1-v_j^q\right)\right)\right)^{\frac{2q}{n(n+1)}}\right)^{1/2}\right)^{1/q}\right)\right\rangle, \tag{14}$$

which is a q-rung orthopair linguistic line Heronian mean operator.

Case 4 When $s=t=1/2$, then the q-ROLHM operator reduces to the followings

$$q\text{-ROLHM}^{\frac{1}{2},\frac{1}{2}}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left\langle s_{\frac{2}{n(n+1)}\sum\limits_{i=1}^{n}\sum\limits_{j=i}^{n}\sqrt{\theta_i\theta_j}},\right.$$
$$\left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\sqrt{u_i^q u_j^q}\right)\right)^{\frac{2}{n(n+1)}}\right)^{1/q},\right.$$
$$\left.\left.\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\sqrt{\left(1-v_i^q\right)\left(1-v_j^q\right)}\right)\right)^{\frac{2}{n(n+1)}}\right)\right\rangle, \tag{15}$$

which is a q-rung orthopair linguistic basic Heronian mean operator.

Case 5 When $q = 2$, then the q-ROLHM operator reduces to the followings,

$$q\text{-ROLHM}^{s,t}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \Bigg\langle s_{\left(\frac{2}{n(n+1)}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\theta_i^s\theta_j^t\right)^{\frac{1}{s+t}}},$$

$$\Bigg(\Bigg(1 - \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1 - u_i^{2s}u_j^{2t}\right)\right)^{\frac{2}{n(n+1)}}\Bigg)^{1/2(s+t)},$$

$$\left(1 - \left(1 - \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1 - \left(1 - v_i^2\right)^s\left(1 - v_j^2\right)^t\right)\right)^{\frac{4}{n(n+1)}}\right)^{1/(s+t)}\right)^{1/2}\Bigg)\Bigg\rangle, \tag{16}$$

which is the Pythagorean linguistic Heronian mean operator.

Case 6 When $q = 1$, then the q-ROLHM operator reduces to the followings,

$$q\text{-ROLHM}^{s,t}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \Bigg\langle s_{\left(\frac{2}{n(n+1)}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\theta_i^s\theta_j^t\right)^{\frac{1}{s+t}}},$$

$$\Bigg(\Bigg(1 - \left(\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1 - u_i^{s}u_j^{t}\right)\right)^{\frac{2}{n(n+1)}}\Bigg)^{\frac{1}{s+t}},$$

$$1 - \left(1 - \left(\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1 - \left(1 - v_i\right)^s\left(1 - v_j\right)^t\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\Bigg)\Bigg\rangle, \tag{17}$$

which is the intuitionistic linguistic Heronian mean operator.

3.2. The q-rung orthopair linguistic weighted Heronian mean (q-ROLWHM) operator

It is noted that the proposed q-ROLHM operator does not consider the weights of the aggregated arguments. Therefore, we put forward the weighted Heronian mean for q-ROLNs.

Definition 12 *Let α_i $(i = 1,2,\ldots,n)$be a collection of q-ROLNs, and s, $t > 0$, $w = (w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ be the weight vector, satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. If*

$$q\text{-}ROLWHM^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}(nw_i\alpha_i)^s\left(nw_j\alpha_j\right)^t\right)^{1/(s+t)}, \quad (18)$$

then q-ROLWHM$^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n)$ is called the q-ROLWHM.

According to the operations for q-ROLNs, the following theorem can be obtained.

Theorem 5 *Let α_i $(i = 1,2,\ldots,n)$ be a collection of q-ROLNs, $w = (w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ be the weight vector, satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then the aggregated value by using q-ROLWHM is also a q-ROLN and*

$$q\text{-}ROLWHM^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left\langle s_{\left(\frac{2}{n(n+1)}\sum\limits_{i=1}^{n}\sum\limits_{j=i}^{n}(nw_i\theta_i)^s\times(nw_j\theta_j)^t\right)^{1/(s+t)}}, \right.$$
$$\left(\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\left(1-u_i^q\right)^{nw_i}\right)^{\frac{2s}{n(n+1)}}\left(1-\left(1-u_j^q\right)^{nw_j}\right)^{\frac{2t}{n(n+1)}}\right)\right)^{1/(s+t)q}\right.,$$
$$\left.\left.\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-v_i^{nw_iq}\right)^s\left(1-v_j^{nw_jq}\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{1/(s+t)}\right)^{1/q}\right)\right\rangle. \quad (19)$$

The proof of Theorem 5 is similar to that of Theorem 1, which is omitted here.

To illustrate the performance of q-ROLWHM operator, we provide an example in the followings.

Example 1 Let $\alpha_1 = \langle s_5,(0.2,0.7)\rangle$, $\alpha_2 = \langle s_1,(0.4,0.6)\rangle$, $\alpha_3 = \langle s_7,(0.3,0.8)\rangle$, and $\alpha_3 = \langle s_6,(0.5,0.9)\rangle$ be four q-ROLNs with the weight vector is $w = (0.2,0.3,0.4,0.1)^{\mathrm{T}}$. Let $\alpha = \langle s_\theta,(u,v)\rangle$ be the comprehensive value and if we utilize q-ROLWHM to aggregate the four q-ROLNs, we can obtain (suppose $q = 3$, $s = 2$, $t = 3$):

$$\theta = \left(\frac{2}{4\times(4+1)}\left(\begin{array}{l}(4\times0.2\times5)^2\times(4\times0.3\times1)^3+(4\times0.2\times5)^2\times(4\times0.4\times7)^3+\\(4\times0.2\times5)^2\times(4\times0.1\times6)^3+(4\times0.3\times1)^2\times(4\times0.4\times7)^3+\\(4\times0.3\times1)^2\times(4\times0.1\times6)^3+(4\times0.4\times7)^2\times(4\times0.1\times6)^3\end{array}\right)\right)^{1/(2+3)}$$
$$= 7.2755.$$

Similarly, we can obtain $u = 0.3784$ and $v = 0.7012$.

It is noted that the parameters s and t play a significant role in the score of the comprehensive value. Details can be found in Fig. 1.

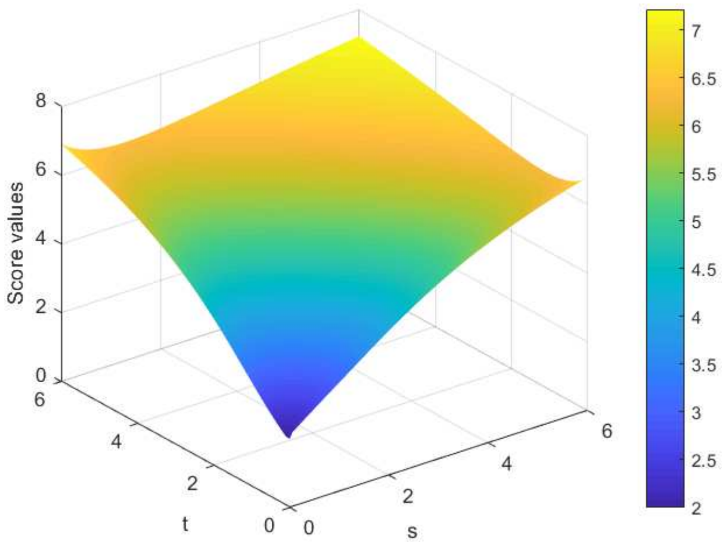

Figure 1: Score values of the alternative when $s,\ t \in [1,6]$ and $q = 3$ using q-ROLWHM operator

Similarly, q-ROLWHM has the following properties.

Theorem 6 (Monotonicity) *Let α_i and β_i $(i = 1,2,\dots,n)$ be two collections of q-ROLNs, if $\alpha_i \leqslant \beta_i$ for all i, then*

$$q\text{-}ROLWHM^{s,t}(\alpha_1,\alpha_2,\dots,\alpha_n) \leqslant q\text{-}ROLWHM^{s,t}(\beta_1,\beta_2,\dots,\beta_n). \tag{20}$$

Theorem 7 (Boundedness) *The q-ROLWHM operator lies between the* max *and* min *operators*

$$\min(\alpha_1,\alpha_2,\dots,\alpha_n) \leqslant q\text{-}ROLWHM^{s,t}(\alpha_1,\alpha_2,\dots,\alpha_n) \leqslant \max(\alpha_1,\alpha_2,\dots,\alpha_n). \tag{21}$$

Evidently, the q-ROLWHM operator does not has the property of idempotency.

3.3. The q-rung orthopair linguistic geometric Heronian mean (q-ROLGHM) operator

Definition 13 *Let α_i $(i = 1,2,\dots,n)$ be a collection of q-ROLNs, and $s,\ t > 0$. If*

$$q\text{-}ROLGHM^{s,t}(\alpha_1,\alpha_2,\dots,\alpha_n) = \frac{1}{s+t}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(s\alpha_i + t\alpha_j\right)^{\frac{2}{n(n+1)}}, \tag{22}$$

then q-ROLGHM^{s,t} is called the q-rung orthopair linguistic geometric Heronian mean (q-ROLGHM) operator.

Similarly, the following theorem can be obtained according to Definition 5.

Theorem 8 *Let α_i $(i=1,2,\ldots,n)$ be a collection of q-ROLNs, then the aggregated value by using q-ROLGHM is also a q-ROLN and*

$$q\text{-}ROLGHM^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\Bigg\langle s_{\frac{1}{s+t}\prod\limits_{i=1}^{n}\prod\limits_{j=i}^{n}(s\theta_i+t\theta_j)^{\frac{2}{n(n+1)}}},$$
$$\left(\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-u_i^q)^s\left(1-u_j^q\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{1/q},\right.$$
$$\left.\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-v_i^{sq}v_j^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{(s+t)q}}\right)\Bigg\rangle. \tag{23}$$

The proof of Theorem 8 is similar to that of Theorem 1. In the following, we present some desirable properties of the q-ROLGHM operator.

Theorem 9 (Idempotency) *Let $\alpha_i=\langle s_{\theta_i},(u_i,v_i)\rangle$ $(i=1,2,\ldots,n)$ be a collection of q-ROLNs, if all the q-ROLNs are equal, i.e. $\alpha_i=\alpha$ for all i, then*

$$q\text{-}ROLGHM^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\alpha. \tag{24}$$

The proof of Theorem 9 is similar to that of Theorem 2.

Theorem 10 (Monotonicity) *Let α_i and β_i $(i=1,2,\ldots,n)$ be two collections of q-ROLNs, if $\alpha_i\leqslant\beta_i$ for all i, then*

$$q\text{-}ROLGHM^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n)\leqslant q\text{-}ROLGHM^{s,t}(\beta_1,\beta_2,\ldots,\beta_n). \tag{25}$$

The proof of Theorem 10 is similar to that of Theorem 3.

Theorem 11 (Boundedness) *Let $\alpha_i=\langle s_{\theta_i},(u_i,v_i)\rangle$ $(i=1,2,\ldots,n)$ be a collection of q-ROLNs, then*

$$\begin{aligned}\min(\alpha_1,\alpha_2,\ldots,\alpha_n)\leqslant q\text{-}ROLGHM^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\ \leqslant\max(\alpha_1,\alpha_2,\ldots,\alpha_n).\end{aligned} \tag{26}$$

The proof of Theorem 11 is similar to that of Theorem 4. In the following, we discuss some special cases of the q-ROLGHM operator.

Case 1 When $t \to 0$, then the q-ROLGHM operator reduces to the followings,

$$q\text{-ROLGHM}^{s,0}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \lim_{t\to 0}\left\langle s_{\frac{1}{s+t}\prod\limits_{i=1}^{n}\prod\limits_{j=i}^{n}\left(s\theta_i+t\theta_j\right)^{\frac{2}{n(n+1)}}}, \right.$$

$$\left(\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-u_i^q\right)^s\left(1-u_j^q\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{1/q},$$

$$\left.\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-v_i^{sq}v_j^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{(s+t)q}}\right)\right\rangle$$

$$= \left\langle s_{\frac{1}{s}\left(\prod\limits_{i=1}^{n}(s\theta_i)^{n+1-i}\right)^{\frac{2}{n(n+1)}}}, \left(\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-u_i^q\right)^s\right)^{n+1-i}\right)^{\frac{2}{n(n+1)}}\right)\right)^{\frac{1}{s}}\right)^{\frac{1}{q}}, \right.$$

$$\left.\left(1-\left(\prod_{i=1}^{n}\left(1-v_i^{sq}\right)^{n+1-i}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{sq}}\right)\right\rangle, \tag{27}$$

which is a q-rung orthopair linguistic generalized geometric linear descending weighted mean operator.

Case 2 When $s \to 0$, then the q-ROLGHM operator reduces to the followings,

$$q\text{-ROLGHM}^{0,t}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \lim_{s\to 0}\left\langle s_{\frac{1}{s+t}\prod\limits_{i=1}^{n}\prod\limits_{j=i}^{n}\left(s\theta_i+t\theta_j\right)^{\frac{2}{n(n+1)}}}, \right.$$

$$\left(\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-u_i^q\right)^s\left(1-u_j^q\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{1/q},$$

$$\left.\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-v_i^{sq}v_j^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{(s+t)q}}\right)\right\rangle$$

$$= \left\langle s_{\frac{1}{t}\left(\prod_{i=1}^{n}(t\theta_j)^i\right)^{\frac{2}{n(n+1)}}}, \left(\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-u_j^q\right)^t\right)^i\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{t}}\right)\right)^{1/q}, \right.$$
$$\left. \left(1-\left(\prod_{i=1}^{n}\left(1-v_j^{tq}\right)^i\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{tq}}\right)\right\rangle, \tag{28}$$

which is a q-rung orthopair linguistic generalized geometric linear ascending weighted mean operator.

Case 3 When $s = t = 1$, then the q-ROLGHM operator reduces to the followings,

$$q\text{-ROLGHM}^{1,1}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left\langle s_{\frac{1}{2}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(\theta_i+\theta_j\right)^{\frac{2}{n(n+1)}}}, \right.$$
$$\left(\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-u_i^q\right)\left(1-u_j^q\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{q}},$$
$$\left. \left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-v_i^q v_j^q\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{2q}}\right)\right\rangle, \tag{29}$$

which is a q-rung orthopair linguistic geometric line Heronian mean operator.

Case 4 When $s = t = 1/2$, then the q-ROLGHM operator reduces to the followings,

$$q\text{-ROLGHM}^{\frac{1}{2},\frac{1}{2}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left\langle s_{\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(\frac{1}{2}\theta_i+\frac{1}{2}\theta_j\right)\right)^{\frac{2}{n(n+1)}}}, \right.$$
$$\left(\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\sqrt{\left(1-u_i^q\right)\left(1-u_j^q\right)}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q}},$$
$$\left. \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\sqrt{v_i^q v_j^q}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q}}\right)\right\rangle, \tag{30}$$

which is a q-rung orthopair linguistic basic geometric Heronian mean operator.

Case 5 When $q = 2$, then the q-ROLGHM operator reduces to the followings

$$q\text{-ROLGHM}^{s,t}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left\langle s_{\frac{1}{s+t}\left(\prod\limits_{i=1}^{n}\prod\limits_{j=1}^{n}(s\theta_i + t\theta_j)\right)^{\frac{2}{n(n+1)}}}, \right.$$
$$\left(\left(1 - \left(1 - \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-u_i^2\right)^s\left(1-u_j^2\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)\right)^{1/2},$$
$$\left.\left(1 - \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1 - v_i^{2s}v_j^{2t}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{2(s+t)}}\right)\right\rangle, \qquad (31)$$

which is the Pythagorean linguistic geometric Heronian mean operator.

Case 6 When $q = 1$, then the q-ROLGHM operator reduces to the followings

$$q\text{-ROLGHM}^{s,t}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left\langle s_{\frac{1}{s+t}\left(\prod\limits_{i=1}^{n}\prod\limits_{j=1}^{n}(s\theta_i + t\theta_j)\right)^{\frac{2}{n(n+1)}}}, \right.$$
$$\left(1 - \left(1 - \left(\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-(1-u_i)^s(1-u_j)^t\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}},\right.$$
$$\left.\left(1 - \left(\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1 - v_i^s v_j^t\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)\right\rangle, \qquad (32)$$

which is the intuitionistic linguistic geometric Heronian mean operator.

3.4. The q-rung orthopair linguistic weighted geometric Heronian mean (q-ROLWGHM) operator

Definition 14 *Let α_i $(i = 1, 2, \ldots, n)$ be a collection of q-ROLNs, and s, $t > 0$, $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ be the weight vector, satisfying $w_i \in [0, 1]$ and $\sum\limits_{i=1}^{n} w_i = 1$. If*

$$q\text{-}ROLWGHM^{s,t}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{s+t}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(sa_i^{nw_i} + ta_j^{nw_j}\right)^{\frac{2}{n(n+1)}}, \qquad (33)$$

then q-ROLWGHM^{s,t} is called the q-ROLWGHM.

The following theorem can be easily obtained.

Theorem 12 *Let α_i $(i = 1,2,\ldots,n)$ be a collection of q-ROLNs, $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ be the weight vector, satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then the aggregated value by using q-ROLWGHM is also a q-ROLN and*

$$q\text{-}ROLWGHM^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left\langle s_{\frac{1}{s+t}\prod\limits_{i=1}^{n}\prod\limits_{j=i}^{n}\left(\theta_i^{nw_i}+t\theta_j^{nw_j}\right)^{\frac{2}{n(n+1)}}}, \right.$$

$$\left(\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-u_i^{nw_iq}\right)^s\left(1-u_j^{nw_jq}\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{1/q},\right.$$

$$\left.\left(1-\prod_{i=1}^{n}\prod_{i=i}^{n}\left(1-\left(1-(1-v_i^q)^{nw_i}\right)^s\left(1-(1-v_j^q)^{nw_j}\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{(s+t)q}}\right)\right\rangle. \tag{34}$$

The proof of Theorem 12 is similar to that of Theorem 1, which is omitted here.

Example 2 We utilize the values provided in Example 1 to demonstrate the performance of q-ROLWGHM operator. Let $\alpha = \langle s_\theta, (u, v)\rangle$ be the comprehensive value and suppose $q = 3$, $s = 2$, $t = 3$. If we utilize q-ROLWGHM operator to aggregate the four q-ROLNs, we can obtain

$$\alpha = \langle s_{4.4682}, (0.3495, 0.7843)\rangle, \qquad S(\alpha) = 2.5036.$$

The calculation process is similar to that of Example 1. Subsequently, we investigate the influence of the parameters s and t on the score function of the comprehensive value. Details can be found in Fig 2.

In addition, the q-ROLWGHM operator has the following properties.

Theorem 13 (Monotonicity) *Let α_i and β_i $(i = 1,2,\ldots,n)$ be two collections of q-ROLNs, if $\alpha_i \leqslant \beta_i$ for all i, then*

$$q\text{-}ROLWGHM^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n) \leqslant q\text{-}ROLWGHM^{s,t}(\beta_1,\beta_2,\ldots,\beta_n). \tag{35}$$

Theorem 14 (Boundedness) *The q-ROLWGHM operator lies between the max and min operators*

$$\begin{aligned}\min(\alpha_1,\alpha_2,\ldots,\alpha_n) \leqslant q\text{-}ROLWGHM^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\ \leqslant \max(\alpha_1,\alpha_2,\ldots,\alpha_n).\end{aligned} \tag{36}$$

Evidently, the q-ROLWGHM operator does not has the property of idempotency.

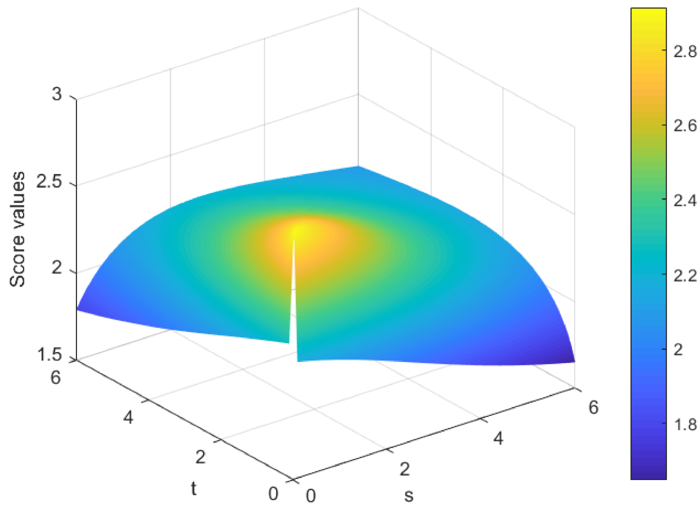

Figure 2: Score values of the alternative when $s,t \in [1,6]$ and $q=3$ using q-ROLWGHM operator

4. A novel approach to MAGDM based on the proposed operators

In this section, we shall propose a novel decision making method with q-rung orthopair linguistic information. Considering a MAGDM process under q-rung orthopair linguistic environment: let $X = \{x_1, x_2, \ldots, x_m\}$ be a set of all alternatives, and $Y = \{y_1, y_2, \ldots, y_n\}$ be a set of attributes with the weight vector being $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$, satisfying $\sum_{i=1}^{n} w_i = 1$. Several experts D_k are organized to make the assessment for every attribute y_j $(j = 1, 2, \ldots, n)$ of all alternatives by q-ROLNs $\alpha_{ij}^k = \left\langle s_{\theta_{ij}}^k, \left(u_{ij}^k, v_{ij}^k\right)\right\rangle$, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is the weight vector of decision makers $\{D_1, D_2, \ldots, D_p\}$. Therefore, the q-rung orthopair linguistic decision matrices can be obtained by $A^k = (\alpha_{ij}^k)_{m\times n}$. The main steps to solve the MAGDM problems based on the proposed operators are given as follows.

Step 1 Standardize the original decision matrices. There are two types of attributes, benefit and cost attributes. Therefore, the original decision matrix should be normalized by

$$\alpha_{ij}^k = \begin{cases} \left\langle s_{\theta_{ij}}^k, \left(u_{ij}^k, v_{ij}^k\right)\right\rangle & y_j \in I_1 \\ \left\langle s_{\theta_{ij}}^k, \left(v_{ij}^k, u_{ij}^k\right)\right\rangle & y_j \in I_2 \end{cases}, \tag{37}$$

where I_1 and I_2 represent the benefit attributes and cost attributes respectively.

Step 2 Utilize the q-ROLWHM operator

$$\alpha_{ij} = q\text{-ROLWHM}^{s,t}\left(\alpha_{ij}^{1}, \alpha_{ij}^{2}, \ldots, \alpha_{ij}^{p}\right), \tag{38}$$

or the q-ROLWGHM operator

$$\alpha_{ij} = q\text{-ROLWGHM}^{s,t}\left(\alpha_{ij}^{1}, \alpha_{ij}^{2}, \ldots, \alpha_{ij}^{p}\right), \tag{39}$$

to aggregate all the decision matrices A^k $(k = 1, 2, \ldots, p)$ into a collective decision matrix $A = (\alpha_{ij})_{m\times n}$.

Step 3 Utilize the q-ROLWHM operator

$$\alpha_{i} = q\text{-ROLWHM}^{s,t}(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}), \tag{40}$$

or the q-ROLWGHM operator

$$\alpha_{i} = q\text{-ROLWGHM}^{s,t}(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}), \tag{41}$$

to aggregate the assessments α_{ij} $(j = 1, 2, \ldots, n)$ for each A_i so that the overall preference values α_i $(i = 1, 2, \ldots, m)$ of alternatives can be obtained.

Step 4 Calculate the score functions of the overall values α_i $(i = 1, 2, \ldots, m)$.

Step 5 Rank all alternatives according to the score functions of the corresponding overall values and select the best one(s).

Step 6 End.

5. Numerical example

In this section, to verify the proposed method, we provide a numerical instance adopted from [36]. An investment company wants invest its money to a company. After primary evaluation, there are four possible companies remained on the candidates list and they are: (1) A_1 is a car company; (2) A_2 is a computer company; (3) A_3 is a TV company; (4) A_4 is a food company. Three experts are invited to evaluate the four candidates under four attributes, they are (1) C_1 is the risk analysis; (2) C_2 is the growth analysis; (3) $C_3$is the social-political impact analysis; (4) C_4 is the environmental impact analysis. Weight vector of the four attribute is $w = (0.32, 0.26, 0.18, 0.24)^{\mathrm{T}}$. The decision makers are required to use the linguistic term set $S = \{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{fair}, s_4 = \text{good}, s_5 = \text{very good}, s_6 = \text{extremely good}\}$to express their preference information. Decision makers' weight vector is $\lambda = (0.4, 0.32, 0.28)^{\mathrm{T}}$. After evaluation, the individual intuitionistic linguistic decision matrix $A^k = \left(\alpha_{ij}^{k}\right)_{4\times 4}$, $(k = 1, 2, 3)$ can be obtained, which are shown in Tables 1, 2 and 3.

Table 1: Intuitionistic linguistic decision matrix R^1

	C_1	C_2	C_3	C_4
A_1	$\langle s_5,(0.2,0.7)\rangle$	$\langle s_2,(0.4,0.6)\rangle$	$\langle s_5,(0.5,0.5)\rangle$	$\langle s_5,(0.2,0.6)\rangle$
A_2	$\langle s_4,(0.4,0.6)\rangle$	$\langle s_5,(0.4,0.5)\rangle$	$\langle s_3,(0.1,0.8)\rangle$	$\langle s_4,(0.5,0.5)\rangle$
A_3	$\langle s_3,(0.2,0.7)\rangle$	$\langle s_4,(0.2,0.7)\rangle$	$\langle s_4,(0.3,0.7)\rangle$	$\langle s_5,(0.2,0.7)\rangle$
A_4	$\langle s_6,(0.5,0.4)\rangle$	$\langle s_2,(0.2,0.8)\rangle$	$\langle s_3,(0.2,0.6)\rangle$	$\langle s_3,(0.3,0.6)\rangle$

Table 2: Intuitionistic linguistic decision matrix R^2

	C_1	C_2	C_3	C_4
A_1	$\langle s_4,(0.1,0.7)\rangle$	$\langle s_3,(0.2,0.7)\rangle$	$\langle s_3,(0.2,0.8)\rangle$	$\langle s_6,(0.4,0.5)\rangle$
A_2	$\langle s_5,(0.4,0.5)\rangle$	$\langle s_3,(0.3,0.6)\rangle$	$\langle s_4,(0.2,0.6)\rangle$	$\langle s_3,(0.2,0.7)\rangle$
A_3	$\langle s_4,(0.2,0.6)\rangle$	$\langle s_4,(0.2,0.7)\rangle$	$\langle s_2,(0.4,0.6)\rangle$	$\langle s_3,(0.3,0.7)\rangle$
A_4	$\langle s_5,(0.3,0.6)\rangle$	$\langle s_4,(0.4,0.5)\rangle$	$\langle s_2,(0.3,0.6)\rangle$	$\langle s_4,(0.2,0.6)\rangle$

Table 3: Intuitionistic linguistic decision matrix R^3

	C_1	C_2	C_3	C_4
A_1	$\langle s_5,(0.2,0.6)\rangle$	$\langle s_3,(0.3,0.7)\rangle$	$\langle s_4,(0.4,0.5)\rangle$	$\langle s_4,(0.2,0.7)\rangle$
A_2	$\langle s_4,(0.3,0.7)\rangle$	$\langle s_5,(0.3,0.6)\rangle$	$\langle s_2,(0.1,0.8)\rangle$	$\langle s_3,(0.4,0.6)\rangle$
A_3	$\langle s_4,(0.2,0.7)\rangle$	$\langle s_5,(0.3,0.6)\rangle$	$\langle s_1,(0.1,0.8)\rangle$	$\langle s_4,(0.2,0.7)\rangle$
A_4	$\langle s_3,(0.2,0.7)\rangle$	$\langle s_3,(0.1,0.7)\rangle$	$\langle s_4,(0.3,0.6)\rangle$	$\langle s_5,(0.4,0.5)\rangle$

5.1. The decision making process

Step 1 As the five attributes are benefit types, the original decision matrices do not need normalization.

Step 2 Utilize Eq. (38) to calculate the comprehensive value α_{ij} of each attribute for every alternative. The collective decision matrix $A = (\alpha_{ij})_{4\times4}$ is shown in Table 4 (suppose $s = t = 1$, $q = 3$).

Table 4: Collective intuitionistic linguistic decision matrix (by q-ROLWHM operator)

	C_1	C_2	C_3	C_4
A_1	$\langle s_{4.7038},(0.1820,0.6711)\rangle$	$\langle s_{2.6020},(0.3377,0.6650)\rangle$	$\langle s_{4.1372},(0.4235,0.5997)\rangle$	$\langle s_{5.0751},(0.3067,0.5992)\rangle$
A_2	$\langle s_{4.3333},(0.3796,0.5992)\rangle$	$\langle s_{4.4066},(0.3515,0.5672)\rangle$	$\langle s_{3.7082},(0.1533,0.7358)\rangle$	$\langle s_{3.4366},(0.4235,0.5999)\rangle$
A_3	$\langle s_{3.6013},(0.2002,0.6686)\rangle$	$\langle s_{4.2846},(0.2396,0.6711)\rangle$	$\langle s_{2.6550},(0.3237,0.6979)\rangle$	$\langle s_{4.1372},(0.2450,0.7011)\rangle$
A_4	$\langle s_{4.9334},(0.4084,0.5621)\rangle$	$\langle s_{2.9382},(0.3018,0.6743)\rangle$	$\langle s_{2.9832},(0.2703,0.6020)\rangle$	$\langle s_{3.8820},(0.3199,0.5710)\rangle$

Step 3 Utilize Eq. (40) to obtain the overall values of each alternative, we can get

$$\alpha_1 = \langle s_{4.1893}, (0.3227, 0.6398)\rangle, \qquad \alpha_2 = \langle s_{3.9518}, (0.3665, 0.6230)\rangle,$$
$$\alpha_3 = \langle s_{3.7683}, (0.2519, 0.6859)\rangle, \qquad \alpha_4 = \langle s_{3.8738}, (0.3476, 0.6064)\rangle.$$

Step 4 Compute the score functions of the overall values, which are shown as follows:

$$S(\alpha_1) = 3.2332, \quad S(\alpha_2) = 3.1908, \quad S(\alpha_3) = 2.6126, \quad S(\alpha_4) = 3.1730.$$

Step 5 Then the rank of the four alternatives is obtained

$$A_1 \succ A_2 \succ A_4 \succ A_3.$$

Therefore, the optimal alternative is A_1.

In step 2, if we utilize Eq. (39) to aggregate the assessments, then we can derive the following collective decision matrix in Table 5 (suppose $s = t = 1$, $q = 3$).

Table 5: Collective intuitionistic linguistic decision matrix (by q-ROLWGHM operator)

	C_1	C_2	C_3	C_4
A_1	$\langle s_{4.7225}, (0.1702, 0.6761)\rangle$	$\langle s_{2.5549}, (0.3041, 0.6655)\rangle$	$\langle s_{4.1010}, (0.3714, 0.6488)\rangle$	$\langle s_{5.0761}, (0.2712, 0.6105)\rangle$
A_2	$\langle s_{4.3324}, (0.3707, 0.6105)\rangle$	$\langle s_{4.3502}, (0.3373, 0.5656)\rangle$	$\langle s_{3.0030}, (0.1391, 0.7531)\rangle$	$\langle s_{3.4309}, (0.3714, 0.6093)\rangle$
A_3	$\langle s_{3.5690}, (0.2061, 0.6726)\rangle$	$\langle s_{4.2763}, (0.2394, 0.6761)\rangle$	$\langle s_{2.3569}, (0.2628, 0.7106)\rangle$	$\langle s_{4.1010}, (0.2400, 0.6998)\rangle$
A_4	$\langle s_{4.8762}, (0.3365, 0.5851)\rangle$	$\langle s_{2.8214}, (0.2278, 0.7082)\rangle$	$\langle s_{2.8973}, (0.2747, 0.6001)\rangle$	$\langle s_{3.7952}, (0.3021, 0.5771)\rangle$

Then we utilize Eq. (41) to obtain the following overall values of alternatives:

$$\alpha_1 = \langle s_{4.1301}, (0.2908, 0.6536)\rangle, \qquad \alpha_2 = \langle s_{3.9618}, (0.3086, 0.6351)\rangle,$$

$$\alpha_3 = \langle s_{3.6986}, (0.2492, 0.6873)\rangle, \qquad \alpha_4 = \langle s_{3.7798}, (0.2944, 0.6249)\rangle.$$

In addition, we calculate the score functions of the overall assessments and we can get

$$S(\alpha_1) = 3.0785, \quad S(\alpha_2) = 3.0636, \quad S(\alpha_3) = 2.5551, \quad S(\alpha_4) = 2.9538.$$

Therefore, the rank of the four alternatives is $A_1 \succ A_2 \succ A_4 \succ A_3$ and the best alternative is A_1.

5.2. The influence of the parameters on the ranking results

The parameters q, s and t pay a significant role in the final ranking results. In the following, we shall investigate the influence of the parameters on the overall assessments of alternatives and the final ranking results. First, we discuss the effects of the parameters q on the ranking results (suppose $s = t = 1$). Details can be found in Figs 3 and 4.

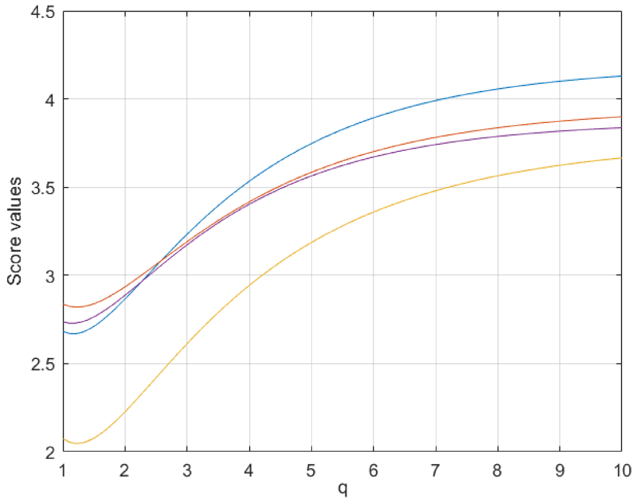

Figure 3: Score values of the alternatives when $q \in [1,10]$, $s = t = 1$ using q-ROLWHM operator

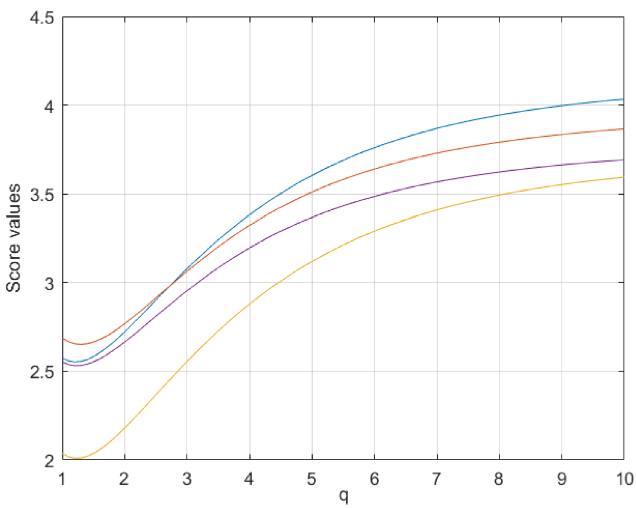

Figure 4: Score values of the alternatives when $q \in [1,10]$, $s = t = 1$ using q-ROLWGHM operator

The prominent feature of the q-ROFS is that the sum of membership and non-membership degrees is allowed to be greater than one with their q-th power of the membership degree and the q-th power of the degree of non-membership being equal to or less than 1. This feature makes q-ROFS more generalized and powerful than IFS and PFS. In addition, q-ROFS can describe and depict wider information range and contain more information than IFS and PFS. The proposed q-ROLS inherits the advantages of q-ROFS. In other word, q-ROLS can obtain more information than ILS and Pythagorean linguistic set (PLS). For instance, the argument $\langle s_5,(0.7,0.8)\rangle$ is not valid for intuitionistic linguistic numbers or Pythagorean linguistic numbers, whereas is it valid for q-ROLNs. Moreover, as seen in Figs 3 and 4, the score values of the overall assessments increase with the increase of the values of q and subsequently result in different ranking results. In addition, the value of q can be viewed as decision makers' attitude to optimism and pessimism. The more optimistic decision makers are, the greater value should be assigned to q, whereas the more pessimistic, the less value should be assigned to q.

In the followings, we investigate influence of the parameters s and t on the score functions and ranking orders respectively (suppose $q = 3$). Details can be found in Figs 5–8.

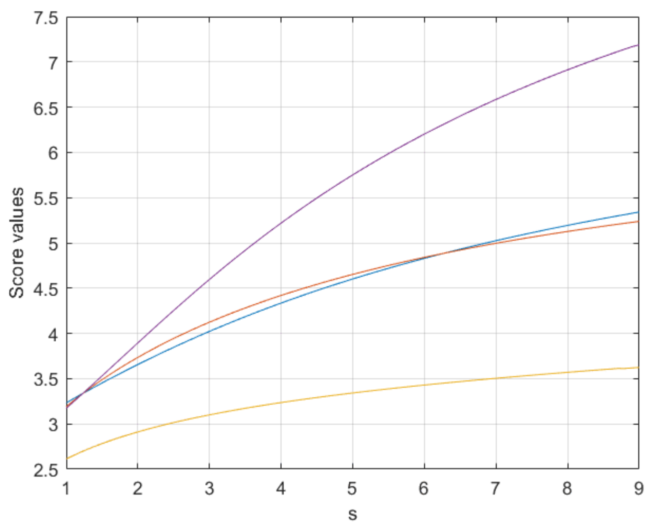

Figure 5: Score values of the alternatives when $s \in [1,9]$, $t = 1$, $q = 3$ using q-ROLWHM operator

As seen in Figs 5–8, different values are assigned to the parameters s and t, resulting in different score values and varying the ranking results. Especially, in q-ROLWHM operator, the increase of the parameters s and t leads to increase of

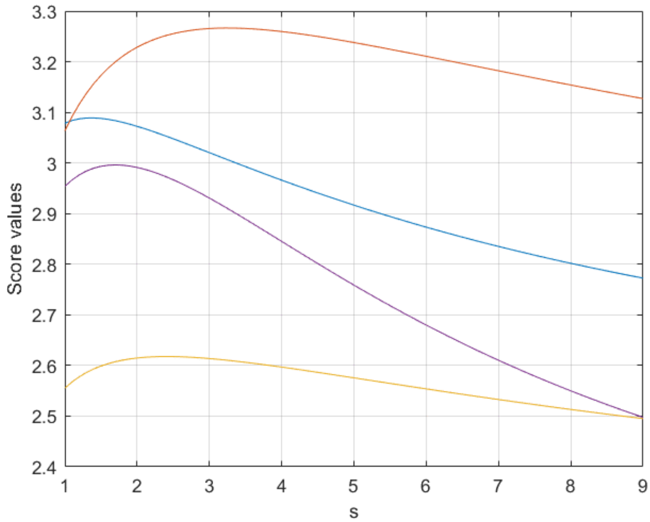

Figure 6: Score values of the alternatives when $s \in [1,9]$, $t = 1$, $q = 3$ using q-ROLWGHM operator

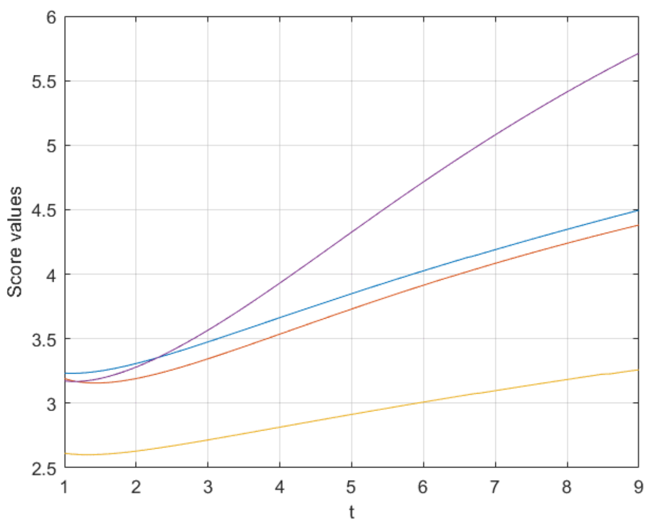

Figure 7: Score values of the alternatives when $t \in [1,9]$, $s = 1$, $q = 3$ using q-ROLWHM operator

the score functions, whereas decrease of the score functions using q-ROLWGHM operator. Therefore, the parameters s and t can be also viewed a decision makers' optimistic or pessimistic attitude to their assessments. This demonstrates the flexibility in the aggregation processes using the proposed operators.

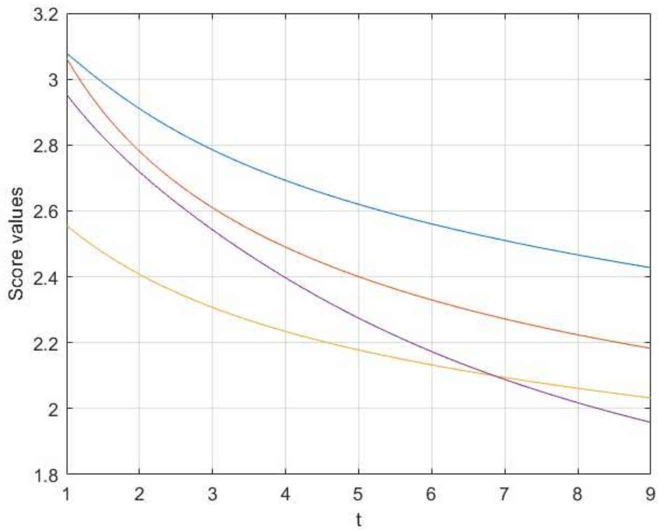

Figure 8: Score values of the alternatives when $t = [1,9]$, $s = 1$, $q = 3$ using q-ROLWGHM operator

5.3. Comparative analysis

In this section, we conduct some comparisons from a quantitative perspective. We utilize some exiting methods to solve the same example and compare their final ranking results. We compare our method with the method proposed by Ju et al. [37] based on weighted intuitionistic linguistic Maclaurin symmetric mean (WILMSM) operator, the method proposed by Wang et al. [38] based on intuitionistic linguistic hybrid (ILH) operator, the method proposed by Liu et al. [39] based on the intuitionistic fuzzy linguistic numbers hybrid geometric (IFLNHG) operator, the method proposed by Liu et al. [40] based on the intuitionistic linguistic weighted Bonferroni mean (ILWBM) operator, and the method proposed by Zhang et al. [41] based on the intuitionistic linguistic generalized weighted Heronian mean (ILGWHM) operator. The score functions and ranking results are shown in Table 6.

First of all, all the methods except our method are based on ILSs. As we mentioned before, ILS is only a special case of q-ROLS (when $q = 1$). Recently, Yager [27] proposed the concept of PFS and if we combine PFS with linguistic variables, we can obtain PLS, which is also a special case of q-ROLS (when $q = 2$). Therefore, our method is more generalized than the other methods.

Ju et al.'s [37] method is based on WILMSM operator and when $k = 2$, the interrelationship between any two arguments can be considered, which is the same as our proposed method. However, our method is based on the q-ROLWHM (or

Table 6: Score functions and ranking results using different methods

Method	Score functions	Ranking result
Ju et al.'s [37] method based on WILMSM operator ($k = 2$)	$S(\alpha_1) = 0.1784$ $S(\alpha_2) = 0.1976$ $S(\alpha_3) = 0.1407$ $S(\alpha_4) = 0.1892$	$A_2 \succ A_4 \succ A_1 \succ A_3$
Wang et al.'s [38] method based on ILH operator	$S(\alpha_1) = 0.1554$ $S(\alpha_2) = 0.1862$ $S(\alpha_3) = 0.1249$ $S(\alpha_4) = 0.1713$	$A_2 \succ A_4 \succ A_1 \succ A_3$
Liu et al.'s [39] method based on IFLNHG operator	$S(\alpha_1) = 0.1691$ $S(\alpha_2) = 0.1916$ $S(\alpha_3) = 0.1586$ $S(\alpha_4) = 0.1894$	$A_2 \succ A_4 \succ A_1 \succ A_3$
Liu et al.'s [40] method based on the ILWBM operator	$S(\alpha_1) = 2.4355$ $S(\alpha_2) = 2.5011$ $S(\alpha_3) = 1.8986$ $S(\alpha_4) = 2.3713$	$A_2 \succ A_1 \succ A_4 \succ A_3$
Zhang et al.'s [41] method based on the ILGWHM operator	$S(\alpha_1) = 2.5760$ $S(\alpha_2) = 2.6877$ $S(\alpha_3) = 2.0378$ $S(\alpha_4) = 2.5539$	$A_2 \succ A_1 \succ A_4 \succ A_3$
The proposed method based on q-ROLWHM operator in this paper	$S(\alpha_1) = 3.2332$ $S(\alpha_2) = 3.1908$ $S(\alpha_3) = 2.6126$ $S(\alpha_4) = 3.1730$	$A_1 \succ A_2 \succ A_4 \succ A_3$
The proposed method based on q-ROLWGHM operator in this paper	$S(\alpha_1) = 3.0785$ $S(\alpha_2) = 3.0636$ $S(\alpha_3) = 2.5551$ $S(\alpha_4) = 2.9538$	$A_1 \succ A_2 \succ A_4 \succ A_3$

q-ROLWGHM) operator, which has two parameters (s and t). The prominent of the method is that we can control the degree of the interactions of attribute values that are emphasized. The increase of values of the parameters (s and t) means the interactions of attribute values are more emphasized. Therefore, the decision making committee can properly select the desirable alternative according to their interests and the actual needs by determining the values of parameters. Moreover, in the WILMSM operator proposed by Ju et al. [37], the balancing coefficient n is not considered, leading to some unreasonable results. In our proposed operator, the coefficient n is considered so that our method is more reliable and reasonable.

Wang et al.'s [38] and Liu et al.'s [39] methods are based on hybrid averaging operator, which cannot consider the interrelationship among attribute values. In most real decision making problems, attributes are correlated so that the interrelationship among attributes should be taken into consideration. Therefore, our proposed method is more reasonable than Wang et al.'s [38] and Liu et al.'s [39] methods.

Liu et al.'s [40] and Zhang et al.'s methods [41] are based on BM and HM respectively, which can cope with the interrelationship between augments. However, as Yan and Wu [42] pointed out that HM has some advantages over BM, our method is better than Liu et al.'s [40] method. As Zhang et al.'s method [41] is based on ILS, which is a special case of q-ROLS $(q = 1)$, our proposed method is also better than Zhang et al.'s [41] method.

6. Conclusions

In this paper, we propose the q-ROLS which is a powerful and effective tool for coping with uncertainty and vagueness. Subsequently, we investigate MAGDM problems in q-rung orthopair linguistic environment. To aggregate q-ROLNs, we extend the HM to q-ROLSs and propose a family of q-rung orthopair linguistic Heronian mean operators, such as the q-ROLHM operator, the q-ROLWHM operator, the q-ROLGHM operator, and the q-ROLWGHM operator. The prominent characteristic of these proposed operators is that they can capture the interrelationship between q-ROLNs. Moreover, we have studied some desirable properties and special cases of the proposed operators. Thereafter, we utilize the proposed operators to establish a novel method to MAGDM problems. To illustrate the validity of the proposed method, we utilize the method to solve an investment project selection problem. In addition, we conduct some comparative analysis to demonstrate the effectiveness and superiorities of the proposed method. In the future, we will utilize the proposed method to solve some other practical decision-making problems.

List of symbols

A	a q-rung orthopair fuzzy set (q-ROFS)
$\widetilde{a}_i$	$(i = 1, 2, \ldots, n)$ a collection of q-rung orthopair fuzzy number (q-ROLN)
a_i	$(i = 1, 2, \ldots, n)$ a set of crisp numbers
$H(\widetilde{a})$	the accuracy function of $\widetilde{a}$
$i,\ j$	index of a control volume
q	power of q-rung orthopair fuzzy set
s	positive number
S	a linguistic term set
$\overline{S}$	a continuous linguistic term set of $S = \{s_i \mid i = 1, 2, \ldots, t\}$
s_i	a linguistic variable in the linguistic term set S
$s_{\theta(x)}$	linguistic variable of $A = \left\{\left\langle x, s_{\theta(x)}, (u_A(x), v_A(x))\right\rangle \mid x \in X\right\}$
s_θ	linguistic variable of the q-ROLN $\alpha = \langle s_\theta, (u, v)\rangle$
s_{θ_1}	linguistic variable of the q-ROLN $\alpha_1 = \left\langle s_{\theta_1}, (u_1, v_1)\right\rangle$
s_{θ_2}	linguistic variable of the q-ROLN $\alpha_2 = \left\langle s_{\theta_2}, (u_2, v_2)\right\rangle$
$S(\widetilde{a})$	the score function of $\widetilde{a}$
t	positive number
u	membership degree of q-ROLN $\alpha = \langle s_\theta, (u, v)\rangle$
$u_A(x)$	membership degree of q-ROFS $A = \{\langle x, u_A(x), v_A(x)\rangle \ \lvert x \in X\}$
u_1	membership degree of q-ROFN $\widetilde{a}_1 = (u_1, v_1)$
v	non-membership degree of q-ROLN $\alpha = \langle s_\theta, (u, v)\rangle$

$v_A(x)$ non-membership degree of q-ROFS $A = \{\langle x, u_A(x), v_A(x)\rangle \ |x \in X\}$
v_1 non-membership degree of q-ROFN $\widetilde{a}_1 = (u_1, v_1)$
w weight vector
x variable in fixed set X
X an ordinary fixed set
λ a positive real number
α, β a q-ROLN

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