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Absolute stability of a class of positive nonlinear continuous-time and discrete-time systems

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The positivity and absolute stability of a class of nonlinear continuous-time and discrete-time systems are addressed. Necessary and sufficient conditions for the positivity of this class of nonlinear systems are established. Sufficient conditions for the absolute stability of this class of nonlinear systems are also given.

Key words: absolute stability, positive, nonlinear, discrete-time, continuous-time, systems

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs and papers [1, 2, 6, 10, 11]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine.

The stability of linear and nonlinear standard and positive fractional systems has been addressed in [3–8, 14, 15, 19–22]. The stabilization of positive descriptor fractional systems has been investigated in [9, 18, 19, 20]. The superstable linear systems have been addressed in [16, 17]. Positive linear systems with different fractional orders have been introduced in [13, 12] and their stability has been analyzed in [3, 19].

In this paper the positivity and absolute stability of a class of nonlinear continuous-time and discrete-time systems will be investigated.

The paper is organized as follows. In section 2 some preliminaries concerning positivity and stability of linear systems are recalled. The positivity and absolute stability of positive continuous-time nonlinear systems is investigated in section 3 and of positive discrete-time nonlinear systems in section 4. Concluding remarks are given in section 5.

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The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n \times m$ real matrices, $\Re_{+}^{n\times m}$ – the set of $n \times m$ real matrices with nonnegative entries and $\Re_{+}^{n} = \Re_{+}^{n\times 1}$, M_n – the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n \times n$ identity matrix, A^T – the transpose of matrix A.

2. Preliminaries

Consider the continuous-time linear system

$$\dot{x} = Ax + Bu, \tag{1a}$$

$$y = Cx, \tag{1b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Definition 1 *[6, 11] The continuous-time linear system (1) is called (internally) positive if $x(t) \in \Re_{+}^{n}$, $y(t) \in \Re_{+}^{p}$, $t \geqslant 0$ for any initial conditions $x(0) \in \Re_{+}^{n}$ and all inputs $u(t) \in \Re_{+}^{m}$, $t \geqslant 0$.*

Theorem 1 *[6, 11] The continuous-time linear system (1) is positive if and only if*

$$A \in M_n, \quad B \in \Re_{+}^{n\times m}, \quad C \in \Re_{+}^{p\times n}. \tag{2}$$

Definition 2 *[6, 11] The positive continuous-time system (1) for $u(t) = 0$ is called asymptotically stable if*

$$\lim_{t\to\infty} x(t) = 0 \quad \textit{for any} \;\; x(0) \in \Re_{+}^{n}. \tag{3}$$

Theorem 2 *[6, 11] The positive continuous-time linear system (1) for $u(t) = 0$ is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

1. *All coefficient of the characteristic polynomial*

$$p_n(s) = \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{4}$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, \ldots, n-1$.

2. *There exists strictly positive vector $\lambda^T = [\lambda_1 \;\; \cdots \;\; \lambda_n]^T$, $\lambda_k > 0$, $k = 1, \ldots, n$ such that*

$$A\lambda < 0 \qquad or \qquad \lambda^T A < 0. \tag{5}$$

If the matrix A is nonsingular then we can choose $\lambda = A^{-1}c$, where $c \in \Re^n$ is strictly positive.

Consider the discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i, \qquad i \in Z_+ = \{0, 1, \dots\}, \tag{6a}$$

$$y_i = Cx_i, \tag{6b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state and input vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Definition 3 *[6, 11] The discrete-time linear system (6) is called (internally) positive if $x_i \in \Re_+^n$, $y_i \in \Re_+^p$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.*

Theorem 3 *[6, 11] The discrete-time linear system (6) is positive if and only if*

$$A \in \Re_+^{n\times n}, \qquad B \in \Re_+^{n\times m}, \qquad C \in \Re_+^{p\times n}. \tag{7}$$

Definition 4 *[6, 11] The positive discrete-time system (6) for $u_i = 0$ is called asymptotically stable if*

$$\lim_{i\to\infty} x_i = 0 \quad \text{for any} \;\; x_0 \in \Re_+^n. \tag{8}$$

Theorem 4 *[6, 11] The positive discrete-time linear system (6) for $u_i = 0$ is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

1. *All coefficient of the characteristic polynomial*

$$p_n(z) = \det[I_n(z+1) - A] = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0 \tag{9}$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, \dots, n-1$.

2. *There exists strictly positive vector $\lambda^T = [\lambda_1 \;\; \cdots \;\; \lambda_n]^T$, $\lambda_k > 0$, $k = 1, \dots, n$ such that*

$$(A - I_n)\lambda < 0 \qquad \text{or} \qquad \lambda^T(A^T - I_n) < 0. \tag{10}$$

If the matrix $(A - I_n)$ is nonsingular then we can choose $\lambda = (A - I_n)^{-1}c$, where $c \in \Re^n$ is strictly positive.

3. Absolute stability of positive continuous-time nonlinear systems

Consider the nonlinear continuous-time system shown in Fig. 1 and described by the equations

$$\dot{x} = Ax + Bu, \qquad u = f(e), \tag{11a}$$

$$y = Cx, \tag{11b}$$

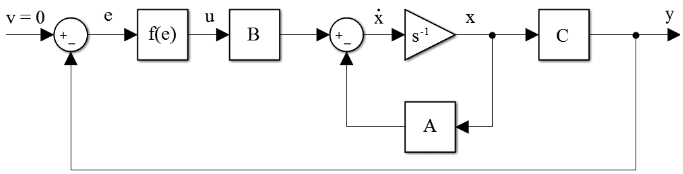

Figure 1: Nonlinear system

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors of the system $A \in \Re^{n\times n}$, $B \in \Re^{n\times 1}$, $C \in \Re^{1\times n}$ and the characteristic $f(e)$ of the nonlinear element (Fig. 2) satisfies the condition

$$0 < f(e) < ke, \qquad 0 < k < \infty. \tag{12}$$

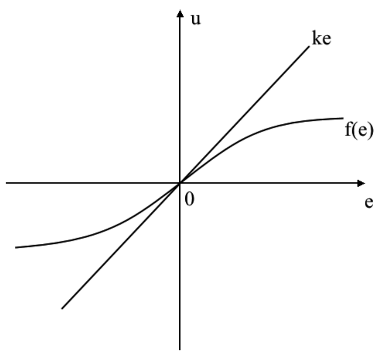

Figure 2: Characteristic of nonlinear element

Definition 5 *The nonlinear system (11) is called (internally) positive if* $x(t) \in \Re_+^n$, $y(t) \in \Re_+^p$, $t \geqslant 0$ *for any initial conditions* $x(0) \in \Re_+^n$ *and all inputs* $u(t) \in \Re_+$, $t \geqslant 0$.

Theorem 5 *The nonlinear system (11) is positive if and only if*

$$A \in M_n, \qquad B \in \Re_+^{n\times 1}, \qquad C \in \Re_+^{1\times n} \tag{13a}$$

and

$$f(e) \geqslant 0 \ \ \textit{for} \ \ e \geqslant 0 \qquad \textit{and} \qquad f(-e) < 0 \ \ \textit{for} \ \ -e < 0. \tag{13b}$$

Proof. It is well-known [11] that if $u = f(e) \in \Re_+$, $t \geqslant 0$ then $x(t) \in \Re_+^n$, $t \geqslant 0$ for $x(0) \in \Re_+^n$ if and only if $A \in M_n$ and $B \in \Re_+^{n\times 1}$. From (11b) for $t = 0$ we have $y(0) = Cx(0) \in \Re_+$ for $x(0) \in \Re_+^n$ if and only if $C \in \Re_+^{1\times n}$. □

Definition 6 *The positive nonlinear system (11) is called absolutely stable if* $x(t) \in \Re_+^n$, $t \geqslant 0$ *and*

$$\lim_{t\to\infty} x(t) = 0 \quad \textit{for any} \;\; x(0) \in \Re_+^n . \tag{14}$$

The Metzler matrix $A \in M_n$ *is called Hurwitz Metzler matrix if its all eigenvalues* λ_k *satisfy the condition* $\mathrm{Re}\lambda_k < 0$, $k = 1, \ldots, n$.

Theorem 6 *The positive nonlinear system (11) is absolutely stable if:*

1. $A \in M_n$ *is the Hurwitz Metzler matrix,*

$$B \in \Re_+^{n\times 1}, \qquad C \in \Re_+^{1\times n} . \tag{15}$$

2. *The nonlinear characteristic* $f(e)$ *satisfy the condition (12).*

Proof. Proof is based on the Lyapunov method for positive systems. As a candidate of the Lyapunov function it is assumed the linear function of the state vector $x(t) \in \Re_+^n$, $t \geqslant 0$

$$V(x(t)) = \lambda^T x(t), \qquad \lambda^T = [\lambda_1 \quad \ldots \quad \lambda_n], \qquad \lambda_k > 0, \quad k = 1, \ldots, n. \tag{16}$$

Using (16) and (11a) we obtain

$$\dot{V}(x) = \lambda^T \dot{x}(t) = \lambda^T [Ax(t) + Bf(e)] < 0 \tag{17}$$

since by (15) and (5)

$$\lambda^T A < 0 \quad \text{and} \quad f(-e) < 0 \quad \text{for} \;\; -e < 0 \quad \text{and} \quad t \geqslant 0. \tag{18}$$

Therefore, the positive nonlinear system (11) is absolutely stable if the conditions 1) and 2) of Theorem 6 are satisfied. □

Remark 1 *The absolute stability of the positive nonlinear system is directly independent of the transfer function of its linear part (and also of its frequency characteristics).*

Example 1. Consider the nonlinear system (11) with

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \tag{19a}$$

for the following two cases:

Case 1.

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_1 = [1 \quad 0]. \tag{19b}$$

Case 2.

$$B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_2 = [0 \quad 1]. \tag{19c}$$

In both cases the nonlinear system is positive since the matrix (19a) is Hurwitz Metzler matrix and its characteristic polynomial

$$\det[I_2 s - A] = \begin{vmatrix} s+1 & -1 \\ -1 & s+2 \end{vmatrix} = s^2 + 3s + 1 \tag{20}$$

has positive coefficients.

By Theorem 6 the nonlinear system with (19) is absolutely stable for all nonlinear element with the characteristic $f(e)$ satisfying (12).

In the Case 1 the transfer function of the linear part has the form

$$T_1(s) = C_1[I_2 s - A]^{-1} B_1 = [1 \quad 0] \begin{bmatrix} s+1 & -1 \\ -1 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2 + 3s + 1} \tag{21}$$

and in Case 2

$$T_2(s) = C_2[I_2 s - A]^{-1} B_2 = [0 \quad 1] \begin{bmatrix} s+1 & -1 \\ -1 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{s+1}{s^2 + 3s + 1}. \tag{22}$$

Substitution $s = j\omega$ in (21) and (22) yields

$$\begin{gathered} T_1(j\omega) = \frac{1}{1 - \omega^2 + j3\omega} = P_1(\omega) + jQ_1(\omega), \\ P_1(\omega) = \frac{1 - \omega^2}{1 + 7\omega^2 + \omega^4}, \qquad Q_1(\omega) = \frac{-3\omega}{1 + 7\omega^2 + \omega^4} \end{gathered} \tag{23}$$

and

$$\begin{gathered} T_2(j\omega) = \frac{1 + j\omega}{1 - \omega^2 + j3\omega} = P_2(\omega) + jQ_2(\omega), \\ P_2(\omega) = \frac{1 + 2\omega^2}{1 + 7\omega^2 + \omega^4}, \qquad Q_2(\omega) = -\frac{2\omega + \omega^3}{1 + 7\omega^2 + \omega^4}. \end{gathered} \tag{24}$$

The frequency characteristics for Case 1 and Case 2 are shown in Fig. 3.

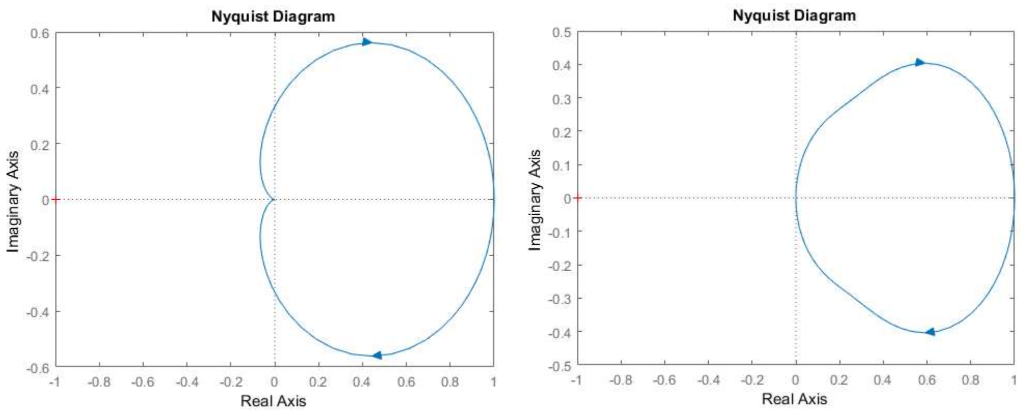

Figure 3: The frequency characteristics

4. Absolute stability of positive discrete-time nonlinear systems

Consider the nonlinear discrete-time system shown in Fig. 4 and described by the equations

$$x_{i+1} = Ax_i + Bu_i, \quad u_i = f(e_i), \quad i \in Z_+ = \{0, 1, \ldots\}, \tag{25a}$$

$$y_i = Cx_i, \tag{25b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors of the system $A \in \Re^{n\times n}$, $B \in \Re^{n\times 1}$, $C \in \Re^{1\times n}$ and the characteristic $f(e_i)$ of the nonlinear element (Fig. 5) satisfies the condition

$$0 < f(e_i) < ke_i, \qquad 0 < k < \infty. \tag{26}$$

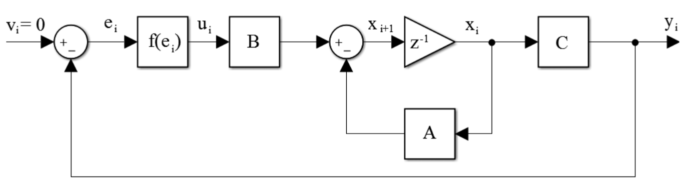

Figure 4: Nonlinear system

Definition 7 *The nonlinear system (25) is called (internally) positive if $x_i \in \Re^n_+$, $y_i \in \Re^p_+$, $i \in Z_+$ for any initial conditions $x_0 \in \Re^n_+$ and all inputs $u_i \in \Re_+$, $i \in Z_+$.*

Theorem 7 *The nonlinear system (25) is positive if and only if*

$$A \in \Re^{n\times n}_+, \qquad B \in \Re^{n\times 1}_+, \qquad C \in \Re^{1\times n}_+ \tag{27a}$$

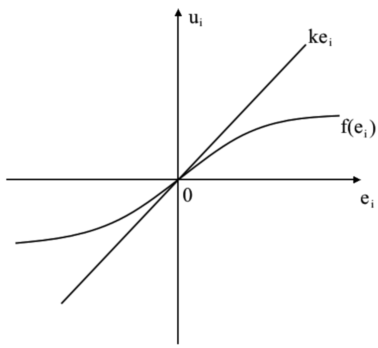

Figure 5: Characteristic of nonlinear element

and

$$f(e_i) \geqslant 0 \quad \textit{for} \quad e_i \geqslant 0 \qquad \textit{and} \qquad f(-e_i) < 0 \;\; \textit{for} \;\; -e_i < 0, \;\; i \in Z_+. \tag{27b}$$

Proof. It is well-known [11] that if $u_i = f(e_i) \in \Re_+$, $i \geqslant 0$ then $x_i \in \Re_+^n$, $i \in Z_+$ for $x_i \in \Re_+^n$ if and only if $A \in \Re_+^{n\times n}$ and $B \in \Re_+^{n\times 1}$. From (25b) for $i = 0$ we have $y_0 = Cx_0 \in \Re_+$ for $x_0 \in \Re_+^n$ if and only if $C \in \Re_+^{1\times n}$. □

Definition 8 *The positive nonlinear system (25) is called absolutely stable if $x_i \in \Re_+^n$, $i \in Z_+$ and*

$$\lim_{i\to\infty} x_i = 0 \qquad \textit{for any} \quad x_0 \in \Re_+^n. \tag{28}$$

The matrix $A \in \Re_+^{n\times n}$ is called Schur matrix if its all eigenvalues z_i satisfy the condition

$$|z_i| < 1, \qquad i = 1, \ldots, n. \tag{29}$$

Theorem 8 *The positive nonlinear system (25) is absolutely stable if:*

1. *$A \in \Re_+^{n\times n}$ is a Schur matrix,*

$$B \in \Re_+^{n\times 1}, \qquad C \in \Re_+^{1\times n}. \tag{30}$$

2. *The nonlinear characteristic $f(e_i)$ satisfies the condition (26).*

Proof. Proof is based on the Lyapunov method for positive discrete-time systems. As a candidate of the Lyapunov function it is assumed the linear function of the state vector $x_i \in \Re_+^n$, $i \in Z_+$

$$V(x_i) = \lambda^T x_i, \qquad \lambda^T = [\lambda_1 \quad \ldots \quad \lambda_n], \qquad \lambda_k > 0, \quad k = 1, \ldots, n. \tag{31}$$

Using (31) and (25a) we obtain

$$\Delta V(x_i) = V(x_{i+1}) - V(x_i) = \lambda^T(x_{i+1} - x_i) = \lambda^T(A - I_n)x_i + Bf(e_i) < 0 \quad (32)$$

since by (30) and (10)

$$\lambda^T(A - I_n) < 0 \quad \text{and} \quad f(-e_i) < 0 \quad \text{for } i \in Z_+ . \quad (33)$$

Therefore, the positive nonlinear system (25) is absolutely stable if the conditions 1. and 2. of Theorem 8 are satisfied. □

Remark 2 *The absolute stability of the positive nonlinear system is directly independent of the transfer function of its linear part.*

Example 2. Consider the positive nonlinear system (25) with

$$A = \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0 & 0.6 & 0.2 \\ 0.1 & 0 & 0.5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad C = [0 \quad 0 \quad 3] \quad (34)$$

and the characteristic of the nonlinear element satisfying the condition (4.3b). The matrix $A \in \Re_+^{3\times 3}$ is Schur matrix since its characteristic polynomial (9) of the form

$$\det[I_3(z+1) - A] = \begin{vmatrix} z+0.8 & -0.1 & -0.3 \\ 0 & z+0.4 & -0.2 \\ -0.1 & 0 & z+0.5 \end{vmatrix} = z^3 + 1.7z^2 + 0.89z + 0.146 \quad (35)$$

has positive coefficients.

The same result can be obtained by the use of the condition (10) since for $\lambda^T = [0.5 \quad 1 \quad 1]$ and

$$A - I_3 = \begin{bmatrix} -0.8 & 0.1 & 0.3 \\ 0 & -0.4 & 0.2 \\ 0.1 & 0 & -0.5 \end{bmatrix} \quad (36)$$

we have

$$\lambda^T(A - I_3) = \begin{bmatrix} -0.3 \\ -0.35 \\ -0.15 \end{bmatrix} < 0. \quad (37)$$

If the characteristic $f(e_i)$ of nonlinear element satisfies the condition (27b) then by Theorem 8 the nonlinear system is absolutely stable.

5. Concluding remarks

The positivity and absolute stability of a class of nonlinear continuous-time and discrete-time systems have been addressed. Necessary and sufficient conditions for the positivity of the nonlinear systems have been established (Theorems 5 and 7). Sufficient conditions for the absolute stability of the nonlinear systems have been also obtained (Theorems 6 and 8). The considerations have been illustrated by numerical examples. The presented results can be extended to multi-inputs multi-outputs nonlinear systems. The considerations can be also extended to fractional nonlinear systems with the same fractional order and with different fractional orders.

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