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An LMI approach to checking stability of 2D positive systems

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Abstract. Two-dimensional (2D) positive systems are 2D state-space models whose state, input and output variables take only nonnegative values. In the paper we explore how linear matrix inequalities (LMIs) can be used to address the stability problem for 2D positive systems. Necessary and sufficient conditions for the stability of positive systems have been provided. The results have been obtained for most popular models of 2D positive systems, that is: Roesser model, both Fornasini-Marchesini models (FF-MM and SF-MM) and for the general model.

1. Introduction

The main distinguishing feature of positive systems is that for nonnegative initial conditions their state variables and outputs assume nonnegative values, provided the inputs are nonnegative [1–4]. A number of quantities, such as for example pressure, sugar concentration in blood, population levels, etc., take only nonnegative values, hence positive systems are frequently encountered in engineering [5–8], medicine and biology [9–15], economics etc. The stability is a crucial feature when we consider dynamic systems of any kind; the positive systems apply to this rule as well. The stability problem for positive system has been considered in many papers, for example, [16–23]. The well known Lyapunov result on the stability of linear systems can be perceived as the beginning of the long history of application of LMIs to the stability checking. For more details on that one is referred to [24]. The LMI framework has been successfully applied for checking stability of positive systems [24,25]. In [26] duality aspects of semidefinite programming are presented as well as the role they play in control theory. The duality results derived from optimization theory presented in [26] provide us with better insight into some problems of control theory. It turns out that some of ideas from [26] may be extended to study the positive systems [27]. In [27] the problems of positive systems stability are addressed by means of LMIs, in particular, alternative formulations of stability criteria are proposed. In this paper, the results from [27] are extended to the 2D positive systems. The most popular models of two dimensional (2D) systems are models introduced by Roesser [28], Fornasini and Merchesini [29,30] and Kurek [31]. The positive 2D Roesser type model has been introduced in [32]. More developments in 2D positive systems theory can be found in [4], [33–37]. 2D positive systems models facilitate better understanding of phenomena whose desription involves two independent variables, for instance river pollution and self-purification process [38], gas absorption, water stream heating, etc.

The paper is organized as follows. In section II basic definitions and lemmas concerning the linear matrix inequalities and positive 2D linear systems are given. Section III, which contains the main results of the paper, studies the asymptotic stability of positive 2D linear systems (for Roesser model, 2D general model, the first and the second Fornasini-Marchesini models, respectively), in particular, the necessary and sufficient conditions in terms of LMI for the asymptotic stability.

All numerical examples provided in the paper have been solved using MATLAB®environment together with SEDUMI®solver and YALMIP®parser. More details on the computational aspects can be found in [39–42].

2. 2D positive systems

2.1. Preliminaries. Let us denote by $\boldsymbol{R}^{m\times n}$ ($\boldsymbol{C}^{m\times n}$) the set of real (complex) matrices with m rows and n columns. Also let $\boldsymbol{R}^m := \boldsymbol{R}^{m\times 1}$ ($\boldsymbol{C}^m := \boldsymbol{C}^{m\times 1}$).

Definition 1. [4] A matrix $A = [a_{ij}] \in \boldsymbol{R}^{n\times m}$ is called nonnegative if $a_{ij} \geq 0$ for $i = 1,\ldots,n$, $j = 1,\ldots,m$.
The set of nonnegative $n\times m$ matrices will be denoted $\boldsymbol{R}_+^{n\times m}$. For the nonnegative matrix A we write $A \geq 0$. Let us note, that nonnegative matrix $A \in \boldsymbol{R}^{n\times m}$ may have all entries equal to zero.

Definition 2. [4] A matrix $A = [a_{ij}] \in \boldsymbol{R}^{n\times m}$ is called positive if $a_{ij} \geq 0$ for $i = 1,\ldots,n$, $j = 1,\ldots,m$, and $a_{ij} > 0$ for at least one pair (i,j);
For the positive matrix A we write $A > 0$.

Definition 3. [4] A matrix $A = [a_{ij}] \in \boldsymbol{R}^{n\times m}$ is called strictly positive if $a_{ij} > 0$ for $i = 1,\ldots,n$, $j = 1,\ldots,m$.
The set of strictly positive $n \times m$ matrices will be denoted $\boldsymbol{R}_{++}^{n\times m}$. For the strictly positive matrix A we write $A \gg 0$.

Definition 4. [4] The matrix $A = [a_{ij}] \in \boldsymbol{R}^{n\times n}$ is called a Metzler matrix if its all off-diagonal entries are nonnegative, i.e., $a_{ij} \geq 0$ for $i \neq j$, $i,j = 1,2,\ldots,n$.
The set of all $n\times n$ Metzler matrices will be denoted by $\boldsymbol{M}^n$.

Definition 5. The matrix $A = [a_{ij}] \in \boldsymbol{R}^{n\times n}$ is called a Hurwitz matrix if it has all eigenvalues with negative real part, i.e., $\sigma(A) \subset \boldsymbol{C}^-$, where $\sigma(\cdot)$ denotes the spectrum of the matrix, and $\boldsymbol{C}^-$ denotes the left open halfplane of the complex plane.

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Definition 6. The matrix $A = [a_{ij}] \in \boldsymbol{R}^{n\times n}$ is called a Schur matrix if it has all eigenvalues with modulii less then one, i.e. $|\lambda_i| < 1$ for $i = 1, 2, \ldots, n$, where λ_i $i = 1, 2, \ldots, n$ are the eigenvalues of A.

Lemma 1. Let $P = [p_{ij}] \in \boldsymbol{R}^{n\times n}$ and $Q = [q_{ij}] \in \boldsymbol{C}^{n\times n}$ be a complex matrix such that $|Q| := [|q_{ij}|] \leq P$. Than

$$\rho(Q) \leq \rho(P),$$

where ρ denotes the spectral radius of a matrix[1].

Proof. See, e.g., [43,44].

Lemma 2. If $A \in \boldsymbol{R}_+^{n\times n}$ is a nonnegative matrix, then $\rho(A)$ is an eigenvalue of A and there is a positive vector $x > 0$, such that $Ax = \rho(A)x$.

Proof. See, e.g., [45].

2.2. Internally positive Roesser model.

The set of integers is denoted $\boldsymbol{Z}$. The set of nonnegative integers is denoted $\boldsymbol{Z}_+$. The 2D Roesser model is a 2D system of the following form [4,28,46]

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{i,j}, \quad (1a)$$

$$y_{i,j} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix} + Du_{i,j}, \quad i, j \in \boldsymbol{Z}_+, \quad (1b)$$

where $x^h_{i,j} \in \boldsymbol{R}^{n_1}$ and $x^v_{i,j} \in \boldsymbol{R}^{n_2}$ are the horizontal and vertical state vectors at the point $(i, j) \in \boldsymbol{Z}_+ \times \boldsymbol{Z}_+$, respectively, $u_{i,j} \in \boldsymbol{R}^m$ and $y_{i,j} \in \boldsymbol{R}^p$ are the input and output vectors, respectively, and $A_{11} \in \boldsymbol{R}^{n_1\times n_1}$, $A_{12} \in \boldsymbol{R}^{n_1\times n_2}$, $A_{21} \in \boldsymbol{R}^{n_2\times n_1}$, $A_{22} \in \boldsymbol{R}^{n_2\times n_2}$, $B_1 \in \boldsymbol{R}^{n_1\times m}$, $B_2 \in \boldsymbol{R}^{n_2\times m}$, $C_1 \in \boldsymbol{R}^{p\times n_2}$, $C_2 \in \boldsymbol{R}^{p\times n_2}$, $D \in \boldsymbol{R}^{p\times m}$, with the following boundary conditions

$$x^h_{0,j} \in \boldsymbol{R}^{n_1}, \text{ for } j \in \boldsymbol{Z}_+ \text{ and } x^v_{i,0} \in \boldsymbol{R}^{n_2}, \text{ for } i \in \boldsymbol{Z}_+.$$

Definition 7. The model given by (1a)–(1b) is said to be a 2D internally positive Roesser model if for any nonnegative boundary conditions

$$x^h_{0,j} \in \boldsymbol{R}_+^{n_1}, \text{ for } j \in \boldsymbol{Z}_+ \text{ and } x^v_{i,0} \in \boldsymbol{R}_+^{n_2}, \text{ for } i \in \boldsymbol{Z}_+ \quad (2)$$

and arbitrary nonnegative inputs $u_{i,j} \in \boldsymbol{R}_+^m$, $i, j \in \boldsymbol{Z}_+$, we have

$$x_{i,j} = \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix} \in \boldsymbol{R}_+^n, \;\; n = n_1 + n_2, \;\; y_{i,j} \in \boldsymbol{R}_+^p \;\; \forall i, j \in \boldsymbol{Z}_+.$$

Lemma 3. The model given by (1a)–(1b) is an internally positive Roesser model if and only if $A_{11} \in \boldsymbol{R}_+^{n_1\times n_1}$, $A_{12} \in \boldsymbol{R}_+^{n_1\times n_2}$, $A_{21} \in \boldsymbol{R}_+^{n_2\times n_1}$, $A_{22} \in \boldsymbol{R}_+^{n_2\times n_2}$, $B_1 \in \boldsymbol{R}_+^{n_1\times m}$, $B_2 \in \boldsymbol{R}_+^{n_2\times m}$, $C_1 \in \boldsymbol{R}_+^{p\times n_2}$, $C_2 \in \boldsymbol{R}_+^{p\times n_2}$, $D \in \boldsymbol{R}_+^{p\times m}$.

Proof. See [4].

2.3. Internally positive general model.

The 2D general model is a 2D system of the following form [4,31,47]

$$\begin{aligned} x_{i+1,j+1} &= A_0 x_{i,j} + A_1 x_{i+1,j} + A_2 x_{i,j+1} \\ &\quad + B_0 u_{i,j} + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \end{aligned} \quad (3a)$$

$$y_{i,j} = Cx_{i,j} + Du_{i,j}, \quad i, j \in \boldsymbol{Z}_+, \quad (3b)$$

where $x_{i,j} \in \boldsymbol{R}^n$ is the state vector at the point $(i, j) \in \boldsymbol{Z}_+ \times \boldsymbol{Z}_+$, $u_{ij} \in \boldsymbol{R}^m$ and $y_{ij} \in \boldsymbol{R}^p$ are the input and output vectors, respectively, and $A_0 \in \boldsymbol{R}^{n\times n}$, $A_1 \in \boldsymbol{R}^{n\times n}$, $A_2 \in \boldsymbol{R}^{n\times n}$, $B_0 \in \boldsymbol{R}^{n\times m}$, $B_1 \in \boldsymbol{R}^{n\times m}$, $B_2 \in \boldsymbol{R}^{n\times m}$, $C \in \boldsymbol{R}^{p\times n}$, $D \in \boldsymbol{R}^{p\times m}$, with the following boundary conditions

$$x_{i,0} \in \boldsymbol{R}^n, \text{ for } i \in \boldsymbol{Z}_+ \text{ and } x_{0,j} \in \boldsymbol{R}^n, \text{ for } j \in \boldsymbol{Z}_+. \quad (4)$$

Definition 8. The model given by (3a)–(3b) is said to be a 2D internally positive general model if for any nonnegative boundary conditions

$$x_{i,0} \in \boldsymbol{R}_+^n, \text{ for } i \in \boldsymbol{Z}_+ \text{ and } x_{0,j} \in \boldsymbol{R}_+^n, \text{ for } j \in \boldsymbol{Z}_+ \quad (5)$$

and arbitrary nonnegative inputs $u_{i,j} \in \boldsymbol{R}_+^m$, $i, j \in \boldsymbol{Z}_+$, we have

$$x_{i,j} \in \boldsymbol{R}_+^n, \text{ and } y_{i,j} \in \boldsymbol{R}_+^p \text{ for all } i, j \in \boldsymbol{Z}_+.$$

Lemma 4. The model given by (3a)–(3b) is an internally positive general model if and only if $A_0 \in \boldsymbol{R}_+^{n\times n}$, $A_1 \in \boldsymbol{R}_+^{n\times n}$, $A_2 \in \boldsymbol{R}_+^{n\times n}$, $B_0 \in \boldsymbol{R}_+^{n\times m}$, $B_1 \in \boldsymbol{R}_+^{n\times m}$, $B_2 \in \boldsymbol{R}_+^{n\times m}$, $C \in \boldsymbol{R}_+^{p\times n}$, $D \in \boldsymbol{R}_+^{p\times m}$.

Proof. See [4].

Let us consider the autonomous general model (3), i.e.,

$$x_{i+1,j+1} = A_0 x_{i,j} + A_1 x_{i+1,j} + A_2 x_{i,j+1}, \quad (6)$$

where the vector $x_{i,j}$ and the matrices A_0, A_1, A_2 are defined as for (3a).

Definition 9. [4] A 2D positive system described by (3a)–(3a) is called asymptotically stable if the free state evolution (i.e., the state trajectory of (6)) corresponding to any set of nonnegative boundary conditions (5) asymptotically tends to zero, i.e.,

$$\lim_{i,j\to\infty} x(i, j) = 0.$$

For the sake of brevity, instead of saying that a system is asymptotically stable we will say that the matrix triple (A_0, A_1, A_2) is asymptotically stable.

Lemma 5. [48] Let (A_0, A_1, A_2) be a triple of $n \times n$ nonnegative matrices. The triple (A_0, A_1, B_1) is asymptotically stable if and only if $\rho(A_0 + A_1 + A_2) < 1$

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[1] $|M| := [|m_{ij}|]$ for any matrix $M \in \boldsymbol{C}^{k\times l}$, this notation is used consistently throughout the paper and should not be mistaken with the determinant of the square matrix

Proof. [48] It is well known that the positive general model (3) is asymptotically stable if and only if

$$\forall (z_1, z_2) \in \{(z_1, z_2) : |z_1| \le 1, |z_2| \le 1\}$$

$$\det\left(I_n - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2\right) \ne 0. \tag{7}$$

Suppose that

$$\rho(A_0 + A_1 + A_2) < 1. \tag{8}$$

Note that for any complex numbers z_1 and z_2 such that $|z_1| \le 1$, $|z_2| \le 1$ we have

$$|A_0 z_1 z_2| + |A_1 z_1| + |A_2 z_2| \le A_0 + A_1 + A_2. \tag{9}$$

From Lamma 1, (8) and (9) one obtains the relation

$$\rho(A_0 z_1 z_2 + A_1 z_1 + A_2 z_2) \le \rho(A_0 + A_1 + A_2) < 1, \tag{10}$$

which with (7) taken into account implies the asymptotic stability of the general model (3).

Let us note, that owing to the positivity assumption on matrix triples, the stability analysis is considerably simpler, then in the case of arbitrary matrix triples. Ascertaining stability of an arbitrary triple $(A_0,\ A_1,\ A_2)$ is a difficult task, since one has to analyze the zeros of the characteristic polynomial of $(A_0,\ A_1,\ A_2)$

$$\Delta_{A_0,A_1,A_2}(z_1, z_2) := \det(I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2).$$

Thus the problem of simplification introduced by positivity constraint is crucial as it suffices to check whether the eigenvalues of the matrix sum $A_0 + A_1 + A_2$ are clustered inside the unit disk of the complex plane. According to Lemma 2 every nonnegative matrix has a positive real eigenvalue whose modulus is greater or equal to the modulus of any other eigenvalue. Thus this eigenvalue is equal to the spectral radius of the matrix. The triple $(A_0,\ A_1,\ A_2)$ is asymptotically stable if and only if

$$\rho(A_0 + A_1 + A_2) < 1, \tag{11}$$

thus it follows immediately that $\rho(A_0) < 1$ and $\rho(A_1) < 1$ and $\rho(A_2) < 1$ is a necessary condition for the matrix triple $(A_0,\ A_1,\ A_2)$ to be asymptotically stable. Indeed, if Q is a nonnegative square matrix than [45]

$$\rho(Q) = \sup\left\{\lambda \in \boldsymbol{R} : \exists x \ge 0 \text{ s.t. } Qx \ge \lambda x\right\}.$$

Now, let x denote a positive eigenvector of A_0 corresponding to the spectral radius $\rho(A_0)$. One has

$$(A_0 + A_1 + A_2)x = \rho(A_0)x + A_1 x + A_2 x \ge \rho(A_0)x,$$

hence $\rho(A_0 + A_1 + A_2) \ge \rho(A_0)$. In the same vein one can show that $\rho(A_0 + A_1 + A_2) \ge \rho(A_1)$ and $\rho(A_0 + A_1 + A_2) \ge \rho(A_2)$.

2.4. The first Fornasini-Marchesini model (FF-MM).

The first Fornasini-Marchesini model (FF-MM) is as follows [4,30,46]

$$x_{i+1,j+1} = A_0 x_{i,j} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B u_{i,j} \tag{12a}$$

$$y_{i,j} = C x_{i,j} + D u_{i,j}, \quad i, j \in \boldsymbol{Z}_+, \tag{12b}$$

where $x_{i,j} \in \boldsymbol{R}^n$ is the state vector at the point $(i,j) \in \boldsymbol{Z}_+ \times \boldsymbol{Z}_+$, $u_{i,j} \in \boldsymbol{R}^m$ and $y_{i,j} \in \boldsymbol{R}^p$ are the input and output vectors, respectively, and $A_0 \in \boldsymbol{R}^{n\times n}$, $A_1 \in \boldsymbol{R}^{n\times n}$, $A_2 \in \boldsymbol{R}^{n\times n}$, $B \in \boldsymbol{R}^{n\times m}$, $C \in \boldsymbol{R}^{p\times n}$, $D \in \boldsymbol{R}^{p\times m}$, with the boundary conditions (4).

Thus it is a particular case of the general model (3a)–(3b) with $B_1 = B_2 = 0$ and $B_0 = B$.

Since the autonomous parts of the general model (3) and FF-MM (12) are the same the whole discussion of positive general model stability applies to the FF-MM.

2.5. The second Fornasini-Marchesini model (SF-MM).

The second Fornasini-Marchesini model (SF-MM) is as follows [4,30,46]

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \tag{13a}$$

$$y_{i,j} = C x_{i,j} + D u_{i,j}, \quad i, j \in \boldsymbol{Z}_+, \tag{13b}$$

where $x_{i,j} \in \boldsymbol{R}^n$ is the state vector at the point $(i,j) \in \boldsymbol{Z}_+ \times \boldsymbol{Z}_+$, $u_{i,j} \in \boldsymbol{R}^m$ and $y_{i,j} \in \boldsymbol{R}^p$ are the input and output vectors, respectively, and $A_0 \in \boldsymbol{R}^{n\times n}$, $A_1 \in \boldsymbol{R}^{n\times n}$, $A_2 \in \boldsymbol{R}^{n\times n}$, $B \in \boldsymbol{R}^{n\times m}$, $C \in \boldsymbol{R}^{p\times n}$, $D \in \boldsymbol{R}^{p\times m}$. with the boundary conditions (4). Thus it is a particular case of the general model (3a)–(3b) with $A_0 = 0$ and $B_0 = 0$.

Let us consider the autonomous part of the SF-MM (13), i.e., autonomuos system of the form

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1}, \tag{14}$$

where the vector $x_{i,j}$ and the matrices A_1, A_2 are defined as for (13a).

One can see that the autonomous part of the SF-MM is a special case of its general model's counterpart with $A_0 = 0$.

Therefore, the results concerning stability of general model can be in a straightforward way applied to SF-MM.

2.6. Linear matrix inequalities.

The set of $n \times n$ symmetric matrices is denoted by $\boldsymbol{S}^n$. We say that $Q \in \boldsymbol{S}^n$ is positive definite (positive semidefinite) if its quadratic form is positive, i.e., $\forall x \in \boldsymbol{R}^n,\ x \ne 0,\ x^{\mathrm{T}} Q x > 0$ (nonnegative, i.e., $\forall x \in \boldsymbol{R}^n\ x^{\mathrm{T}} Q x \ge 0$). We denote this fact by $Q \succ 0$ ($Q \succeq 0$). The negative definiteness (negative semidefiniteness) is defined in a similar way.

Definition 10. [26] A linear matrix inequality (LMI) in the variable x is an inequality of the form

$$\mathcal{F}(x) + F \succeq 0, \tag{15}$$

where the variable x takes values in the real vector space $\boldsymbol{V}$, the mapping $\mathcal{F} : \boldsymbol{V} \to \boldsymbol{S}^n$ is linear, and $F \in \boldsymbol{S}^n$.

We say that the LMI (15) is *feasible* if there exist an $x \in \boldsymbol{V}$ such that the inequality (**??**) is satisfied. If an LMI is not feasible then we say it is infeasible. In our considerations we discriminate the following three kinds of feasibility:

1) Strict feasibility: $\exists x \in \boldsymbol{V}$ with $\mathcal{F}(x) + F_0 \succ 0$.
2) Nonzero feasibility: $\exists x \in \boldsymbol{V}$ with $\mathcal{F}(x) + F_0 \succneqq 0$ (i.e., positive semidefinite and nonzero).
3) Feasibility: $\exists x \in \boldsymbol{V}$ with $\mathcal{F}(x) + F_0 \succeq 0$.

Lemma 6. [27] Suppose that A is a Metzler matrix, i.e., $A \in \boldsymbol{M}^n$. The matrix $A \in \boldsymbol{M}^n$ is Hurwitz if and only if the following LMIs are infeasible with respect to the matrix variable Y

$$Y = Y^{\mathrm{T}} \succneqq 0, \tag{16a}$$

$$I \circ [AY] \succeq 0, \tag{16b}$$

where I stands for identity matrix of appropriate dimensions and the symbol $\circ$ denotes the Hadamard product of two matrices (i.e., entrywise multiplication). In other words, $A \in \boldsymbol{M}^n$ has at least one eigenvalue with nonegative real part if and only if LMIs (16a-)-(16b) are feasible.

Proof. See [27].

Lemma 7. [27] Suppose that A is a nonnegative matrix, i.e., $A \in \boldsymbol{R}_+^{n\times n}$. The matrix $A \in \boldsymbol{R}_+^{n\times n}$ is a Schur matrix if and only if the following LMIs are infeasible with respect to the matrix variable Y

$$Y = Y^{\mathrm{T}} \succneqq 0, \tag{17a}$$

$$I \circ \left[AYA^{\mathrm{T}} - Y\right] \succeq 0, \tag{17b}$$

where I stands for identity matrix of appropriate dimensions. In other words, $A \in \boldsymbol{R}_+^{n\times n}$ has at least one eigenvalue with modulus greater or equal to 1 if and only if LMIs (17a)–(17b) are feasible.

Proof. See [27].

Lemma 8. Suppose that $A \in \boldsymbol{R}^{n\times n}$ is a Metzler matrix, i.e., $A \in \boldsymbol{M}^n$. Then A is a Hurwitz matrix if and only if there exists a strictly positive vector $\lambda \in \boldsymbol{R}_{++}^n$ such that $A\lambda \ll 0$.

Proof. See, e.g, [3,4].

Lemma 9. A Metzler matrix $A \in \boldsymbol{M}^n$ is a Hurwitz matrix if and only if the following LMI are feasible with respect to the diagonal matrix variable P

$$\begin{bmatrix} -(A^{\mathrm{T}}P + PA) & 0 \\ 0 & P \end{bmatrix} \succ 0. \tag{18}$$

Proof. See, e.g., [3,4].

From Perron-Frobenus theorem [3,4,45] it follows that any nonnegative square matrix is a Schur matrix if and only if $(A - I)$ is a Hurwitz matrix. Since under assumption of nonnegativity of A, the matrix $(A - I)$ is a Metzler matrix, then with Lemma 9 we can conclude that the matrix $A \in \boldsymbol{R}_+^{n\times n}$ is a Schur matrix if and only if there exists a positive definite diagonal matrix $P \succ 0$ (of appropriate dimensions) such that

$$(A - I)^{\mathrm{T}}P + P(A - I) \prec 0$$

holds. Thus one has the following corollary.

Corollary 1. A nonnegative matrix $A \in \boldsymbol{R}_+^{n\times n}$ is a Schur matrix if and only if the following LMI are feasible with respect to the diagonal matrix variable P

$$\begin{bmatrix} (I - A)^{\mathrm{T}}P + P(I - A) & 0 \\ 0 & P \end{bmatrix} \succ 0. \tag{19}$$

On the other hand, with the above reasoning in mind, and with Lemma 6 taken into account we obtain the following corollary.

Corollary 2. Suppose that A is a nonnegative matrix, i.e., $A \in \boldsymbol{R}_+^{n\times n}$. The matrix $A \in \boldsymbol{R}_+^{n\times n}$ is Schur if and only if the following LMIs are infeasible with respect to the matrix variable Y

$$Y = Y^{\mathrm{T}} \succneqq 0, \tag{20a}$$

$$I \circ [AY - Y] \succeq 0, \tag{20b}$$

where I stands for identity matrix of appropriate dimensions.

Lemma 10. A nonnegative matrix $A \in \boldsymbol{R}_+^{n\times n}$ is a Schur matrix if and only if the following LMI are feasible with respect to the diagonal matrix variable P [3,4]

$$\begin{bmatrix} P - A^{\mathrm{T}}PA & 0 \\ 0 & P \end{bmatrix} \succ 0. \tag{21}$$

Proof. See, e.g., [3,4].

The inequality $P - A^{\mathrm{T}}PA \succ 0$ which is Lyapunov inequality for discrete-time systems is also called Stein inequality.

3. LMI approach to the stability of 2D positive systems

3.1. The general model. Proposition 1. The 2D positive system of the form (3a)–(3b), i.e, the general model is asymptotically stable if and only if one of the following equivalent conditions holds

1) There exists a strictly positive vector $\lambda \in \boldsymbol{R}_{++}^n$ such that

$$(A_0 + A_1 + A_2)\,\lambda \ll \lambda. \tag{22}$$

2) The following LMI is feasible with respect to the diagonal matrix variable P

$$\begin{bmatrix} \hat{P} & 0 \\ 0 & P \end{bmatrix} \succ 0. \tag{23}$$

where

$$\hat{P} = 2P - \sum_{i=0}^{2}(A_i^{\mathrm{T}}P + PA_i).$$

3) The following LMI is feasible with respect to the diagonal matrix variable P

$$\begin{bmatrix} \hat{P} & 0 \\ 0 & P \end{bmatrix} \succ 0. \tag{24}$$

where

$$\hat{P} = P - \sum_{i,j=0}^{2} (A_i^{\mathrm{T}} P A_i).$$

4) The following LMIs are infeasible with respect to the matrix variable Y

$$Y = Y^{\mathrm{T}} \succneqq 0, \tag{25a}$$

$$I \circ [A_0 Y + A_1 Y + A_2 Y - Y] \succeq 0. \tag{25b}$$

5) The following LMIs are infeasible with respect to the matrix variable Y

$$Y = Y^{\mathrm{T}} \succneqq 0, \tag{26a}$$

$$I \circ \left[\left(\sum_{i=0}^{2} A_i \right) Y \left(\sum_{i=0}^{2} A_i^{\mathrm{T}} \right) - Y \right] \succneqq 0. \tag{26b}$$

Remark 1. The conditions 1), 2) and 3) are well known. The condition 1) is in fact an LP problem, thus it can be regarded as a special case of LMI [24].

Proff.

Ad 1) The inequality (22) can be rewritten as

$$(A_0 + A_1 + A_2 - I)\lambda \ll 0$$

thus the proof follows immediately by Lemmas 5 and 8.

Ad 2) The inequality (23) can be rewritten as

$$\begin{bmatrix} (I - \sum_{i=0}^{2} A_i)^{\mathrm{T}} P + P(I - \sum_{i=0}^{2} A_i) & 0 \\ 0 & P \end{bmatrix} \succ 0.$$

thus the proof follows immediately by Lemma 5 and Corollary 1.

Ad 3) The inequality (24) can be rewritten as

$$\begin{bmatrix} P - \left(\sum_{i=0}^{2} A_i \right)^{\mathrm{T}} P \left(\sum_{i=0}^{2} A_i \right) & 0 \\ 0 & P \end{bmatrix} \succ 0.$$

thus the proof follows by Lemma 5 and Lemma 10

Ad 4) The inequality (25a) can be rewritten as

$$I \circ \left[\left(\sum_{i=0}^{2} A_i \right) Y \right] \succeq 0,$$

thus the proof follows by Lemma 5 and Corollary 2.

Ad 5) The inequality (26a) follows by Lemmas 5 and 7.

Example 1. Let us consider the general positive model (3) with the state matrices

$$A_0 = \begin{bmatrix} 0.10 & 0.10 & 0.20 \\ 0.02 & 0.01 & 0.25 \\ 0 & 0.30 & 0.20 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.10 & 0.10 & 0.02 \\ 0.01 & 0.10 & 0.25 \\ 0.01 & 0.03 & 0.02 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.10 & 0.10 & 0.20 \\ 0.30 & 0.10 & 0.07 \\ 0.10 & 0.10 & 0.10 \end{bmatrix}.$$

Since $\rho(A_0 + A_1 + A_2) = 0.9804 < 1$ the considered system is asymptotically stable. Indeed, one can check that the inequality (22) holds for

$$\lambda = \begin{bmatrix} 5.3030 & 5.6402 & 4.5715 \end{bmatrix}^{\mathrm{T}} \gg 0,$$

the LMIs (23) hold for

$$P = \mathbf{diag} \begin{bmatrix} 10.5465 & 13.0984 & 20.1838 \end{bmatrix},$$

and the LMIs (24) hold for

$$P = \mathbf{diag} \begin{bmatrix} 7.0628 & 8.5972 & 13.0293 \end{bmatrix}.$$

The LMIs (25) and (26) are infeasible.

Example 2. Let us consider the general positive model (3) with the state matrices

$$A_0 = \begin{bmatrix} 0.10 & 0.10 & 1.20 \\ 0.02 & 0.61 & 0.25 \\ 0 & 0.30 & 0.50 \end{bmatrix}$$

and A_1 and A_2 are as in Example 1. Since in this case $\rho(A_0 + A_1 + A_2) = 1.4902 > 1$, the considered system is not stable. Indeed, one can check that the LMIs (22), (23) and (24) are infeasible. On the other hand one can easily verify that the LMIs (25) are feasible, one possible solution is

$$Y = \begin{bmatrix} 2.0829 & 1.8311 & 1.3416 \\ 1.8311 & 4.1130 & 2.2997 \\ 1.3416 & 2.2997 & 1.9669 \end{bmatrix},$$

and the LMIs (26) are feasible, one possible solution is

$$Y = \begin{bmatrix} 1.9021 & 0.8251 & 0.4462 \\ 0.8251 & 2.5897 & 0.8706 \\ 0.4462 & 0.8706 & 0.8936 \end{bmatrix}.$$

3.2. The second Fornasini-Marchesini model.

Proposition 2. The 2D positive system of the form (13a)–(13b), i.e, the SF-MM is asymptotically stable if and only if one of the following conditions holds.

1) There exists a strictly positive vector $\lambda \in \boldsymbol{R}^n_{++}$ such that

$$(A_1 + A_2)\,\lambda \ll \lambda. \tag{27}$$

2) The following LMI is feasible with respect to the diagonal matrix variable P

$$\begin{bmatrix} \hat{P} & 0 \\ 0 & P \end{bmatrix} \succ 0. \tag{28}$$

where

$$\hat{P} = 2P - A_1^{\mathrm{T}} P - P A_1 - A_2^{\mathrm{T}} P - P A_2.$$

3) The following LMI is feasible with respect to the diagonal matrix variable P

$$\begin{bmatrix} \hat{P} & 0 \\ 0 & P \end{bmatrix} \succ 0. \tag{29}$$

where

$$\hat{P} = P - A_1^{\mathrm{T}} P A_1 - A_1^{\mathrm{T}} P A_2 - A_2^{\mathrm{T}} P A_1 - A_2^{\mathrm{T}} P A_2.$$

4) The following LMIs are infeasible with respect to the matrix variable Y

$$Y = Y^{\mathrm{T}} \gneqq 0, \tag{30a}$$

$$I \circ [A_1 Y + A_2 Y - Y] \succeq 0. \tag{30b}$$

5) The following LMIs are infeasible with respect to the matrix variable Y

$$Y = Y^{\mathrm{T}} \gneqq 0, \tag{31a}$$

$$I \circ \left[(A_1 + A_2) Y (A_1^{\mathrm{T}} + A_2^{\mathrm{T}}) - Y \right] \gneqq 0. \tag{31b}$$

Proof. The proof follows immediately from Proposition 1 with $A_0 = 0$.

3.3. Roesser model.

Let us consider the positive 2D Roesser model given by (1a)-(1b). We say that a positive 2D Roesser model is asymptotically stable if its autonomous part

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix}, \tag{32}$$

where vectors $x^h_{i,j}$, $x^v_{i,j}$, and matrices A_{11}, A_{12}, A_{21}, A_{22} are defined as in (1a), is stable.

Proposition 3. The 2D positive system of the form (1a)–(1b) is asymptotically stable if and only if one of the following equivalent conditions holds

1) There exists a strictly positive vector $0 \ll \lambda \in \boldsymbol{R}^n_{++}$ such that

$$\begin{bmatrix} A_{11} - I & A_{12} \\ A_{21} & A_{22} - I \end{bmatrix} \lambda \ll 0. \tag{33}$$

2) The following LMI is feasible with respect to the diagonal matrix variables P_1 and P_2

$$\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} & 0 & 0 \\ \hat{P}_{12}^{\mathrm{T}} & \hat{P}_{22} & 0 & 0 \\ 0 & 0 & P_1 & 0 \\ 0 & 0 & 0 & P_2 \end{bmatrix} \succ 0, \tag{34}$$

where

$$\begin{aligned} \hat{P}_{11} &= 2P_1 - A_{11}^{\mathrm{T}} P_1 - P_1 A_{11}, \\ \hat{P}_{12} &= -A_{21}^{\mathrm{T}} P_2 - P_1 A_{12}, \\ \hat{P}_{22} &= 2P_2 - A_{22}^{\mathrm{T}} P_2 - P_2 A_{22}. \end{aligned}$$

3) The following LMI is feasible with respect to the diagonal matrix variables P_1 and P_2

$$\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} & 0 & 0 \\ \hat{P}_{12}^{\mathrm{T}} & \hat{P}_{22} & 0 & 0 \\ 0 & 0 & P_1 & 0 \\ 0 & 0 & 0 & P_2 \end{bmatrix} \succ 0, \tag{35}$$

where

$$\begin{aligned} \hat{P}_{11} &= P_1 - A_{11}^{\mathrm{T}} P_1 A_{11} - A_{21}^{\mathrm{T}} P_2 A_{21}, \\ \hat{P}_{12} &= -A_{11}^{\mathrm{T}} P_1 A_{12} - A_{21}^{\mathrm{T}} P_2 A_{22}, \\ \hat{P}_{22} &= P_2 - A_{12}^{\mathrm{T}} P_1 A_{12} - A_{22}^{\mathrm{T}} P_2 A_{22}. \end{aligned}$$

4) The following LMIs are infeasible with respect to the matrix variables $Y_{11} = Y_{11}^{\mathrm{T}}$, Y_{12}, and $Y_{22} = Y_{22}^{\mathrm{T}}$

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathrm{T}} & Y_{22} \end{bmatrix} \gneqq 0, \qquad I \circ \begin{bmatrix} \hat{P}_{11} & 0 \\ 0 & \hat{P}_{22} \end{bmatrix} \succeq 0, \tag{36}$$

where

$$\begin{aligned} \hat{P}_{11} &= (A_{11} - I) Y_{11} + A_{12} Y_{12}^{\mathrm{T}}, \\ \hat{P}_{22} &= (A_{22} - I) Y_{22} + A_{21} Y_{12}. \end{aligned}$$

5) The following LMIs are infeasible with respect to the matrix variables $Y_{11} = Y_{11}^{\mathrm{T}}$, Y_{12}, and $Y_{22} = Y_{22}^{\mathrm{T}}$

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathrm{T}} & Y_{22} \end{bmatrix} \gneqq 0, \qquad I \circ \begin{bmatrix} \hat{P}_{11} & 0 \\ 0 & \hat{P}_{22} \end{bmatrix} \succeq 0, \tag{37}$$

where

$$\begin{aligned} \hat{P}_{11} &= A_{11}(Y_{11} A_{11}^{\mathrm{T}} + Y_{12} A_{12}^{\mathrm{T}}) \\ &\quad + A_{12}(Y_{12}^{\mathrm{T}} A_{11}^{\mathrm{T}} + Y_{22} A_{12}^{\mathrm{T}}) - Y_{11}, \\ \hat{P}_{22} &= A_{21}(Y_{11} A_{21}^{\mathrm{T}} + Y_{12} A_{22}^{\mathrm{T}}) \\ &\quad + A_{22}(Y_{12}^{\mathrm{T}} A_{21}^{\mathrm{T}} + Y_{22} A_{22}^{\mathrm{T}}) - Y_{22}. \end{aligned}$$

Proof. The positive Roesser model (1) is equivalent to the positive SF-MM (13). We restrict our considerations to the autonomous systems, details can be found in [4]. Indeed, from (1a) one has

$$\begin{aligned} x^h_{i+1,j+1} &= A_{11}x^h_{i,j+1} + A_{12}x^v_{i,j+1}, \\ x^v_{i+1,j+1} &= A_{21}x^h_{i+1,j} + A_{22}x^v_{i+1,j}. \end{aligned}$$

The equations can be rewritten in the form

$$\begin{aligned} \begin{bmatrix} x^h_{i+1,j+1} \\ x^v_{i+1,j+1} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h_{i+1,j} \\ x^v_{i+1,j} \end{bmatrix} \\ &+ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x^h_{i,j+1} \\ x^v_{i,j+1} \end{bmatrix}, \end{aligned} \tag{38}$$

With the following definitions in mind

$$\begin{aligned} x_{ij} := \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, \\ A_2 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{39}$$

we may rewrite (38) as

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1}, \tag{40}$$

i.e., the SF-MM model. If the autonomous Roesser model (32) is positive then the matrices in (40) are positive and the autonomous SF-MM is also positive. Now, with (39) taken into account, the proof of Proposition 3 follows readily by virtue of Lemma 5 and Corollary 2.

Example 3. Let us consider the positive Roesser model (3) with the state matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} 0.10 & 0.10 & 0.01 & 0.10 \\ 0.30 & 0.10 & 0.10 & 0.20 \\ \hline 0.30 & 0.05 & 0.20 & 0.10 \\ 0.10 & 0.40 & 0.10 & 0.30 \end{array}\right].$$

Since $\sigma(A) = \{0.6393,\ -0.0893,\ 0.0750 \pm 0.0307j\}$, the considered system is asymptotically stable. Indeed, one can check that the inequality (33) holds for

$$\lambda = \begin{bmatrix} 0.1865 & 0.2840 & 0.2588 & 0.3688 \end{bmatrix}^{\mathrm{T}} \gg 0,$$

the LMIs (34) hold for

$$P_1 = \begin{bmatrix} 1.7152 & 0 \\ 0 & 1.6545 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.6574 & 0 \\ 0 & 1.7754 \end{bmatrix},$$

and the LMIs (35) hold for

$$P_1 = \begin{bmatrix} 2.2733 & 0 \\ 0 & 2.0782 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.8732 & 0 \\ 0 & 1.9608 \end{bmatrix}.$$

The LMIs (36) and (37) are infeasible.

Example 4. Let us consider the general positive model (1) with the state matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} 0.10 & 0.10 & 0.90 & 0.10 \\ 0.30 & 0.10 & 0.10 & 0.50 \\ \hline 0.30 & 0.50 & 0.20 & 0.10 \\ 0.10 & 0.40 & 0.10 & 0.30 \end{array}\right].$$

Since $\sigma(A) = \{1.0431,\ 0.3082,\ -0.3257 \pm 0.2293j\}$, the considered system is not stable. Indeed, one can check that the LMIs (33), (34) and (35) are infeasible. On the other hand one can easily that the LMIs (36) are feasible, one possible solution is

$$Y_{11} = \begin{bmatrix} 14.1561 & 10.5081 \\ 10.5081 & 8.1975 \end{bmatrix},$$

$$Y_{12} = \begin{bmatrix} 12.4165 & 8.8952 \\ 9.3440 & 6.9423 \end{bmatrix},$$

$$Y_{22} = \begin{bmatrix} 11.2742 & 8.0011 \\ 8.0011 & 6.1502 \end{bmatrix},$$

and the LMIs (37) are feasible, one possible solution is

$$Y_{11} = \begin{bmatrix} 13.3092 & 9.6567 \\ 9.6567 & 7.6565 \end{bmatrix},$$

$$Y_{12} = \begin{bmatrix} 11.1427 & 8.2962 \\ 8.5102 & 6.3024 \end{bmatrix},$$

$$Y_{22} = \begin{bmatrix} 10.3859 & 7.2745 \\ 7.2745 & 5.7033 \end{bmatrix}.$$

3.4. The first Fornasini-Marchesini model.

Let us consider the FF-MM model given by (12).

We say that the positive FF-MM model is asymptotically stable if its autonomous part

$$x_{i+1,j+1} = A_0 x_{i,j} + A_1 x_{i,j+1} + A_2 x_{i+1,j}, \tag{41}$$

where vector $x_{i,j}$ and matrices A_0, A_1 and A_2 are defined as in (12), is asymptotically stable. Since the autonomous part of the positive FF-MM (12) is the same as that of general positive model (3), Proposition 1 can be applied directly to checking stability of FF-MM. Nevertheless, two another propositions are provided with regard to the stability problem for the FF-MM.

Proposition 4. The 2D positive system (12) (the FF-MM) is asymptotically stable if and only if one of the following equivalent conditions holds

1) There exist strictly positive vectors $\lambda_1 \in \boldsymbol{R}_{++}^n$, $\lambda_2 \in \boldsymbol{R}_{++}^n$ such that

$$A_2\lambda_1 + [A_0 + A_2A_1]\,\lambda_2 \ll \lambda_1, \tag{42a}$$

$$\lambda_1 + A_1\lambda_2 \ll \lambda_2. \tag{42b}$$

2) The following LMIs are feasible with respect to the diagonal matrix variables P_1 and P_2

$$\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} & 0 & 0 \\ \hat{P}_{12}^{\mathrm{T}} & \hat{P}_{22} & 0 & 0 \\ 0 & 0 & P_1 & 0 \\ 0 & 0 & 0 & P_2 \end{bmatrix} \succ 0, \tag{43}$$

where

$$\hat{P}_{11} = 2P_1 - A_2^{\mathrm{T}}P_1 - P_1A_2,$$
$$\hat{P}_{12} = -P_2 - P_1\,[A_0 + A_2A_1]\,,$$
$$\hat{P}_{22} = 2P_2 - A_1^{\mathrm{T}}P_2 - P_2A_1.$$

3) The following LMI is feasible with respect to the diagonal matrix variables P_1 and P_2

$$\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} & 0 & 0 \\ \hat{P}_{12}^{\mathrm{T}} & \hat{P}_{22} & 0 & 0 \\ 0 & 0 & P_1 & 0 \\ 0 & 0 & 0 & P_2 \end{bmatrix} \succ 0, \tag{44}$$

where

$$\hat{P}_{11} = P_1 - P_2 - A_2^{\mathrm{T}}P_1A_2,$$
$$\hat{P}_{12} = -A_2^{\mathrm{T}}P_1\,[A_0 + A_2A_1 - P_2A_1]\,,$$
$$\hat{P}_{22} = P_2 - A_1^{\mathrm{T}}P_2A_1 - [A_0 + A_2A_1]^{\mathrm{T}}\,P_1\,[A_0 + A_2A_1]\,.$$

4) The following LMIs are infeasible with respect to the matrix variables $Y_{11} = Y_{11}^{\mathrm{T}}$, Y_{12}, and $Y_{22} = Y_{22}^{\mathrm{T}}$

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathrm{T}} & Y_{22} \end{bmatrix} \succneqq 0, \quad I \circ \begin{bmatrix} \hat{P}_{11} & 0 \\ 0 & \hat{P}_{22} \end{bmatrix} \succeq 0, \tag{45}$$

where

$$\hat{P}_{11} = A_2Y_{11} + [A_0 + A_2A_1]\,Y_{12}^{\mathrm{T}} - Y_{11},$$
$$\hat{P}_{22} = Y_{12} + A_1Y_{22} - Y_{22}.$$

5) The following LMIs are infeasible with respect to the matrix variables $Y_{11} = Y_{11}^{\mathrm{T}}$, Y_{12}, and $Y_{22} = Y_{22}^{\mathrm{T}}$

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathrm{T}} & Y_{22} \end{bmatrix} \succneqq 0, \quad I \circ \begin{bmatrix} \hat{P}_{11} & 0 \\ 0 & \hat{P}_{22} \end{bmatrix} \succeq 0, \tag{46}$$

where

$$\hat{P}_{11} = A_2\,[Y_{11}A_2 + Y_{12}] + [A_0 + A_2A_1]\,[Y_{12}^{\mathrm{T}}A_2 + Y_{22}] - Y_{11},$$
$$\hat{P}_{22} = Y_{11}\,[A_0 + A_2A_1] - Y_{12}A_1 + A_1Y_{12}^{\mathrm{T}}\,[A_0 + A_2A_1] - A_1Y_{22}A_1 - Y_{22}.$$

Proof. The positive FF-MM model (12) is equivalent to the positive Roesser model (1). We restrict our considerations to the autonomous systems, details can be found in [4]. Indeed, let us consider Eq. 41

$$x_{i+1,j+1} = A_0x_{i,j} + A_1x_{i+1,j} + A_2x_{i,j+1},$$

defining

$$x_{ij}^h := x_{i,j+1} + A_1x_{i,j}, \quad \text{and} \quad x_{ij}^v := x_{ij}$$

one can write

$$\begin{aligned} x_{i+1,j}^h &= x_{i+1,j+1} - A_1x_{i+1,j} \\ &= A_0x_{i,j}^v + A_2\left[x_{ij}^h + A_1x_{ij}^v\right] \\ &= [A_0 + A_2A_1]\,x_{i,j}^v + A_2x_{ij}^h, \\ x_{i,j+1}^v &= x_{ij}^h + A_1x_{ij}^v, \end{aligned}$$

this yields

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} A_2 & A_0 + A_2A_1 \\ I & A_1 \end{bmatrix} \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix} \tag{47}$$

Equation (47) describes the Roesser model. If the autonomous FF-MM model (41) is positive then the state matrix in (47) is positive and the autonomous Roesser model (47) is also positive. Now the proof of Proposition 4 follows by virtue of Lemma 5 and Proposition 3.

Example 5. Let us consider the positive FF-MM (12) with the state matrices A_0, A_1, A_2 are defined as in Example 1. One can check that the inequalities (42) hold for

$$\lambda_1 = \begin{bmatrix} 6.5697 \\ 6.1786 \\ 6.8055 \end{bmatrix} \gg 0, \quad \lambda_2 = \begin{bmatrix} 8.5901 \\ 9.1310 \\ 7.4136 \end{bmatrix} \gg 0,$$

the LMIs (43) hold for

$$P_1 = \mathbf{diag}\begin{bmatrix} 60.5271 & 79.3351 & 91.6313 \end{bmatrix},$$
$$P_2 = \mathbf{diag}\begin{bmatrix} 19.7376 & 38.8056 & 62.1317 \end{bmatrix},$$

and the LMIs (35) hold for

$$P_1 = \mathbf{diag}\begin{bmatrix} 50.0106 & 67.6583 & 78.2404 \end{bmatrix},$$
$$P_2 = \mathbf{diag}\begin{bmatrix} 12.9309 & 31.6203 & 52.0119 \end{bmatrix}.$$

The LMIs (45) as well as (46) are infeasible.

Example 6. Let us consider the positive FF-MM (12) with the state matrices A_0, A_1, A_2 defined as in Example 2.

One can check that the LMIs (42), (43) and (44) are infeasible. On the other hand one can easily that the LMIs (45) are feasible, one possible solution is

$$Y_{11} = \begin{bmatrix} 7.1247 & 5.3169 & 4.9448 \\ 5.3169 & 6.9283 & 4.9691 \\ 4.9448 & 4.9691 & 4.9753 \end{bmatrix},$$

$$Y_{12} = \begin{bmatrix} 2.9242 & 5.4586 & 4.3338 \\ 2.5034 & 6.1025 & 4.4049 \\ 2.2284 & 5.1482 & 4.2020 \end{bmatrix},$$

$$Y_{22} = \begin{bmatrix} 2.4431 & 2.4885 & 2.0780 \\ 2.4885 & 7.0262 & 4.5679 \\ 2.0780 & 4.5679 & 4.0422 \end{bmatrix},$$

and the LMIs (46) are feasible, one possible solution is

$$Y_{11} = \begin{bmatrix} 3.0665 & 1.4620 & 0.8935 \\ 1.4620 & 3.5576 & 1.0386 \\ 0.8935 & 1.0386 & 1.9247 \end{bmatrix},$$

$$Y_{12} = \begin{bmatrix} 0.3666 & 2.0717 & 1.2340 \\ 0.4109 & 2.4042 & 1.5603 \\ 0.3164 & 1.4635 & 0.9166 \end{bmatrix},$$

$$Y_{22} = \begin{bmatrix} 1.8710 & 0.5159 & 0.3434 \\ 0.5159 & 4.0780 & 1.9504 \\ 0.3434 & 1.9504 & 1.5628 \end{bmatrix},$$

Thus the system under consideration is not stable.

Proposition 5. The 2D positive system (12) (the FF-MM) is asymptotically stable if and only if one of the following equivalent conditions holds

1) There exist strictly positive vectors $\lambda_1 \in \boldsymbol{R}^n_{++}$, $\lambda_2 \in \boldsymbol{R}^n_{++}$ such that

$$(A_1 + A_2)\lambda_1 - A_0\lambda_2 \ll \lambda_1, \tag{48a}$$

$$\lambda_1 \ll \lambda_2. \tag{48b}$$

2) The following LMIs are feasible with respect to the diagonal matrix variables P_1 and P_2

$$\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} & 0 \\ \hat{P}_{12}^{\mathrm{T}} & \hat{P}_{22} & 0 \\ 0 & 0 & P_1 \end{bmatrix} \succ 0, \tag{49}$$

where

$$\hat{P}_{11} = 2P_1 - \sum_{i=1}^{2}(A_i^{\mathrm{T}}P_1 - P_1A_i),$$
$$\hat{P}_{12} = -P_2 - P_1A_0,$$
$$\hat{P}_{22} = 2P_2.$$

3) The following LMI is feasible with respect to the diagonal matrix variables P_1 and P_2

$$\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} & 0 & 0 \\ \hat{P}_{12}^{\mathrm{T}} & \hat{P}_{22} & 0 & 0 \\ 0 & 0 & P_1 & 0 \\ 0 & 0 & 0 & P_2 \end{bmatrix} \succ 0, \tag{50}$$

where

$$\hat{P}_{11} = P_1 - P_2 - \sum_{i,j=1}^{2} A_i^{\mathrm{T}}P_1A_j,$$
$$\hat{P}_{12} = -(A_1 + A_2)^{\mathrm{T}}P_1A_0,$$
$$\hat{P}_{22} = P_2 - A_0^{\mathrm{T}}P_1A_0.$$

4) The following LMIs are infeasible with respect to the matrix variables $Y_{11} = Y_{11}^{\mathrm{T}}$, Y_{12}, and $Y_{22} = Y_{22}^{\mathrm{T}}$

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathrm{T}} & Y_{22} \end{bmatrix} \succneqq 0, \quad I \circ \begin{bmatrix} \hat{P}_{11} & 0 \\ 0 & \hat{P}_{22} \end{bmatrix} \succeq 0, \tag{51}$$

where

$$\hat{P}_{11} = (A_1 + A_2 - I)Y_{11} + A_0Y_{12}^{\mathrm{T}},$$
$$\hat{P}_{22} = Y_{12} - Y_{22}.$$

5) The following LMIs are infeasible with respect to the matrix variables $Y_{11} = Y_{11}^{\mathrm{T}}$, Y_{12}, and $Y_{22} = Y_{22}^{\mathrm{T}}$

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathrm{T}} & Y_{22} \end{bmatrix} \succneqq 0, \quad I \circ \begin{bmatrix} \hat{P}_{11} & 0 \\ 0 & \hat{P}_{22} \end{bmatrix} \succeq 0, \tag{52}$$

where

$$\hat{P}_{11} = \left[(A_1 + A_2)Y_{11} + A_0Y_{12}^{\mathrm{T}}\right](A_1^{\mathrm{T}} + A_2^{\mathrm{T}})$$
$$\qquad \left[(A_1 + A_2)Y_{12} + A_0Y_{22}\right]A_0^{\mathrm{T}} - Y_{11},$$
$$\hat{P}_{22} = Y_{11} - Y_{22}.$$

Proof. The autonomous part of the positive FF-MM (12) is equivalent to the autonomous part of the positive SF-MM (13). We restrict our considerations to the autonomous systems, details can be found in [48]. Indeed, let us consider equation (41)

$$x_{i+1,j+1} = A_0x_{i,j} + A_1x_{i+1,j} + A_2x_{i,j+1},$$

defining

$$\bar{x}_{i,j} := \begin{bmatrix} x_{i,j} \\ x_{i-1,j} \end{bmatrix} \quad \text{or} \quad \hat{x}_{i,j} := \begin{bmatrix} x_{i,j} \\ x_{i-1,j} \end{bmatrix}$$

one obtains corresponding SF-MM models

$$\bar{x}_{i+1,j+1} = \begin{bmatrix} A_1 & A_0 \\ 0 & 0 \end{bmatrix}\bar{x}_{i+1,j} + \begin{bmatrix} A_2 & 0 \\ I & 0 \end{bmatrix}\bar{x}_{i,j+1},$$

$$\hat{x}_{i+1,j+1} = \begin{bmatrix} A_1 & 0 \\ I & 0 \end{bmatrix}\hat{x}_{i+1,j} + \begin{bmatrix} A_2 & A_0 \\ 0 & 0 \end{bmatrix}\hat{x}_{i,j+1}.$$

Thus the proof of Proposition 5 follows by virtue of Lemma 5 and Proposition 2.

Example 7. Let us consider the positive FF-MM (12) with the state matrices A_0, A_1, A_2 defined as in Example 1. One can check that the inequalities (42) hold for

$$\lambda_1 = \begin{bmatrix} 7.4429 \\ 7.9063 \\ 6.4242 \end{bmatrix} \gg 0, \quad \lambda_2 = \begin{bmatrix} 7.5429 \\ 8.0063 \\ 6.5242 \end{bmatrix} \gg 0,$$

the LMIs (43) hold for

$$P_1 = \mathbf{diag}\begin{bmatrix} 64.7571 & 76.6419 & 114.6352 \end{bmatrix},$$
$$P_2 = \mathbf{diag}\begin{bmatrix} 17.6390 & 38.0350 & 59.0075 \end{bmatrix},$$

and the LMIs (44) hold for

$$P_1 = \mathbf{diag}\begin{bmatrix} 52.6809 & 62.8472 & 97.3178 \end{bmatrix},$$
$$P_2 = \mathbf{diag}\begin{bmatrix} 10.8177 & 29.7220 & 48.3693 \end{bmatrix},$$

The LMIs in the conditions (45) and (46) are infeasible.

Example 8. Let us consider the positive FF-MM (12) with the state matrices A_0, A_1, A_2 defined as in Example 2.

One can check that the inequalities (48) as well as the LMIs (49) and (50) are infeasible. On the other hand one can easily that the LMIs (51) are feasible, one possible solution is

$$Y_{11} = \begin{bmatrix} 9.3429 & 7.7067 & 5.7105 \\ 7.7067 & 10.8778 & 6.3899 \\ 5.7105 & 6.3899 & 5.0396 \end{bmatrix},$$
$$Y_{12} = \begin{bmatrix} 3.2741 & 5.6313 & 4.5771 \\ 2.9018 & 7.1252 & 5.1525 \\ 2.0999 & 4.7561 & 3.9136 \end{bmatrix},$$
$$Y_{22} = \begin{bmatrix} 2.2814 & 2.0764 & 1.8015 \\ 2.0764 & 6.1176 & 3.8207 \\ 1.8015 & 3.8207 & 3.5134 \end{bmatrix},$$

and the LMIs in the conditions (52) are feasible, one possible solution is

$$Y_{11} = \begin{bmatrix} 3.7119 & 2.1594 & 1.2177 \\ 2.1594 & 4.7863 & 1.4241 \\ 1.2177 & 1.4241 & 1.8809 \end{bmatrix},$$
$$Y_{12} = \begin{bmatrix} 0.1789 & 2.2800 & 1.3348 \\ 0.1940 & 2.6868 & 1.6092 \\ 0.1514 & 1.5255 & 0.9257 \end{bmatrix},$$
$$Y_{22} = \begin{bmatrix} 1.8585 & 0.1945 & 0.1544 \\ 0.1945 & 3.7456 & 1.6853 \\ 0.1544 & 1.6853 & 1.3697 \end{bmatrix},$$

Thus the system under consideration is not stable.

4. Conclusions and open problems

The problem of stability of positive 2D systems has been considered. Necessary and sufficient conditions for the stability of the general model as well as FF-MM, SF-MM and Roesser models in the LMI framework have been provided. The considerations have been illustrated wit numerical examples. Generalization of the proposed results onto singular 2D positive systems remains an open problem.

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