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Non-classical operational calculus applied to certain linear discrete time-system

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Abstract. In the paper there has been made an advantage of the non-classical operational calculus to determination of the response of the certain discrete time-systems. The Z-transform is often used to analysis of the stationary discrete time-systems. However, the use of the Z-transform to determination of the response especially of the non-stationary discrete time-systems is doubtful or may cause complications. This method leads to differential equations of n-th order of variable coefficients, whose solutions are very difficult or impossible. The non-classical operational calculus can be used to analysis both of the stationary and non-stationary discrete time-systems. The presented method with the use of the Heaviside operator soon leads to the target without unnecessary differential equations.

Key words: discrete time-system, Z-transform, non-classical operational calculus, Heaviside operator, analysis of the discrete time-system.

1. Introduction

Discrete time-systems, discrete dynamical systems are now of immense practical significance. Therefore the processes of their analysis and synthesis are indispensable.

The method of Z-transform is often used for this purpose. This transform proves in the case of stationary control systems described by applying a difference operation

$$\Delta^n \{x_k\} = \left\{ \sum_{i=0}^{n} (-1)^i \binom{n}{i} x_{n+k-i} \right\}, \quad n = 1, 2, 3, ...$$

where $\{x_k\} \in C(N)$ – space of real sequences. That is in the case of control systems of the form

$$a_n \Delta^n x_k + a_{n-1} \Delta^{n-1} x_k + ... + a_1 \Delta x_k + a_0 x_k = u_k, \quad (1)$$

where $a_i \in R$ for $i = 0, 1, ..., n$. (R is a set of real numbers.) In Eq. (1) x_k is a discrete output signal, while u_k is a discrete input signal in space $C(N)$. The basic characteristics of the discrete systems are described in the Z-transform language.

REMARK. In Eq. (1) to simplify the denotation in defining the sequence the braces have been omitted.
Moreover, in the denotation of the sequence for simplicity of the notation the braces will be omitted if no ambiguous situation arises.

In the paper careful note will be taken of the problems that are likely to arise, for example, while determining the response of a particular non-stationary control system (dynamical system)

$$a_2(k+2)(k+1)\Delta^2 x_k + (a_1 + a_2)(k+1)\Delta x_k + a_0 x_k = u_k, \quad (2)$$

where $a_0, a_1, a_2 \in R$.

System (2) may be treated as non-stationary algorithm of time processing of discrete signal. In the analysis the discrete system is so assumed that it is immediately possible to take advantage of the Z- transform properties of [1,2]. As point 2 indicates, as a consequence of using Z-transform in the control system (2), in order to determine its response it is necessary to solve a second order differential equation of variable coefficients, which as a rule causes numerous problems, or is simply impossible to solve. It should be pointed out that in general this will lead to differential equations of n-th order of variable coefficients. Even if $n = 3$ the solution of the differential equation is very difficult.

If the control system (1) or (2) is governed by initial conditions different from

$$x_i = x_{i,0}, \quad i = 0, 1, 2, ..., n-1, \quad (3)$$

e.g. conditions of the form

$$x_i = x_{i,0}, \quad i = k_0, k_0+1, ..., k_0+n-1, \quad k_0 > 0, \quad (4)$$

then the use of the Z-transform becomes doubtful, or may cause complications.

Taking into consideration the above facts, in the paper an attempt has been made to apply such methods that will eliminate the difficulty. It turns out, as it was proved at point 6, that the application of a non-classical operational calculus makes it possible to easily determine the response of the control systems of type (2), as a special case of such control systems as

$$a_2 \alpha_k \alpha_{k+1} \Delta^2 x_k + \alpha_k (a_2 \Delta \alpha_k + a_1) \Delta x_k + a_0 x_k = u_k, \quad (5)$$

here $a_0, a_1, a_2 \in R$ and $\alpha_k \neq 0$ for $k = 0, 1, 2, ...$ without using the differential equations.

The non-classical operational calculus mentioned above enables us also to analyze the stability of such control systems. However, in this respect the Z-transform does not offer such opportunities.

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2. Determination of the response by using the *Z*-transform

Let us take into consideration the non-stationary discrete time-system – (2). Let us determine its response in the case if

$$a_2 = 2, a_1 = 3, a_0 = 1, \{u_k\} = \{1\}$$

and the system is encumbered with conditions

$$x_0 = a, x_1 = b. \tag{6}$$

Taking advantage of *Z*-transform in system (2) we have

$$2z^2(z-1)^2 X''(z) + z(z-1)(4z-1)X'(z) = \frac{z}{z-1}.$$

Thus, we have obtained a differential equation in the form of

$$2z(z-1)X''(z) + (4z-1)X'(z) = \frac{1}{(z-1)^2} \tag{7}$$

for the transform of response x_k being looked for. The differential Eq. (7) should be provided with the conditions for the transform corresponding to conditions (6). They have the following form

$$\lim_{z\to\infty} X(z) = x_0 = a, \lim_{z\to\infty}(-z^2X'(z)) = x_1 = b. \tag{8}$$

From the differential Eq. (7) with conditions (8) (after long and troublesome calculations) we have obtained the transform of the sought response in the form

$$X(z) = (2b-2)\sqrt{\frac{z}{z-1}} + \frac{1}{z-1} + a - 2b + 2,$$

then

$$x_k = Z^{-1}[X(z)],$$

therefore

$$x_k = (2b-2)Z^{-1}\left[\sqrt{\frac{z}{z-1}}\right] + Z^{-1}\left[\frac{1}{z-1}\right] + (a-2b+2)Z^{-1}[1].$$

From the transforms table we obtain

$$x_k = \begin{cases} a & \text{for } k=0 \\ (2b-2)\frac{1\cdot3\cdot5\cdot\ldots\cdot(2k-1)}{2^k k!} + 1 & \text{for } k = 1,2,3,\ldots \end{cases}. \tag{9}$$

Let us now consider system (2) where $a_2 = -1$, $a_1 = 3$, $a_0 = 4$, $\{u_k\} = \{1\}$, with conditions (6).

Having *Z*-transform we obtain the following differential equations for the transform $X(z) = Z[x_k]$

$$z(z-1)X''(z) + 2(z+2)X'(z) = -\frac{1}{(z-1)^2},$$

whose solution with conditions (8) is

$$X(z) = \frac{b}{z-1} + \left(b - \frac{1}{4}\right) \cdot \left(\frac{2}{(z-1)^2} + \frac{2}{(z-1)^3} + \frac{1}{(z-1)^4} + \frac{\frac{1}{5}}{(z-1)^5}\right) + a.$$

To wind x_k one should calculate $Z^{-1}[X(z)]$. At this point it is necessary to explain that this causes a problem. Being equipped with the properties of the Z^{-1} transformation and the transforms tables it is impossible to carry out the operation.

Another case shows that the problems to be solved by using the Z-transform can occur already at some earlier stages. For this purpose let us consider the system

$$(k+2)(k+1)\Delta^2 x_k - 2(k+1)\Delta x_k + 2x_k = u_k \tag{10}$$

with conditions (6). (Here $a_2 = 1$, $a_1 = -3$, $a_0 = 2$, $\{u_k\} = \{1\}$.) The application of the Z-transform leads to the differential equation

$$z^2(z-1)^2X''(z) + 2z(z-1)(z+2)X'(z) + 6X(z) = \frac{z}{z-1},$$

which cannot be solved analytically.

Moreover, let us note that if the system has the form

$$a_3(k+3)(k+2)(k+1)\Delta^3 x_k + a_2(k+2)(k+1)\Delta^2 x_k + a_1(k+1)\Delta x_k + a_0 x_k = u_k,$$

then the use of the Z-transform will lead us to a differential equation of third order of variable coefficients the solution of which, in general, is impossible.

The applications of the Z-transform to some selected discrete systems, as has been outlined briefly before, show some evident problems to cope with. They are in each case, troublesome and time-consuming calculations, and pose problems connected with the determination of the inverse transform for $X(z)$, or create difficulties that start already at the outset of determining the $X(z)$ transform from the differential equation which as a rule has variable coefficients.

Taking into consideration the problems noted here it follows that one should try to look for methods that will avoid them, will be less difficult in the calculations, and will relatively soon lead to the target of our pursuit. One of such methods has been presented at point 6 of the paper. Points 3, 4 and 5 are an introduction to the method.

3. Non-classical operational calculus

The non-classical operational calculus owing to its generality can be used to describe and analyze the dynamical systems. In this paper advantage will be taken of the essential knowledge relating to the theory of the operational calculus mentioned earlier, that is necessary to analyze the discrete time-varying system (5). As has been mentioned before it will turn out that the use of this tool in the analysis of the discrete system (2) is more advantageous than the application of the Z-transform, as can be seen at point 2.

Definition 1. [3–5]. Axioms of non-classical operational calculus. Non-classical operational calculus is a set of $(L^0, L^1, S, T_q, s_q, Q)$, where $L^1 \subset L^0$ and L^0, L^1 are linear spaces. Linear operation $S: L^1 \to L^0, S(L^1) = L^0$, is called the generalized derivative. The set Q is a set of indices q for the linear operation $T_q : L^0 \to L^1$ and for the linear operation $s_q : L^1 \to KerS$ such that

$$ST_q f = f$$

$$T_q Sx = x - s_q x,$$

where $f \in L^0, x \in L^1, q \in Q$.
Operation T_q is called the generalized integral. Operation s_q is called the limit condition.

REMARK. One can also talk about families of operations $\{T_q\}_{q\in Q}$ and $\{s_q\}_{q\in Q}$ which satisfy some specified conditions.

Elements of the kernel of operation S, that is the set of

$$KerS \doteq \{c \in L^1 : Sc = 0\},$$

are called constants.
The elements of the set of constants constitution a linear subspace of space L^1.
Thus, we have

$$KerS \subset L^1 \subset L^0.$$

REMARK. Operation T_q is an injection.

EXAMPLE 1. Let $C(N)$ be a space of real or complex sequences $x = \{x_k\} = (x_0, x_1, x_2, ...)$. In paper [4,8] the generalized derivative (difference derivative) $S = \Delta$ is defined by the formula

$$\Delta\{x_k\} \doteq \{x_{k+1} - x_k\}, \{x_k\} \in C(N). \tag{11}$$

The derivative Δ is adequate to the operation T_{k_0} and the limit condition s_{k_0}. These operations are described by the formulas

$$T_{k_0}\{f_k\} \doteq \{r_k\}, \quad \text{where}$$

$$r_k \doteq \begin{cases} 0 & \text{for } k = k_0 \\ f_{k_0} + f_{k_0+1} + ... + f_{k-1} & \text{for } k > k_0, \{f_k\} \in C(N), \\ -f_{k_0-1} - f_{k_0-2} - ... - f_k & \text{for } k < k_0 \end{cases} \tag{12}$$

$$s_{k_0}\{x_k\} \doteq \{x_{k_0}\}, \{x_k\} \in C(N). \tag{13}$$

Formulas (11), (12), (13) indicate that in this case $L^1 = L^0 = C(N)$.

The derivative $S = \Delta$ has numerous properties – [7,8]. Among them is the following

$$\Delta\{f_k g_k\} = \{f_{k+1}\}\Delta\{g_k\} + (\Delta\{f_k\})\{g_k\},$$
$$\{f_k\}, \{g_k\} \in C(N).$$

For more models look at [3,5,6].

To be able to talk about the iteration of the derivative S, let us introduce the definition of space $L^m, m = 2, 3, ...$

DEFINITION 2.

$$L^m \doteq \left\{x \in L^{m-1} : Sx \in L^{m-1}\right\}, m = 2, 3, 4, ...$$

On the assumption that spaces L^0, L^1 are commutative algebras with unity it is possible to formulate the following theorem.

THEOREM 1. [5]. If the operational calculus is given

$$(L^0, L^1, S, T_q, s_q, Q),$$

where L^0, L^1 are commutative algebras with unity, therefore with the use of an invertible element $\alpha \in L^0$, the operations

$$\bar{S}x \doteq \alpha Sx, \quad x \in L^1$$
$$\bar{T}_q f \doteq T_q(\alpha^{-1} f), \quad f \in L^0$$
$$\bar{s}_q x \doteq s_q x, \quad x \in L^1$$

are also satisfied by axioms of the operational calculus.

COROLLARY 1. The set $C(N)$ with simple operations: summation and multiplication is an algebra. From the last theorem and from Example 1 it follows that the three operations

$$\bar{\Delta}\{x_k\} \doteq \{\alpha_k\}\Delta\{x_k\}, \tag{14}$$

$$\bar{T}_{k_0}\{f_k\} \doteq T_{k_0}\left\{\frac{f_k}{\alpha_k}\right\}, \tag{15}$$

$$\bar{s}_{k_0}\{x_k\} \doteq s_{k_0}\{x_k\}, \tag{16}$$

if $\{\alpha_k\} \in C(N), \alpha_k \neq 0$ for $k = 0, 1, 2, ...$ are also fulfilled by the axioms of the non-classical operational calculus.

The operational calculus constructed in this manner, together with its properties is used in the paper at point 6 for the analysis of the discrete system of the form (5), which after the use of operation $\bar{\Delta}$ and properties of operation Δ can be written as follows

$$a_2\bar{\Delta}^2 x_k + a_1\bar{\Delta}x_k + a_0 x_k = u_k \tag{17}$$

4. Non-classical Heaviside operator and its properties

One of the methods to determine the response of the generalized dynamical systems is the application of the non-classical Heaviside operator – [3,4,5]. To define non-classical Heaviside operator is necessary to accept the notions of the result and the operator.

DEFINITION 3. [4,5]. By semi-group of commutative endomorphisms of linear space X over field K is meant a fixed set $\prod(X)$ of endomorphisms of space X, such that for any arbitrary $U_1, U_2, U_3 \in \prod(X)$ the following conditions are fulfilled

$$(U_1U_2)U_3 = U_1(U_2, U_3)$$
$$U_1U_2 = U_2U_1$$
$$(U_1 x = 0) \Rightarrow (x = 0) \text{ for } x \in X.$$

DEFINITION 4. [4,5]. Fractions given below will be referred to as the results

$$\varsigma \doteq \frac{f}{U}, \quad \text{where} \quad f \in X, U \in \prod(X),$$

using the following definitions of equalities and operations

$$\frac{f}{U} \doteq \frac{g}{V} \quad \text{if and only if} \quad Vf = Ug$$
$$\frac{f}{U} + \frac{g}{V} \doteq \frac{Vf + Ug}{UV}$$
$$\tau\frac{f}{U} \doteq \frac{\tau f}{U}$$
$$V\left(\frac{f}{U}\right) \doteq \frac{Vf}{U}$$

for $f, g \in X$ and $U, V \in \prod(X), \tau \in K$.

REMARK. The elements of space X can be identified with such elements as

$$\frac{f}{I}$$

from the results space, because the mapping

$$f \mapsto \frac{f}{I}$$

is isomorphism, where $I = id \in \prod(X)$ is an identity operation. Therefore space X can be regarded as a linear subspace of the results space and elements of space X can be treated as the results. It should be pointed out, following the information of [4,5] that the results space is richer in elements than the outlet space X.

Let W be endomorphism of space X commutative with injections $U \in \Pi(X)$.

DEFINITION 5. [4,5]. By operator $\mu = \frac{W}{U}$ of the results space is meant an operation described by the formula

$$\mu\left(\frac{f}{V}\right) \doteq \frac{Wf}{UV},$$

where $f \in X$ and $U, V \in \prod(X)$.

REMARK. From the definition of the operator it follows that the sum of the operators, the product of the operator by a number, and the composition of the operators is again an operator. If the division is carried out by operators $\frac{U_1}{U_2}$ whose numerators U_1 re injections as multiplication by the inverse of the operator, then we will also obtain an operator.

DEFINITION 6. [4,5]. The operator p_q with the generalized integral T_q

$$p_q \doteq \frac{I}{T_q}$$

is known as the generalized Heaviside operator. From the definition of the Heaviside operator it follows that

$$T_q = \frac{I}{p_q}.$$

Operator p_q is defined correctly, due to the assumption made at point 3, generalized integral T_q is an injection. The theorem given below expresses the relation between the generalized derivative S and the Heaviside operator.

THEOREM 2. If $x \in L^n$, then

$$S^n x = p_q^n x - p_q^n x_{0,q} - p_q^{n-1} x_{1,q} - ... - p_q x_{n-1,q},$$

where $x_{i,q} = s_q S^i x, i = 0, 1, ..., n-1$. The proof of the theorem can be found in [4].

COROLLARY 2. For $n = 1$ we have

$$Sx = p_q x - p_q x_{0,q}, \quad x \in L^1.$$

For $n = 2$ we have

$$S^2 x = p_q^2 x - p_q^2 x_{0,q} - p_q x_{1,q}, \quad x \in L^2.$$

For $n = 3$ we have

$$S^3 x = p_q^3 x - p_q^3 x_{0,q} - p_q^2 x_{1,q} - p_q x_{2,q}, \quad x \in L^3.$$

The relation given in Theorem 2, or its particular cases used in Corollary 2 enable us at point 6 to determine the response of the discrete system in an operational form.

5. Generalized exponential functions

In order to determine the generalized exponential functions necessary to analyze the systems use will be made of an indispensable definition and an auxiliary theorem.

If endomorphism R is a logarithm (see [4,5]), then the solution of the abstract differential equation

$$Sx = Rx, x \in L^1 \tag{18}$$

with condition

$$s_q x = c, c \in KerS \tag{19}$$

exists or it does not, but if it exists, then there is only one solution (see [4,5]).

Let R be a logarithm and let a solution of Eq. (18) with condition (19) exist. The solution will be denoted by

$$x \doteq e^{Rt_q} c.$$

Function $e^{Rt_q} : KerS \to L^1$ given by formula $c \mapsto e^{Rt_q} c$ is called generalized exponential function (see [4,5]). The generalized exponential function can be presented in operational form. This fact results from the theorem.

THEOREM 3. [4,5]. The generalized exponential function can be expressed in terms of operator $\frac{I}{I - T_q R} = \frac{p_q}{p_q - R}$ in the following way (provided that the result $\frac{c}{I - T_q R}$ is an element of space L^0)

$$e^{Rt_q} c = \frac{c}{I - T_q R} = \frac{p_q}{p_q - R} c, \quad c \in KerS.$$

Let us consider some examples of the generalized exponential functions for selected models of operational calculus.

EXAMPLE 2. The generalized exponential function for the model from Example 1 is defined by the formula

$$e^{at_q}\{x_{k_0}\} = \left\{(1 + a)^{k - k_0} x_{k_0}\right\}.$$

Endomorphism R is multiplication by a real or complex number a.

(For $k_0 = 0$ element a is arbitrary, for $k_0 \geqslant 1$ element $a \neq -1$.)

EXAMPLE 3. The generalized exponential function used in the model of the operational calculus with derivative $\bar{\Delta}$ from Corollary 1 described on the basis of formulas (18), (19) has the form

$$e^{at_q}\{x_0\} = \left\{x_0 \prod_{i=0}^{k-1}\left(1 + \frac{a}{\alpha_i}\right)\right\}, \quad \prod_{i=0}^{-1}(\cdot) \doteq 1, \tag{20}$$

where endomorphism R is multiplication by a real or complex number a.

On the basis of Theorem 3 this function can be expressed by the use of the Heaviside operator p_q corresponding to integral $\bar{T}_{k_0}, k_0 = 0$, defined by Corollary 1 by formula (15). The above denotation is as follows

$$e^{at_q}\{x_0\} = \frac{p_q}{p_q - aI}\{x_0\}. \tag{21}$$

The obtained relationships (20), (21) are significant on account of the analysis of the discrete system (17).

6. Application of the generalized Heaviside operator to the solution of the problem

The determination of the response of the system (2) with $a_2 = 2, a_1 = 3, a_0 = 1, \{u_k\} = \{1\}$ and conditions (6) will be carried out by non-classical methods of the operational calculus. The above system and its conditions must be written in the language used for the operational calculus. For this purpose advantage will be taken of the operational calculus model given by Corollary 1 with derivative $\bar{\Delta}$ defined by formula (14), where $\alpha_k \doteq k + 1$. According to formula (17) it is possible to present it in the form

$$2\bar{\Delta}^2 x_k + 3\bar{\Delta} x_k + x_k = u_k, \ \ \{u_k\} = \{1\}. \qquad (22)$$

Conditions (6) are satisfied by the ones formulated by the use of the limit condition (16) for $k_0 = 0$. They are as follows

$$\bar{s}_{k_0} x_k = x_0 = a \qquad (23)$$

$$\begin{aligned} \bar{s}_{k_0} \bar{\Delta} x_k &= s_{k_0}((k+1)\Delta x_k) = s_{k_0}(k+1) s_{k_0} \Delta x_k \\ &= s_{k_0}(x_{k+1} - x_k) = x_1 - x_0 = b - a. \end{aligned} \qquad (24)$$

REMARK. It is worth mentioning that the values obtained from formulas (23) and (24) mean constant sequences. How it was said before in point 1 instead of writing sequence $\{x_k\}$ we will often write x_k.

Making use of the equalities of Corollary 2 system (22) with conditions (23) and (24) can be recorded in the following operational form

$$2(p_q^2 x_k - p_q^2 a - p_q(b-a)) + 3(p_q x_k - p_q a) + x_k = 1, \quad (25)$$

where p_q is the Heaviside operator corresponding to the integral defined by formula (15) for $q = k_0 = 0$ and $\alpha_k = k+1$. (Here $x_{0,q} = a, \ \ x_{1,q} = b - a$.) On the basis of equation (25) it follows that the response of our system can be expressed by operator form

$$x_k = \frac{2p_q^2 a + 2p_q(b-a) + 3p_q a + 1}{2p_q^2 + 3p_q + I},$$

that is, after some transformation we have

$$\begin{aligned} x_k = a\frac{p_q^2}{(p_q+1)(p_q+\frac{1}{2})}1 + \frac{2b+a}{2}\frac{p_q}{(p_q+1)(p_q+\frac{1}{2})}1 \\ + \frac{1}{2}\frac{I}{(p_q+1)(p_q+\frac{1}{2})}1. \end{aligned} \qquad (26)$$

REMARK. The expression $p_q + d$ should be understood as $p_q + dI$.

Making use of the tables [5] the values appearing in formula (26) can be written by means of some generalized exponential functions. They will take the form

$$\frac{p_q^2}{(p_q+1)\left(p_q+\frac{1}{2}\right)} = \frac{-e^{t_q} + \frac{1}{2}e^{-\frac{1}{2}t_q}}{-1+\frac{1}{2}},$$

$$\frac{p_q}{(p_q+1)\left(p_q+\frac{1}{2}\right)} = \frac{e^{-t_q} - e^{-\frac{1}{2}t_q}}{-1+\frac{1}{2}},$$

$$\frac{I}{(p_q+1)\left(p_q+\frac{1}{2}\right)} = \frac{-\frac{1}{2}e^{-t_q} + e^{-\frac{1}{2}t_q}}{-1\left(-\frac{1}{2}\right)\left(-1+\frac{1}{2}\right)} + \frac{1}{(-1)\left(-\frac{1}{2}\right)}.$$

After putting the last relations into formula (26) and some transformations we obtain the formula for the response expressed in terms of generalized exponential functions. The form of the response is as follows

$$\{x_k\} = (a - 2b + 1)e^{-t_q}\{1\} + (2b-2)e^{-\frac{1}{2}t_q}\{1\} + \{1\}. \quad (27)$$

According to formula (20) we have

$$e^{-t_q}\{1\} = \left\{\prod_{i=0}^{k-1}\left(1 - \frac{1}{i+1}\right)\right\} = \{d_k\},$$

$$e^{-\frac{1}{2}t_q}\{1\} = \left\{\prod_{i=0}^{k-1}\left(1 - \frac{1}{2(i+1)}\right)\right\}.$$

REMARK. Sequence $\{d_k\}$ denotes a sequence of the form (1, 0, 0, 0, ...).

Inserting the last values to (27) and transforming them yields

$$x_k = \begin{cases} (a - 2b + 1)\,1 + 2(b-1)1 + 1 \ \text{for} \quad k = 0 \\ 2(b-1)\prod_{i=0}^{k-1}\left(1 - \frac{1}{2(i+1)}\right) + 1 \ \text{for} \ k = 1, 2, \ldots \end{cases}.$$

Finally

$$x_k = \begin{cases} a & \text{for} \quad k = 0 \\ (b-1)\frac{3\cdot 5\cdot \ldots \cdot (2k-1)}{2^{k-1}k!+1} & \text{for} \ k = 1, 2, \ldots \end{cases} \qquad (28)$$

is the response of system (2) with conditions (6), if $a_2 = 2$, $a_1 = 3$, $a_0 = 1$, $\{u_k\} = \{1\}$.

The graph of the discrete signal x_k for $a = 2, b = -1$ is shown in Fig. 1.

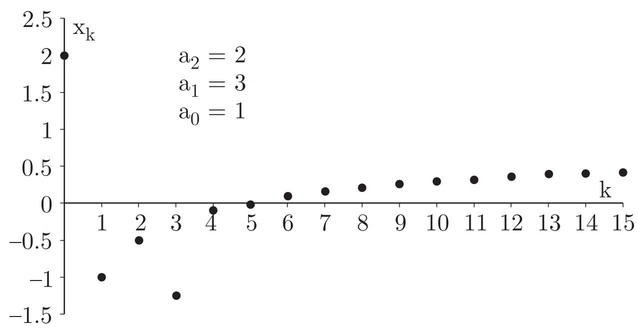

Fig. 1. The diagram of the discrete signal $x_k(a = 2, b = -1)$

Using the same procedure it is possible to determine the response of system (2) with conditions (6) where $a_2 = -1$, $a_1 = 3$, $a_0 = 4$, $\{u_k\} = \{1\}$. This system can be written in the form

$$-\bar{\Delta}^2 x_k + 3\bar{\Delta} x_k + 4x_k = 1$$

with conditions (23), (24). Taking advantage of the Heaviside operator p_q the relation can be expressed by

$$-(p_q^2 x_k - p_q^2 a - p_q(b-a)) + 3(p_q x_k - p_q a) + 4x_k = 1,$$

from which it follows that

$$\{x_k\} = \frac{-p_q^2 a - p_q(b-a) + 3p_q a + 1}{-(p_q+1)(p_q-4)} \{1\}$$
$$= a\frac{p_q^2}{(p_q+1)(p_q-4)}\{1\} + (b-4a)\frac{p_q}{(p_q+1)\,(p_q-4)}\{1\}$$
$$- \frac{I}{(p_q+1)\,(p_q-4)}\{1\}.$$

Furthermore from the tables we have

$$\{x_k\} = a\frac{-e^{-t_q} - 4e^{4t_q}}{-1-4}\{1\} + (b-4a)\frac{e^{-t_q} - e^{4t_q}}{-1-4}\{1\}$$
$$- \left(\frac{4e^{-t_q} + e^{4t_q}}{-1\cdot 4(-1-4)}\{1\} - \frac{1}{4}\{1\}\right) = \left(a - \frac{1}{5}b - \frac{1}{5}\right)$$
$$\cdot\, e^{-t_q}\{1\} + \left(\frac{1}{5}b - \frac{1}{20}\right) e^{4t_q}\{1\} + \frac{1}{4}\{1\}.$$

After making use of formula (20) the equality takes the form

$$x_k = \left(a - \frac{1}{5}b - \frac{1}{5}\right) d_k + \left(\frac{1}{5}b - \frac{1}{20}\right)\prod_{i=0}^{k-1}\left(1 + \frac{4}{i+1}\right) + \frac{1}{4}.$$

The graph of the discrete signal x_k for $a = 2, b = 1$ is shown in Fig. 2.

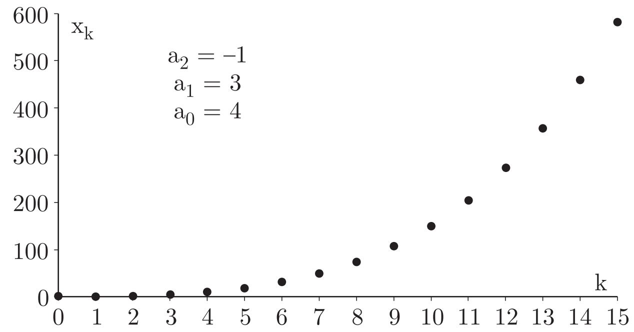

Fig. 2. The diagram of the discrete signal x_k $(a = 2, b = 1)$

At this place it is worthwhile mentioning that in the case of applying the Z-transform to the system, there occurred the problem of determining the response x_k, since it was impossible to define the inverse transform. Here the determination of the response turned out well.

Another system that caused problems was system (10) with conditions (6). The application of the operational calculus to this case also appeared very quickly to be useful to reach our target because the system can be expressed by

$$\bar{\Delta}^2 x_k - 3\bar{\Delta} x_k + 2x_k = 1$$

with conditions (23), (24). Thus, we obtain

$$\{x_k\} = a\frac{p_q^2}{(p_q-2)(p_q-1)}\{1\}$$
$$+ (b-4a)\frac{p_q}{(p_q-2)\,(p_q-1)}\{1\}$$
$$+ \frac{I}{(p_q-2)\,(p_q-1)}\{1\}\ ,$$

that is

$$\{x_k\} = \left(b - 2a + \frac{1}{2}\right) e^{2t_q}\{1\} + (3a-b-1)e^{t_q}\{1\} + \frac{1}{2}\{1\},$$

and so

$$x_k = \left(b - 2a + \frac{1}{2}\right)\prod_{i=0}^{k-1}\frac{i+3}{i+1} + (3a-b-1)\prod_{i=0}^{k-1}\frac{i+2}{i+1} + \frac{1}{2}.$$

The graph of the discrete signal x_k for $a = 2, b = 1$ is shown in Fig. 3.

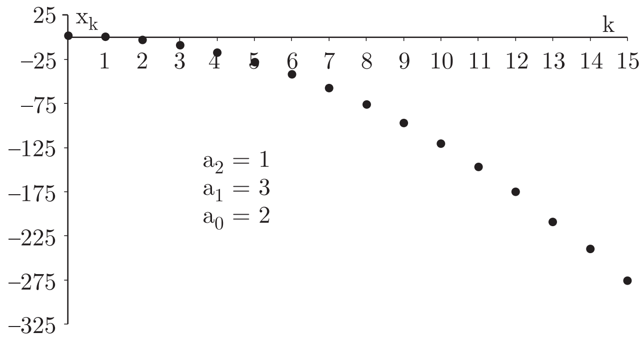

Fig. 3. The diagram of the discrete signal x_k $(a = 2, b = 1)$

As can be seen the presented method with the use of the Heaviside operator is effective, and soon leads us to the target without unnecessary differential equations whose solution might cause problems. This approach can perfectly be applied to each case when dealing with system (5), with any $\alpha_k \neq 0$ for $k = 0, 1, 2, \ldots$. Our analysis includes systems in which $\alpha_k = k+1$ for the purpose of comparing the proposed method with the Z-transform procedure. Using an arbitrary α_k the Z-transform procedure will not work. Moreover, the given conditions may be of the form

$$x_{k_0} = c,\ \ x_{k_0+1} = d,\ \ k_0 \geqslant 1.$$

It is worthy of mention that the application of the Heaviside operator has also proved effective in the case of a control system of third order in which the use of the Z-transform leads to a differential equation of third order of variable coefficients. This can be observed in the system

$$(k+3)(k+2)(k+1)\Delta^3 x_k + 6(k+2)(k+1) \cdot \Delta^2 x_k - 2(k+1)\Delta x_k - 8x_k = 1 \tag{29}$$

with conditions

$$x_0 = a,\ \ x_1 = b,\ \ x_2 = c.$$

The use of the derivative $\bar{\Delta}$ gives the following system

$$\bar{\Delta}^3 x_k + 3\bar{\Delta}^2 x_k - 6\bar{\Delta} x_k - 8x_k = 1$$

with conditions (23), (24) provided with the requirement

$$\bar{s}_{k_0}\bar{\Delta}^2 x_k = a - 3b + 2c.$$

Having made use of the Heaviside operator the response takes the form

$$x_k = \begin{cases} a & \text{for } k = 0 \\ b & \text{for } k = 1 \\ c & \text{for } k = 2 \\ \frac{8}{9}b + c + \frac{1}{6} & \text{for } k = 3 \\ -\frac{1}{8} + \left(\frac{1}{9}b + \frac{1}{9}c + \frac{1}{36}\right)\frac{(k+1)(k+2)}{2} & \text{for } k = 4, 5, 6, \ldots \end{cases}.$$

7. Summary and conclusions

The use of the non-classical operational (calculus) methods make it possible to determine the response of the discrete systems also in situations where the application of the Z-transform is limited or impossible.

In other cases the use of the generalized Heaviside operator to find the response is compatible with respect to the Z-transform. An advantage of the Heaviside operator is the short time it takes and ease of calculations. Moreover, the method can be employed when the conditions in the given values do not necessarily refer to the initial terms of the sequence - see formula (4). The input signal $\{u_k\} = \{1\}$ used for the considerations does not introduce any limitations due to the fact that in the case of an arbitrary signal $\{u_k\}$ in a model with derivative Δ it is only necessary to make use of a convolution [4] or to apply it directly. For example the response of system (10) with conditions (6), where discrete input signal $\{u_k\} = (0, 0, 1, 1, 1, \ldots)$, has the following form

$$x_k = \left(b - 2a + \frac{1}{12}\right)\prod_{i=0}^{k-1}\frac{i+3}{i+1} + (3a - b - \frac{1}{3})\prod_{i=0}^{k-1}\frac{i+2}{i+1} + \frac{1}{12}\prod_{i=0}^{k-1}\frac{i-1}{i+1} - \frac{1}{3}\prod_{i=0}^{k-1}\frac{i}{i+1} + \frac{1}{2}.$$

By the application of the methods presented in the paper it is possible to determine, in compliance with the principle adopted in [5], the generalized transmittance of the system.

The research apparatus described in the paper can be utilized in the stability study of systems concerned using for this purpose the results give in [5]. As it follows from the results, the asymptotical stability of the system is determined in this case by generalized exponential functions associated with the system. This indicates that if the generalized exponential functions related to the system are asymptotically stable, then the system is also asymptotically stable. Thus, the asymptotical stability of our systems depends on the exponential functions that have form (20). On the basis of this formula it is possible to assert that: if

$$\left|1 + \frac{a}{\alpha_i}\right| < 1 \text{ for } i \geqslant i_0, \tag{30}$$

then the generalized exponential functions are bounded sequences.

The free system corresponding to:

- system (2) is asymptotically stable (when $a_2 = 2$, $a_1 = 3$, $a_0 = 1$) due to the fact that $a = -1$ or $a = \frac{1}{2}$,
- system (2) is asymptotically unstable (when $a_2 = -1$, $a_1 = 3$, $a_0 = 4$) because $a = -1$ or $a = 4$,
- system (10) is asymptotically unstable for the reason that $a = 1$ or $a = 2$,
- system (29) is asymptotically unstable because $a = -1$ or $a = -4$ or $a = 2$.

In all the above cases $\alpha_k = k + 1$.

It is obvious that stability can be determined without defining the response of the system and, as it was possible to prove, the method of its analysis is quite simple and depends in this case only on the determination of values a occurring in formula (21) or (20). For instance, system

$$4^{k+1}\Delta^2 x_k + (2 \cdot 4^k + 3 \cdot 2^k)\Delta x_k + x_k = 0$$

is stable, because it is equivalent to system (5)

$$\left(\alpha_k = 2^k, \;\; a_2 = 1, \;\; a_1 = \frac{3}{2}, \;\; a_0 = \frac{1}{2}\right)$$

and the generalized exponential functions, as it follows from condition (30), are bounded. In this case

$$a = -1 \quad \text{or} \quad a = -\frac{1}{2}.$$

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