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Positivity of fractional descriptor linear continuous-time systems

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Abstract. The positivity of fractional descriptor linear continuous-time systems is investigated. The solution to the state equation of the systems is derived. Necessary and sufficient conditions for the positivity of fractional descriptor linear continuous-time systems are established. The considerations are illustrated by numerical examples.

Key words: fractional, descriptor, linear, continuous-time, system, stability, solution, positivity.

1. Introduction

A dynamical system is called positive if its state variables take nonnegative values for all nonnegative inputs and nonnegative initial conditions. The positive linear systems have been investigated in [1–3] and positive nonlinear systems in [4–8].

Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems with different fractional orders have been addressed in [9–11]. Descriptor (singular) linear systems have been analyzed in [6, 12–16] and the stability of a class of nonlinear fractional-order systems in [4, 17, 18]. Fractional positive continuous-time linear systems and their reachability have been addressed in [19]. Application of Drazin inverse to analysis of descriptor fractional discrete-time linear systems has been presented in [20]. The robust stabilization of discrete-time positive switched systems with uncertainties has been addressed in [21]. Comparison of three methods of analysis of the descriptor fractional systems has been presented in [22]. Stability of linear fractional order systems with delays has been analyzed in [23] and simple conditions for practical stability of positive fractional systems have been proposed in [24]. The stability of interval positive continuous-time linear systems has been addressed in [25].

In this paper the positivity of fractional descriptor continuous-time linear systems will be investigated.

The paper is organized as follows. In section 2 the basic definitions of Drazin inverse of matrices are recalled and the solution to the state equation of the systems is derived. The necessary and sufficient conditions for the positivity of the fractional descriptor linear continuous-time systems are established in section 3. Concluding remarks are given in section 4.

The following notations will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Fractional descriptor linear continuous-time systems

Consider the fractional descriptor linear system

$$E\frac{d^\alpha x}{dt^\alpha} = Ax + Bu, \ 0 < \alpha < 1, \tag{1a}$$

$$y = Cx, \tag{1b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors and $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$ and

$$\frac{d^\alpha x}{dt^\alpha} = \frac{1}{\Gamma(\alpha-1)}\int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha}dt, \ \dot{x}(\tau) = \frac{dx(\tau)}{d\tau} \tag{1c}$$

is the Caputo derivative of the order α,

$$\Gamma(x) = \int_0^\infty e^{-t}t^{\alpha-1}dt, \tag{1d}$$

is the gamma function.

It is assumed that

$$\det[Es - A] \neq 0 \ \text{ for some } \ s \in \mathbf{C} \tag{2}$$

where $\mathbf{C}$ is the field of complex numbers.

In this case the equation (1a) has unique solution [16, 22].

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Definition 1. For any matrix $\bar{E} \in [E\lambda - A]^{-1}E \in \Re^{n\times n}$ there exists a unique Drazin inverse $\bar{E}^D \in \Re^{n\times n}$ defined by the conditions

$$\bar{E}^D\bar{E} = \bar{E}\bar{E}^D, \tag{3a}$$

$$\bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D, \tag{3b}$$

$$\bar{E}^D\bar{E}^{\mu+1} = \bar{E}^{\mu}, \tag{3c}$$

where μ is the smallest nonnegative integer such that

$$\operatorname{rank} \bar{E}^{\mu} = \operatorname{rank} \bar{E}^{\mu+1} \tag{3d}$$

and λ is chosen so that the matrix $[E\lambda - A]$ is invertible. It is well-known [12, 22] that for

$$P = \bar{E}^D\bar{E} \quad \text{and} \quad \hat{A} = \bar{E}^D[E\lambda - A]^{-1}A \tag{4}$$

the following relations hold

$$P^k = P \quad \text{for} \quad k = 2, 3, \dots, \tag{5a}$$

$$P\hat{A} = \hat{A}P = \hat{A}, \tag{5b}$$

$$Px(t) = x(t), \quad t \geq 0. \tag{5c}$$

Premultiplying (1a) by the matrix $\bar{E}^D[E\lambda - A]^{-1}$ we obtain

$$P\frac{d^{\alpha}x}{dt^{\alpha}} = \hat{A}x + \hat{B}u, \tag{6a}$$

where

$$\hat{B} = \bar{E}^D[E\lambda - A]^{-1}B. \tag{6b}$$

Applying Laplace transform ($\mathcal{L}$) to the equation (6a) and taking into account that

$$\mathcal{L}\left[\frac{d^{\alpha}x}{dt^{\alpha}}\right] = s^{\alpha}X(s) - s^{\alpha-1}x(0), \quad x(0) = Pc = \operatorname{im} P, \quad c \in \Re^n \text{ – arbitrary} \tag{7}$$

we obtain

$$\left[Ps^{\alpha} - \hat{A}\right]X(s) = Ps^{\alpha-1}x(0) + \hat{B}U(s), \tag{8}$$

where

$$X(s) = \mathcal{L}\left[x(t)\right] = \int_0^{\infty} x(t)e^{-st}dt, \quad U(s) = \mathcal{L}\left[u(t)\right]. \tag{9}$$

Note that

$$\begin{aligned}\left[Ps^{\alpha} - \hat{A}\right]X(s) &= \left[I_n s^{\alpha} - \hat{A}\right]PX(s) = \\ &= \left[I_n s^{\alpha} - \hat{A}\right]X(s),\end{aligned} \tag{10}$$

since by (5b) and (5c) $\hat{A} = \hat{A}P$ and $PX(s) = X(s)$.

Taking into account (10) from (8) we obtain

$$\begin{aligned}X(s) = &\left[I_n s^{\alpha} - \hat{A}\right]^{-1}Ps^{\alpha-1}x(0) + \\ &+ \left[I_n s^{\alpha} - \hat{A}\right]^{-1}\hat{B}U(s).\end{aligned} \tag{11}$$

It is easy to show that [17]

$$\left[I_n s^{\alpha} - \hat{A}\right]^{-1} = \sum_{k=0}^{\infty} \hat{A}^k s^{-(k+1)\alpha}. \tag{12}$$

Substituting (12) into (11) we obtain

$$X(s) = \sum_{k=0}^{\infty} \hat{A}^k s^{-(k\alpha+1)}x(0) + \sum_{k=0}^{\infty} \hat{A}^k \hat{B} s^{-(k+1)\alpha}U(s) \tag{13}$$

since $Px(0) = x(0)$.

Applying to (13) the inverse Laplace transform $\left(\mathcal{L}^{-1}\right)$ and the convolution theorem we obtain

$$x(t) = \Phi_0(t)Pc + \int_0^t \Phi(t-\tau)\hat{B}u(\tau)d\tau, \tag{14a}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} \hat{A}\mathcal{L}^{-1}\left[s^{-k(\alpha+1)}\right] = \sum_{k=0}^{\infty} \frac{\hat{A}^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{14b}$$

$$\Phi(t) = \sum_{k=0}^{\infty} \hat{A}\mathcal{L}^{-1}\left[s^{-(k+1)\alpha}\right] = \sum_{k=0}^{\infty} \frac{\hat{A}^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}, \tag{14c}$$

since $\mathcal{L}[t^{\alpha}] = \Gamma(\alpha+1)s^{-(\alpha+1)}$.
Therefore, the following theorem has been proved.

Theorem 1. The solution $x(t)$ of the equation (1a) is given by (14).

Example 1. Consider the fractional descriptor system (1) with

$$\begin{gathered}E = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ x(t) = \begin{cases} 1 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases} \quad x(0) \in \Re^2_+, \quad 0 < \alpha < 1.\end{gathered} \tag{15}$$

The system satisfies the assumption (2) since

$$\det\left[Es - A\right] = \begin{vmatrix} 0 & -1 \\ -s & s \end{vmatrix} = -s. \tag{16}$$

Choosing $\lambda = -1$ and using (15) we obtain

$$\bar{E} = [E\lambda - A]^{-1}E = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix},$$
$$\bar{A} = [E\lambda - A]^{-1}A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}. \tag{17}$$

In this case the Drazin inverse matrix $\bar{E}^D$ of $\bar{E}$ given by (17) has the form

$$\bar{E}^D = \bar{E} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}. \tag{18}$$

Using (4a), (6b), (17) and (18) we obtain

$$P = \bar{E}^D\bar{E} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \tag{19a}$$

and

$$\hat{A} = \bar{E}^D\bar{A} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\hat{B} = \bar{E}^D[E\lambda - A]^{-1}B = \tag{19b}$$
$$= \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Taking into account (19) from (14) we obtain

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{\hat{A}^k t^{k\alpha}}{\Gamma(k\alpha+1)} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{20a}$$

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{\hat{A}^k t^{k\alpha}}{\Gamma(k\alpha+1)} = I_2 \frac{t^{\alpha-1}}{\Gamma(\alpha)} = \begin{bmatrix} \frac{t^{\alpha-1}}{\Gamma(\alpha)} & 0 \\ 0 & \frac{t^{\alpha-1}}{\Gamma(\alpha)} \end{bmatrix}, \tag{20b}$$

and

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)\hat{B}u(\tau)d\tau =$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_0 + \int_0^t \begin{bmatrix} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} & 0 \\ 0 & \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} d\tau = \tag{21}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_0 + \int_0^t \begin{bmatrix} \frac{-(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \\ 0 \end{bmatrix} d\tau.$$

3. Positivity of fractional descriptor linear systems

The following lemma will be used in the further considerations.

Lemma 1. For the fractional linear system

$$\frac{d^\alpha z}{dt^\alpha} = Mz, \ M \in \Re^{n\times n}, \ 0 < \alpha < 1 \tag{22}$$

the following implication holds true

$$Fz(0) \in \Re_+^p \ \text{then} \ Fz(t) \in \Re_+^p$$
$$\text{for} \ F \in \Re^{p\times n} \ \text{and} \ t \geq 0 \tag{23}$$

if and only if there exists a Metzler matrix $H \in M_p$ such that

$$FM = HF. \tag{24}$$

Proof. Premultiplying (22) by the matrix F we obtain

$$F\frac{d^\alpha z}{dt^\alpha} = FMz. \tag{25}$$

The equation (25) has the solution $Fz \in \Re_+^p$ for $t \geq 0$ if and only if (24) holds true. In this case the equation

$$F\frac{d^\alpha z}{dt^\alpha} = HFz \tag{26}$$

has the solution $Fz \in \Re_+^p$, $t \geq 0$ if and only if $H \in M_p$. □

First let us consider the autonomous fractional descriptor system

$$E\frac{d^\alpha z}{dt^\alpha} = Ax \tag{27}$$

obtained from (1a) for $Bu = 0$.

Definition 2. The autonomous fractional descriptor system (27) is called (internally) positive if v for $t \geq 0$ and any admissible initial conditions $x(0) \in \Re_+^n (x(0) \in \Re_+^n)$.

Theorem 2. The fractional descriptor system (27) is positive if and only if there exists a matrix $G \in \Re^{n\times n}$ such that

$$H = \hat{A} + G(I_n - P) \in M_n, \tag{28}$$

where $\hat{A}$ and P are defined by (4).

Proof. By Lemma 1 the system (27) is positive if and only if there exists a Metzler matrix $H \in M_n$ such that

$$\hat{A} = HP. \tag{29}$$

The solution of equation (29) is given by (28) since by (5b) and (5a) $\hat{A}P = \hat{A}$, $P^2 = P$ and

$$HP = \hat{A}P + G(I_n - P)P = \hat{A}P = \hat{A}. \tag{30}$$

This completes the proof. □

Note that the system (27) can be positive even when the matrix $\hat{A}$ is not a Metzler matrix.

If $\hat{A} \in M_n$ then we have the following corollary.

Corollary 1. The fractional descriptor system (27) is positive if $\hat{A} \in M_n$. In this case we may choose in (28) $G = 0$.

Example 1. Consider the system (27) with

$$E = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \; 0 < \alpha < 1. \tag{31}$$

The assumption (2) is satisfied and for $\lambda = 0$ we have

$$\begin{gathered} \bar{E} = [-A]^{-1}E = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \\ \bar{A} = [-A]^{-1}A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \; \bar{E}^D = \bar{E} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}. \end{gathered} \tag{32}$$

Using (4) and (32) we obtain

$$\begin{gathered} P = \bar{E}^D\bar{E} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \\ \hat{A} = \bar{E}^D\bar{A} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}. \end{gathered} \tag{33}$$

Note that the matrix $\hat{A}$ defined by (33) is a Metzler matrix. Therefore, by Corollary 1 the system (27) with (31) is positive.

Example 2. Consider the system (27) with

$$E = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \; 0 < \alpha < 1. \tag{34}$$

The assumption (2) is satisfied and for $\lambda = 0$ we have

$$\begin{gathered} \bar{E} = [-A]^{-1}E = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \\ \bar{A} = [-A]^{-1}A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \; \bar{E}^D = \bar{E} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}. \end{gathered} \tag{35}$$

Using (4) and (35) we obtain

$$P = \bar{E}^D\bar{E} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \; \hat{A} = \bar{E}^D\bar{A} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}. \tag{36}$$

Note that the matrix $\hat{A}$ given by (36) is not a Metzler matrix. Using (28) and (36) we choose the matrix

$$G = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}. \tag{37}$$

so that the matrix

$$\begin{gathered} H = \hat{A} + G(I_n - P) = \\ = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \end{gathered} \tag{38}$$

is a Metzler matrix.

In general case the positivity of the fractional descriptor system (1) is defined as follows.

Definition 3. The fractional descriptor system (1) is called (internally) positive if $x(t) = \Re_+^n$ and $y(t) \in \Re_+^p$, $t \geq 0$ for any admissible initial conditions $x(0) \in \Re_+^n (x(0) \in \operatorname{im} P)$ and $u(t) \in \Re_+^m$, $t \geq 0$.

Theorem 3. The fractional descriptor system (1) is positive if and only if there exists a matrix $G \in \Re^{n \times n}$ such that (28) holds true and

$$\hat{B} \in \Re_+^{n \times m}, \;\; C \in \Re_+^{p \times n}. \tag{39}$$

Proof. The proof of (28) is the same as of Theorem 2. Note that

$$\int_0^t \Phi(t-\tau)\hat{B}u(\tau)d\tau \in \Re_+^n, \;\; t \geq 0 \tag{40}$$

if and only if $\hat{B} \in \Re_+^{n \times m}$ since $u(t) \in \Re_+^m$, $t \geq 0$ is arbitrary. Similarly, $y(t) \in \Re_+^p$, $t \geq 0$ if and only if $C \in \Re_+^{p \times n}$ since $x(t) = \Re_+^n$, $t \geq 0$ can be arbitrary. □

4. Concluding remarks

The positivity of fractional descriptor linear discrete-time has been investigated. The solution to the state equation of the fractional descriptor linear continuous-time systems has been derived (Theorems 1). Necessary and sufficient conditions for the positivity of the fractional descriptor linear continuous-time systems has been established (Theorems 2 and 3). The considerations have been illustrated by numerical examples.

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