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A separation principle for Takagi-Sugeno control fuzzy systems

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An important application of state estimation is feedback control: an estimate of the state (typically the mean estimate) is used as input to a state-feedback controller. This scheme is known as observer based control, and it is a common way of designing an output-feedback controller (i.e. a controller that does not have access to perfect state measurements). In this paper, under the fact that both the estimator dynamics and the state feedback dynamics are stable we propose a separation principle for Takagi-Sugeno fuzzy control systems with Lipschitz nonlinearities. The considered nonlinearities are Lipschitz or meets an integrability condition which have no influence on the LMI to prove the stability of the associated closed-loop system. Furthermore, we give an example to ullistrate the applicability of the main result.

Key words: T-S model, PDC controller, fuzzy observer, separation principle

1. Introduction

Takagi-Sugeno fuzzy models [16] are nonlinear systems described by a set of if-then rules which gives local linear approximations of an underlying system. Such models can approximate or describe a wide class of nonlinear systems. Hence, it is important to study their stability or the synthesis of stabilizing controllers. For a few years, some systematic design algorithms have been developed to guarantee the control performance and system stability for the T-S fuzzy-model based controllers [2, 10, 14–16]. The authors in [12, 13, 24] studied fuzzy observer designs for fuzzy control systems for designing stabilizing output fuzzy controllers. In general, the T-S fuzzy systems based control technique is effective in the control of complex systems with nonlinearities [17–20].

A natural approach to the design of stabilizing state feedback controllers is to use the linear subsystems in the if-then rules considered by Tanaka-Sugeno [18]. Observers for fuzzy systems are also important when we wish to control systems

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using the output available. A certain form of observers is proposed and sufficient conditions for the asymptotic convergence have been given by [1–6].

In all these methods, the fuzzy model is usually obtained by the sector nonlinearity approach. In this case, it is assumed that the state variables belong to a compact set, and all nonlinearities are therefore bounded (see [8]). Moreover, it is assumed that the nonlinearities depend only on measurable variables in order to take benefit from a separation principle [3, 7, 21–24].

In this paper, we prove that the stability of the closed-loop can be simplified in two particular cases. In the first case, the non linearities may depend on the time and may be unbounded and they can depend on unmeasurable variables. Provided an integrability condition, some new results are obtained in particular a separation principle is verified. The new nonlinearities do not have any impact on the classical LMIs condition associated to the rest of the T-S fuzzy model, and if they are added to the structure of the observer, then they can be completely ignored. We show that, these results are true even if the variables of the nonlinearities depend on unmeasured variables. The second case concerns Lipschitz nonlinearities. It appears that all the previous results are valid at the cost of two small LMI conditions added to the classical LMI conditions.

These three results may enable to simplify the design of a T-S fuzzy model, by reducing the number of rules to describe the original system. It is also expected that the LMI conditions associated to the rest of the fuzzy model are easier to solve. The stability analysis in such schemes is performed by using the Lyapunov synthesis approach. This paper is organized as follows: In section 2 we recal the standard conditions of stability for the controller and the observer of a classical fuzzy model. These conditions are written as LMIs. Section 3 deals with the case of the integrability condition. First the new class of T-S fuzzy models is investigated. Then it is shown successively that the classical LMI condition are true for the controller and the observer, and then the separation principle is studied. Section 4 studies the case of Lipschitz non linearities. Section 5 gives an example to show the validity of the proposed approach.

2. A separation principle

We will use separation principle to denote the fact that the combination of a stable state estimator with a stable state-feedback controller yields a stable closed-loop fuzzy system. The observer-based controller design for a T-S fuzzy system subject to Lypschitz perturbation is investigated. First, a stabilizing fuzzy controller is constructed and an observer is designed to estimate the unknown system states. Then, the separation principle for a T-S fuzzy system subject to sufficient conditions on the Lipschitz constant is proposed. With the help of the improved separation principle, the observer and controller can be combined together to make the fuzzy system stable in closed-loop.

2.1. Preliminaries

Consider the following T.S fuzzy dynamic model:

$$\dot{x} = \sum_{i=1}^{r} \mu_i(z)(A_i x + B_i u), \tag{1}$$

$$y = \sum_{i=1}^{r} \mu_i(z) C_i x, \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $y \in \mathbb{R}^q$ is the output. The matrices A_i, B_i and C_i are of appropriate dimension, $r \geqslant 2$ is the number of rules, z is the premise vector which may include unmeasurable variables. It is assumed that $\mu_i(z) \geqslant 0$, for all $i = 1, \dots, r$ and $\sum_{i=1}^{r} \mu_i(z) = 1$, for all $t \geqslant 0$.

Many published results, concerning the control of the fuzzy system, are based on the parallel distributed compensation (PDC) principle [10, 12, 15]). The fuzzy system is assumed to be locally controllable and for most papers the premise vector z is assumed to depend only on measurable variables. Indeed, in this case a separation principle is available. The controller is defined as:

$$u = -\sum_{i=1}^{r} \mu_i(z) K_i x, \tag{3}$$

where $K_i \in \mathbb{R}^{n \times m}$ is the gain matrix.

The design work can be transformed into a convex problem which is efficiently solved by linear matrix inequalities optimization. Of course there are many papers that add performance criteria or use complex matrix properties to extend the results.

In the rest of the paper the type of fuzzy models is as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(z)\big(A_i x + B_i u + f_i(t, x, u)\big). \tag{4}$$

With the output y defined as in (2).

2.2. The integrability condition

The functions f_i are known. Provided some conditions detailed below, these functions can be neglected in the LMIs associated to the stability of the closed-loop, in contrast to a classical fuzzy modeling that would otherwise have to include them.

In this section the assumptions are:

$(\mathcal{A}_1)$

$$\|f_i(t,x,u) - f_i(t,y,u)\| \leqslant \alpha_i(t)\|x-y\|, \qquad i = 1,2,\ldots,r, \tag{5}$$

for all $t \geqslant 0$, $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ and $u \in \mathbb{R}^m$, where α_i are nonnegative continuous functions, such that for all $t \geqslant 0$,

$$\alpha(t) = \sum_{i=1}^{r} \alpha_i(t)$$

meets an integrability condition with a constant $M_\alpha > 0$,

$(\mathcal{A}_2)$

$$\int_0^{+\infty} \alpha(s)\,\mathrm{d}s < M_\alpha. \tag{6}$$

In this section 2, the functions f_i may be unbounded in time. Section 3, deals with the case where $(\mathcal{A}_1)$ is true for constant α_i (and so $(\mathcal{A}_2)$ is false).

Example 1. As an example of such unbounded term, let consider

$$f(t,x,u) = \alpha(t)\cos(u)x,$$

with

$$\alpha(t) = \begin{cases} 0 & \text{if } \ t \in \left[0, 2 - \dfrac{1}{8}\right], \\ n^4 t + (n - n^5) & \text{if } \ t \in \left[n - \dfrac{1}{n^3}, n\right], \quad n \geqslant 2, \\ -n^4 t + (n + n^5) & \text{if } \ t \in \left[n, n + \dfrac{1}{n^3}\right], \quad n \geqslant 2, \\ 0 & \text{if } \ t \in \left[n + \dfrac{1}{n^3}, (n+1) - \dfrac{1}{(n+1)^3}\right], \quad n \geqslant 2. \end{cases}$$

We deduce that

$$\int_0^{+\infty} \alpha(t)\,\mathrm{d}t = \sum_{n \geqslant 2} \frac{1}{n^2} < \infty$$

and

$$\|f(t,x,u) - f(t,y,u)\| \leqslant \alpha(t)\|x-y\|.$$

Therefore, assumption $(\mathcal{A}_2)$ is satisfied.

In the rest of the paper, we will study successively the design of the controller and the observer. Then, a separation principle is proved.

2.3. Construction of the controller

The closed-loop system with respect to the fuzzy control (3) is given by

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i^2 G_{ii} x + 2 \sum_{i<j} \mu_i \mu_j G_{ij} + \sum_{i=1}^{r} \mu_i f_i(t, x, u), \tag{7}$$

where

$$G_{ii} = A_i - B_i K_i$$

and

$$G_{ij} = \frac{1}{2}(A_i - B_i K_j + A_j - B_j K_i).$$

Now, is presented the condition to prove the stability of the closed-loop without an observer.

Theorem 1 *Suppose that* $(\mathcal{A}_1)$ *hold and there exist symmetric and positive definite matrices* P*, and* Q*, and some matrices* K_i*,* $i = 1, \ldots, r$*, such that the following inequalities hold,*

$$G_{ii}^T P + P G_{ii} < -Q, \qquad i, j = 1, \ldots, r, \tag{8}$$

and

$$G_{ij}^T P + P G_{ij} < -Q, \qquad 1 \leqslant i < j \leqslant r, \tag{9}$$

then the fuzzy closed-loop system (7) is guaranteed to be globally uniformly exponentially stable.

For the next, we define $\lambda_0 = \lambda_{\min}(Q)$, $\lambda_{\min}$ denoting the smallest eigenvalue of the matrix.

Remark 1 *(8) and (9) can be written as LMIs by a simple congruence as in [17], with* $X = P^{-1}$*,* $K_j = M_j P$ *and* $H = XQX$*.*

Proof. Consider the Lyapunov function candidate $V(x) = x^T P x$. It's derivative along the trajectories of system (7) is given by,

$$\dot{V}(x) = \sum_{i=1}^{r} \mu_i^2 x^T \left(G_{ii}^T P + P G_{ii} \right) x + 2 \sum_{i<j} \mu_i \mu_j x^T \left(G_{ij}^T P + P G_{ij} \right) x$$
$$+ 2 x^T P \sum_{i=1}^{r} \mu_i f_i(t, x(t), u(t)).$$

The first two terms on the right-hand side constitute the derivative of the Lyapunov function $V(x)$ with respect to the nominal system, while the third term is the effect of the unbounded time-varying term. On the one hand, we have

$$x^T\left(G_{ii}^T P + PG_{ii}\right)x \leqslant -\lambda_0\|x\|^2, \qquad i = 1, 2, \ldots, r,$$

and

$$x^T\left(G_{ij}^T P + PG_{ij}\right)x \leqslant -\lambda_0\|x\|^2, \qquad 1 \leqslant i < j \leqslant r.$$

It follows that,

$$\dot{V}(x) \leqslant -\lambda_0\|x\|^2 \sum_{i=1}^{r}\sum_{i=1}^{r} \mu_i\mu_j + 2x^T P \sum_{i=1}^{r} \mu_i f_i(t, x, u).$$

Since,

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j = 1,$$

then, we have

$$\dot{V}(x) \leqslant -\lambda_0\|x\|^2 + 2x^T P \sum_{i=1}^{r} \mu_i f_i(t, x, u).$$

On the other hand, by $(\mathcal{A}_1)$ we have

$$\left\|\sum_{i=1}^{r} \mu_i f_i(t, x, u)\right\| \leqslant \sum_{i=1}^{r} \mu_i\alpha_i(t)\|x\|.$$

Taking into account the above expressions, it follows that

$$\dot{V}(x) \leqslant -\lambda_0\|x\|^2 + 2\|x\|\|P\| \sum_{i=1}^{r} \mu_i\alpha_i(t)\|x\|.$$

Thus,

$$\dot{V}(x) \leqslant -\lambda_0\|x\|^2 + 2\|P\|\alpha(t)\|x\|^2.$$

Now, by taking $\|P\| = \lambda_{\max}(P)$, yields

$$\frac{\dot{V}(x)}{V(x)} \leqslant -\frac{\lambda_0}{\lambda_{\max}(P)} + 2\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\alpha(t). \tag{10}$$

Integrating between t_0 and t, one obtains for all $t \geqslant t_0$,

$$\int_{t_0}^{t} \frac{\mathrm{d}V(x(s))}{V(x(s))} \leqslant -\frac{\lambda_0}{\lambda_{\max}(P)}(t - t_0) + 2\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \int_{t_0}^{t} \alpha(s)\,\mathrm{d}s.$$

By using simple computations, one gets the following estimation of $\|x(t)\|$,

$$\|x(t)\| \leqslant \frac{\lambda_{\max}^{1/2}(P)}{\lambda_{\min}^{1/2}(P)} \|x(t_0)\| e^{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} M_\alpha} e^{-\frac{\lambda_0}{2\lambda_{\max}(P)}(t-t_0)}, \quad \text{for all} \quad t \geqslant t_0. \tag{11}$$

Hence, the system (7) is globally uniformly exponentially stable. □

Remark 2 *The inequality (6) in ($\mathcal{A}_1$) allows us to negotiate the case when the time varying nonlinearity is not necessarily uniformly bounded in t. This is in contrast with previous papers (see [11]), in which the terms f_i must be vanishing ($f_i(t,.) \to 0$ when $t \to \infty$).*

Now we investigate the observer part of the closed-loop, before providing a separation principle.

2.4. Observer design

In many practical control problems, the physical state variables of systems are partially or fully unavailable for measurement, since the state variables are not accessible by sensing devices and transducers are not available or very expensive. In such cases, observer based control schemes should be designed to estimate the state.

Let consider for (4) an observer of the form:

$$\dot{\hat{x}} = \sum_{i=1}^{r} \mu_i(z)\left(A_i\hat{x} + B_i u + f_i(t,\hat{x},u)\right) - \sum_{i=1}^{r} \mu_i(z) L_i(\hat{y} - y). \tag{12}$$

In this part u depends on x, the separation principle will directly prove the stability of the closed-loop. We also have $\hat{y}$ given by

$$\hat{y} = \sum_{i=1}^{r} \mu_i(z) C_i \hat{x}.$$

Subtracting (4) from (12), we have the system error

$$\dot{e} = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(z)\mu_j(z)(A_i - L_j C_i)e + \sum_{i=1}^{r} \mu_i\left(f_i(t,\hat{x},u) - f_i(t,\hat{x}-e,u)\right). \tag{13}$$

Thus,

$$\dot{e} = \sum_{i=1}^{r} \mu_i^2 \Upsilon_{ij} e + 2\sum_{i<j} \mu_i\mu_j(z)\Upsilon_{ij} e + \sum_{i=1}^{r} \mu_i(z)\left(f_i(t,\hat{x},u) - f_i(t,\hat{x}-e,u)\right),$$

where

$$\Upsilon_{ii} = A_i - L_i C_i,$$

and

$$\Upsilon_{ij} = \frac{1}{2}\left(A_i - L_j C_i + A_j - L_j C_i\right).$$

Then let consider the following theorem.

Theorem 2 *Suppose that* $(\mathcal{A}_1)$ *hold and there exist positive symmetric definite matrices* $\widetilde{P}$, $\widetilde{Q}$ *and some matrices* L_i, $i = 1, \ldots, r$, *such that the following inequalities hold,*

$$\Upsilon_{ii}^T \widetilde{P} + \widetilde{P}\Upsilon_{ii} < -\widetilde{Q}, \qquad i = 1, \ldots, r, \tag{14}$$

and

$$\Upsilon_{ij}^T \widetilde{P} + \widetilde{P}\Upsilon_{ij} < -\widetilde{Q}, \qquad 1 \leqslant i < j \leqslant r, \tag{15}$$

then the system error (13) *is guaranteed to be globally uniformly exponentially stable.*

We define also $\widetilde{\lambda}_0 = \lambda_{\min}(\widetilde{Q})$, $\lambda_{\min}$ denoting the smallest eigenvalue of the matrix.

Remark 3 *(14) and (15) can be written as LMIs by a simple congruence as in [17], with the terms* $\widetilde{X} = \widetilde{P}^{-1}$, $L_j = \widetilde{P}N_j$ *and* $\widetilde{H} = \widetilde{X}\widetilde{Q}\widetilde{X}$.

Proof. Consider the Lyapunov function candidate $V(x) = e^T\widetilde{P}e$. It's derivative with respect to time is given by,

$$\begin{aligned}\dot{V}(e) = &\sum_{i=1}^{r} \mu_i^2 e^T\left(\Upsilon_{ii}^T\widetilde{P} + \widetilde{P}\Upsilon_{ii}\right)e + 2\sum_{i<j}\mu_i\mu_j e^T\left(\Upsilon_{ij}^T\widetilde{P} + \widetilde{P}\Upsilon_{ij}\right)e \\ &+ 2e^T\widetilde{P}\sum_{i=1}^{r}\mu_i\left(f_i(t,\hat{x},u) - f_i(t,\hat{x}-e,u)\right).\end{aligned}$$

On the one hand, we have

$$e^T\left(\Upsilon_{ii}^T\widetilde{P} + \widetilde{P}\Upsilon_{ii}\right)e \leqslant -\widetilde{\lambda}_0\|e\|^2, \qquad i = 1, \ldots, r,$$

and

$$e^T\left(\Upsilon_{ij}^T\widetilde{P} + \widetilde{P}\Upsilon_{ij}\right)e \leqslant -\widetilde{\lambda}_0\|e\|^2, \qquad 1 < i < j < r.$$

Then, one gets

$$\dot{V}(e) \leqslant -\widetilde{\lambda}_0\|e\|^2\sum_{i=1}^{r}\sum_{i=1}^{r}\mu_i\mu_j + 2e^T\widetilde{P}\sum_{i=1}^{r}\mu_i\left(f_i(t,\hat{x},u) - f_i(t,\hat{x}-e,u)\right).$$

It follows that

$$\dot{V}(e) \leqslant -\widetilde{\lambda}_0\|e\|^2 + 2e^T\widetilde{P}\sum_{i=1}^{r}\mu_i\big(f_i(t,\hat{x},u) - f_i(t,\hat{x}-e,u)\big).$$

On the other hand, we have

$$\left\|\sum_{i=1}^{r}\mu_i\left(f_i(t,\hat{x},u) - f_i(t,\hat{x}-e,u)\right)\right\| \leqslant \sum_{i=1}^{r}\alpha_i(t)\|e\|.$$

Taking into account the above expressions, it follows that

$$\dot{V}(e) \leqslant -\widetilde{\lambda}_0\|e\|^2 + 2\|\widetilde{P}\|\alpha(t)\|e\|^2.$$

Then, similar to the controller case, we deduce that (12) is an observer for (4).□

2.5. Observer based controller

Now, to get a separation principle let consider the following theorem.

Theorem 3 *Under assumptions* $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, *and if conditions of theorems* 1 *and* 2 *hold, then the T.S fuzzy composite system*

$$\dot{\hat{x}} = \sum_{i=1}^{r}\mu_i(A_i\hat{x} + B_iu(\hat{x}) + f_i(t,\hat{x},u(\hat{x}))) - \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_jL_jC_ie, \tag{16}$$

$$\dot{e} = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(A_i - L_jC_i)e + \sum_{i=1}^{r}\mu_i\big(f_i(t,\hat{x},u(\hat{x})) - f_i(t,\hat{x}-e,u(\hat{x}))\big), \tag{17}$$

where $u(\hat{x}) = \sum_{i=1}^{r}\mu_iK_i\hat{x}$, *is globally uniformly exponentially stable.*

Proof. Let consider the Lyapunov function candidate

$$V(\hat{x},e) = V_1(\hat{x}) + \nu V_2(e),$$

for the composite system (16) and (17) where $V_1(\hat{x}) = \hat{x}^TP\hat{x}$ and $V_2(e) = e^T\widetilde{P}e$, and ν is a positive constant which will be chosen later.

On the one hand, the derivative of V along the trajectories of (16) and (17) is given by,

$$\dot{V}(\hat{x},e) = \nabla V_1(\hat{x})\left(\sum_{i=1}^{r}\mu_i\left(A_i\hat{x} + B_iu(\hat{x}) + f_i(t,\hat{x},u(\hat{x}))\right)\right)$$
$$- \nabla V_1(\hat{x})\left(\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_jL_jC_ie\right) + \nu\dot{V}_2(e).$$

Since

$$\|\nabla V_1(\hat{x})\| \leqslant 2\lambda_{\max}(P)\|\hat{x}\|,$$

thus, using the properties of V_1 and V_2 given in the above proofs, one gets

$$\begin{aligned}\dot{V}(\hat{x}, e) \leqslant& -\lambda_0\|\hat{x}\|^2 + 2\lambda_{\max}(P)\alpha(t)\|\hat{x}\|^2 + 2\lambda_{\max}(P)\sum_{i=1}^{r}\sum_{j=1}^{r}\|L_j\|\|C_i\|\|e\|\|\hat{x}\| \\ &- \nu\widetilde{\lambda}_0\|e\|^2 + 2\nu\lambda_{\max}(\widetilde{P})\alpha(t)\|e\|^2.\end{aligned}$$

Since for all $\varepsilon > 0$, we have

$$\|\hat{x}\|\|e\| \leqslant \frac{1}{2\varepsilon}\|\hat{x}\|^2 + \frac{\varepsilon}{2}\|e\|^2.$$

Then one gets

$$\begin{aligned}\dot{V}(\hat{x}, e) \leqslant& -\frac{\lambda_0}{\lambda_{\max}(P)}V_1(\hat{x}) + 2\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\alpha(t)V_1(\hat{x}) \\ &+ \frac{\lambda_{\max}(P)}{\varepsilon\lambda_{\min}(P)}\sum_{i=1}^{r}\sum_{j=1}^{r}\|L_j\|\|C_i\|V_1(\hat{x}) \\ &+ \frac{\varepsilon\lambda_{\max}(P)}{\lambda_{\min}(\widetilde{P})}\sum_{i=1}^{r}\sum_{j=1}^{r}\|L_j\|\|C_i\|V_2(e) - \frac{\nu\widetilde{\lambda}_0}{\lambda_{\max}(\widetilde{P})}V_2(e) \\ &+ 2\nu\frac{\lambda_{\max}(\widetilde{P})}{\lambda_{\min}(\widetilde{P})}\alpha(t)V_2(e).\end{aligned}$$

It follows that

$$\begin{aligned}\dot{V}(\hat{x}, e) \leqslant& -\left(\frac{\lambda_0}{\lambda_{\max}(P)} - \frac{\lambda_{\max}(P)}{\varepsilon\lambda_{\min}(P)}\sum_{i=1}^{r}\sum_{j=1}^{r}\|L_j\|\|C_i\|\right)V_1(\hat{x}) \\ &-\left(\frac{\widetilde{\lambda}_0}{\lambda_{\max}(\widetilde{P})} - \frac{\varepsilon\lambda_{\max}(P)}{\nu\lambda_{\min}(\widetilde{P})}\sum_{i=1}^{r}\sum_{j=1}^{r}\|L_j\|\|C_i\|\right)\nu V_2(e) \\ &+ 2\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\alpha(t)V_1(\hat{x}) + 2\nu\frac{\lambda_{\max}(\widetilde{P})}{\lambda_{\min}(\widetilde{P})}\alpha(t)V_2(e).\end{aligned}$$

Let

$$\varepsilon = 2\frac{\lambda_{\max}^2(P)}{\lambda_0\lambda_{\min}(P)}\sum_{i=1}^{r}\sum_{j=1}^{r}\|L_j\|\|C_i\|,$$

then

$$\begin{aligned}\dot{V}(\hat{x},e) \leqslant & -\frac{\lambda_0}{2\lambda_{\max}(P)}V_1(\hat{x}) \\ & -\left(\frac{\widetilde{\lambda}_0}{\lambda_{\max}(\widetilde{P})} - \frac{2\lambda_{\max}^3(P)}{\nu\lambda_0\lambda_{\min}(P)\lambda_{\min}(\widetilde{P})}\left[\sum_{i=1}^{r}\sum_{j=1}^{r}\|L_j\|\|C_i\|\right]^2\right)\nu V_2(e) \\ & + 2\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\alpha(t)V_1(\hat{x}) + 2\nu\frac{\lambda_{\max}(\widetilde{P})}{\lambda_{\min}(\widetilde{P})}\alpha(t)V_2(e).\end{aligned}$$

Let take

$$\nu = \frac{4\lambda_{\max}^3(P)\lambda_{\max}(\widetilde{P})\left[\sum_{i=1}^{r}\sum_{j=1}^{r}\|L_j\|\|C_i\|\right]^2}{\lambda_0\widetilde{\lambda}_0\lambda_{\min}(P)\lambda_{\min}(\widetilde{P})}.$$

Therefore, one obtains

$$\begin{aligned}\dot{V}(\hat{x},e) \leqslant & -\frac{\lambda_0}{2\lambda_{\max}(P)}V_1(\hat{x}) - \frac{\widetilde{\lambda}_0}{2\lambda_{\max}(\widetilde{P})}\nu V_2(e) \\ & + 2\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\alpha(t)V_1(\hat{x}) + 2\nu\frac{\lambda_{\max}(\widetilde{P})}{\lambda_{\min}(\widetilde{P})}\alpha(t)V_2(e).\end{aligned}$$

It follows that,

$$\begin{aligned}\dot{V}(\hat{x},e) \leqslant & -\min\left(\frac{\lambda_0}{2\lambda_{\max}(P)}, \frac{\widetilde{\lambda}_0}{2\lambda_{\max}(\widetilde{P})}\right)V(\hat{x},e) \\ & + 2\max\left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}, \frac{\lambda_{\max}(\widetilde{P})}{\lambda_{\min}(\widetilde{P})}\right)\alpha(t)V(\hat{x},e).\end{aligned}$$

Let

$$a = \min\left(\frac{\lambda_0}{2\lambda_{\max}(P)}, \frac{\widetilde{\lambda}_0}{2\lambda_{\max}(\widetilde{P})}\right) \quad \text{and} \quad b = \max\left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}, \frac{\lambda_{\max}(\widetilde{P})}{\lambda_{\min}(\widetilde{P})}\right).$$

The above expression gives also the following estimate:

$$\dot{V}(\hat{x},e) \leqslant -aV(\hat{x},e) + 2b\alpha(t)V(\hat{x},e).$$

Integrating between t_0 and t, one obtains for all $t \geqslant t_0$,

$$\int_{t_0}^{t} \frac{\mathrm{d}V(\hat{x},e)}{V(\hat{x},e)} \leqslant -a(t-t_0) + 2b \int_{t_0}^{t} \alpha(s)\,\mathrm{d}s.$$

It follows that for all $t \geqslant t_0$, we have

$$\|(\hat{x}(t), e(t))\| \leqslant \frac{\max^{1/2}(\lambda_{\max}(P), \nu\lambda_{\max}(\widetilde{P}))}{\min^{1/2}(\lambda_{\min}(P), \nu\lambda_{\min}(\widetilde{P}))} \|(\hat{x}(t_0), e(t_0))\| \, e^{bM_\alpha} e^{-\frac{1}{2}a(t-t_0)}.$$

Hence, the composite system (16) and (17) is globally uniformly exponentially stable. □

3. Case of Lipschitiz condition

In this section, as said previously, the function f_i satisfies the following assumption:
($\mathcal{A}_3$) there exists a positive constant $k > 0$, such that

$$\|f_i(t,x,u) - f_i(t,y,u)\| \leqslant k\|x-y\|, \qquad i = 1,2,\ldots,r, \tag{18}$$

for all $t \geqslant 0$, $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ and $u \in \mathbb{R}^m$. It means that the function f_i is globally Lipschitiz uniformly on t, and u with respect to x. For the stabilization of (7), we have the following result.

Theorem 4 *Suppose that* ($\mathcal{A}_3$) *hold and there exist symmetric and positive definite matrices P, and Q, and some matrices K_i, $i = 1,\ldots,r$, such that the inequalities* (8) *and* (9) *hold with*

$$P < \frac{1}{2k}Q,$$

then the fuzzy closed-loop system (7) *is guaranteed to be globally uniformly exponentially stable.*

For the next, we define $\lambda_0 = \lambda_{\min}(Q)$, $\lambda_{\min}$ denoting the smallest eigenvalue of the matrix.

Remark 4 *The matrices P, Q and K_i can be obtained using the same LMIs as in remark 1 with*

$$X < \frac{1}{2k}H. \tag{19}$$

See [9] for more details. Next, for the conception of the observer, we prove the following result.

Theorem 5 *Suppose that* $(\mathcal{A}_3)$ *hold and there exist positive symmetric definite matrices* $\widetilde{P}$, $\widetilde{Q}$ *and some matrices* L_i, $i = 1, \ldots, r$, *such that the inequalities* (14) *and* (15) *hold with*

$$\widetilde{P} < \frac{1}{2k}\widetilde{Q},$$

then the system error (13) *is guaranteed to be globally uniformly exponentially stable.*

We define also $\widetilde{\lambda}_0 = \lambda_{\min}(\widetilde{Q})$, $\lambda_{\min}$ denoting the smallest eigenvalue of the matrix.

Remark 5 *The matrices* $\widetilde{P}$, $\widetilde{Q}$ *and* L_i *can be obtained using the same LMIs as in remark 3 with*

$$\widetilde{X} < \frac{1}{2k}\widetilde{H}. \tag{20}$$

Proof. This proof is similar to the controller case. □

Now, we can give a separation principle.

Now, to get a separation principle let consider the following theorem.

Theorem 6 *Under assumption* $(\mathcal{A}_3)$, *and if conditions of theorems* 4 *and* 5 *hold, then the T-S fuzzy composite system*

$$\dot{\hat{x}} = \sum_{i=1}^{r} \mu_i \left(A_i \hat{x} + B_i u(\hat{x}) + f_i(t, \hat{x}, u(\hat{x}))\right) - \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j L_j C_i e \tag{21}$$

$$\dot{e} = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j (A_i - L_j C_i) e + \sum_{i=1}^{r} \mu_i \left(f_i(t, \hat{x}, u(\hat{x})) - f_i(t, \hat{x} - e, u(\hat{x}))\right), \tag{22}$$

where $u(\hat{x}) = \sum_{i=1}^{r} \mu_i K_i \hat{x}$, *is globally uniformly exponentially stable.*

Proof. Let consider the Lyapunov function candidate

$$V(\hat{x}, e) = V_1(\hat{x}) + \nu V_2(e),$$

for the composite system (21) and (22) where $V_1(\hat{x}) = \hat{x}^T P \hat{x}$ and $V_2(e) = e^T \widetilde{P} e$, and ν is a positive constant which will be chosen later.

On the one hand, the derivative of V along the trajectories of (21) and (22) is given by,

$$\dot{V}(\hat{x},e) = \nabla V_1(\hat{x})\left(\sum_{i=1}^{r}\mu_i\left(A_i\hat{x} + B_i u(\hat{x}) + f_i(t,\hat{x},u(\hat{x}))\right)\right) - \nabla V_1(\hat{x})\left(\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j L_j C_i e\right) + \nu\dot{V}_2(e).$$

Since

$$\|\nabla V_1(\hat{x})\| \leqslant 2\lambda_{\max}(P)\|\hat{x}\|,$$

thus, using the properties of V_1 and V_2 given in the above proofs, one gets

$$\dot{V}(\hat{x},e) \leqslant -\lambda_0\|\hat{x}\|^2 + 2k\lambda_{\max}(P)\|\hat{x}\|^2 + 2\lambda_{\max}(P)\sum_{i=1}^{r}\sum_{j=1}^{r}\|L_j\|\|C_i\|\|e\|\|\hat{x}\| - \nu\widetilde{\lambda}_0\|e\|^2 + 2\nu k\lambda_{\max}(\widetilde{P})\|e\|^2.$$

Since for all $\varepsilon > 0$, we have

$$\|\hat{x}\|\|e\| \leqslant \frac{1}{2\varepsilon}\|\hat{x}\|^2 + \frac{\varepsilon}{2}\|e\|^2.$$

Then one gets

$$\dot{V}(\hat{x},e) \leqslant -\left(\lambda_0 - 2k\lambda_{\max}(P) + \frac{\lambda_{\max}(P)}{\varepsilon}\sum_{i=1}^{r}\sum_{j=1}^{r}\|L_j\|\|C_i\|\right)\|\hat{x}\|^2 - \left(\widetilde{\lambda}_0 - 2k\lambda_{\max}(\widetilde{P}) + \frac{\varepsilon\lambda_{\max}(P)}{\nu}\sum_{i=1}^{r}\sum_{j=1}^{r}\|L_j\|\|C_i\|\right)\nu\|e\|^2.$$

Let

$$\varepsilon = \frac{2\lambda_{\max}(P)}{\lambda_0 - 2k\lambda_{\max}(P)}\sum_{i=1}^{r}\sum_{j=1}^{r}\|L_j\|\|C_i\|,$$

then

$$\dot{V}(\hat{x},e) \leqslant -\frac{1}{2}(\lambda_0 - 2k\lambda_{\max}(P))\|\hat{x}\|^2 - \left(\widetilde{\lambda}_0 - 2k\lambda_{\max}(\widetilde{P}) - \frac{2\lambda^2_{\max}(P)}{\nu(\lambda_0 - 2k\lambda_{\max}(P))}\left[\sum_{i=1}^{r}\sum_{j=1}^{r}\|L_j\|\|C_i\|\right]^2\right)\nu\|e\|^2.$$

Let take

$$\nu = \frac{4\left[\lambda_{\max}(P)\sum_{i=1}^{r}\sum_{j=1}^{r}\|L_j\|\|C_i\|\right]^2}{(\lambda_0 - 2k\lambda_{\max}(P))(\widetilde{\lambda}_0 - 2k\lambda_{\max}(\widetilde{P}))}.$$

Therefore, one obtains

$$\dot{V}(\hat{x}, e) \leqslant -\frac{1}{2}(\lambda_0 - 2k\lambda_{\max}(P))\,\|\hat{x}\|^2 - \frac{1}{2}\left(\widetilde{\lambda}_0 - 2k\lambda_{\max}(\widetilde{P})\right)\nu\|e\|^2.$$

It follows that,

$$\begin{aligned}\dot{V}(\hat{x}, e) \leqslant &-\frac{1}{2\lambda_{\max}(P)}(\lambda_0 - 2k\lambda_{\max}(P))\,V_1(\hat{x})\\ &-\frac{1}{2\lambda_{\max}(\widetilde{P})}\left(\widetilde{\lambda}_0 - 2k\lambda_{\max}(\widetilde{P})\right)\nu V_2(e).\end{aligned}$$

Let

$$a = \min\left(\frac{1}{2\lambda_{\max}(P)}(\lambda_0 - 2k\lambda_{\max}(P)),\ \frac{1}{2\lambda_{\max}(\widetilde{P})}\left(\widetilde{\lambda}_0 - 2k\lambda_{\max}(\widetilde{P})\right)\right).$$

The above expression gives the following estimate:

$$\dot{V}(\hat{x}, e) \leqslant -aV(\hat{x}, e).$$

Integrating between t_0 and t, one obtains for all $t \geqslant t_0$,

$$\int_{t_0}^{t}\frac{\mathrm{d}V(\hat{x}, e)}{V(\hat{x}, e)} \leqslant -a(t - t_0).$$

It follows that for all $t \geqslant t_0$, we have

$$\|(\hat{x}(t), e(t))\| \leqslant \frac{\max^{1/2}\left(\lambda_{\max}(P), \nu\lambda_{\max}(\widetilde{P})\right)}{\min^{1/2}\left(\lambda_{\min}(P), \nu\lambda_{\min}(\widetilde{P})\right)}\,\|(\hat{x}(t_0), e(t_0))\|\, e^{-\frac{1}{2}a(t-t_0)}.$$

Hence, the composite system (21) and (22) is globally uniformly exponentially stable. □

We give then an example to illustrate application of the above theorems.

4. Example

Consider the following nonlinear fuzzy planar system,

$$\begin{cases} \dot{x} = \sum_{i=1}^{2} \mu_i(z) \left(A_i x + B_i u + f_i(t,x,u)\right), \\ y = \sum_{i=1}^{2} \mu_i(z) C_i x, \end{cases}$$

where $x(t) = [x_1(t) \;\; x_2(t)]^T$, is the state vector, $u(t)$ is the input vector, $y(t)$ is the output vector.

$$z = \sin(x_1), \qquad A_1 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C_1 = C_2 = \begin{bmatrix} -1 & 1 \end{bmatrix},$$

$$f_1(t,x,u) = \begin{bmatrix} -\alpha(t)\cos(u)x_2 \\ -\alpha(t)\left(x_1^2 + x_2^2\right)^{1/2} \end{bmatrix}, \qquad f_2(t,x,u) = \begin{bmatrix} -\alpha(t)\cos(u)x_2 \\ \alpha(t)\left(x_1^2 + x_2^2\right)^{1/2} \end{bmatrix}.$$

We define the membership functions as:

$$\mu_1(t) = \frac{1 - \sin(x_1(t))}{2} \qquad \text{and} \qquad \mu_2(t) = 1 - \mu_1(t).$$

A classical T-S fuzzy model would require 8 rules for 3 nonlinearities, and it would depend on unmeasured variables. Using an LMI optimization algorithm, we obtain the following feedback gains:

$$K_1 = [-2.9378 \;\; 11.6734] \qquad \text{and} \qquad K_2 = [-0.8202 \;\; 5.6921].$$

Now concerning the observer, let suppose the following fuzzy observer:

$$\begin{cases} \dot{\hat{x}} = \sum_{i=1}^{2} \mu_i(\hat{z}) \left(A_i \hat{x} + B_i u + f_i(t,\hat{x},u) - L_i(\hat{y} - y)\right), \\ \hat{y} = \sum_{i=1}^{2} \mu_i(\hat{z}) C_i \hat{x}. \end{cases}$$

Then, we obtain the following observer gain:

$$L_1 = [422.7991 \;\; 483.2676]^T \qquad \text{and} \qquad L_2 = [85.9898 \;\; 101.2911]^T.$$

We take $\alpha(t)$ the same as in example 1. Therefore, conditions of theorems 3 and 6 are satisfied, then the T-S fuzzy cascaded system

$$\begin{cases} \dot{\hat{x}} = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\left(A_i\hat{x} + u(\hat{x}) + f_i(t,\hat{x},u(\hat{x}))\right) - \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j L_j C_i e, \\ \dot{e} = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(A_i - L_jC_i)e + \sum_{i=1}^{r}\mu_i\left(f_i(t,\hat{x},u(\hat{x})) - f_i(t,\hat{x}-e,u(\hat{x}))\right) \end{cases}$$

with $u(\hat{x}) = \sum_{i=1}^{2}\mu_i K_i\hat{x}$ is globally uniformly exponentially stable.

5. Conclusion

In this paper a new way to simplify the design of controllers and observers for T-S fuzzy models is presented. It concerns the cases of non linearity that either meets an integrability condition or are simply Lipschitz. Some results are obtained: classical LMI conditions can be used for both the observer design and the controller. It is shown that, they can be designed separately and a separation principle is given. The effectiveness of the proposed observer based controller is illustrated by a theoretical example.

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