10.24425/acs.2019.129383

Archives of Control Sciences
Volume 29(LXV), 2019
No. 2*, pages 289–305*

Interval state estimation for linear time-varying (LTV) discrete-time systems subject to component faults and uncertainties

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This paper deals with the design of an interval state estimator for linear time-varying (LTV) discrete-time systems subject to component faults and uncertainties. These component faults and uncertainties are assumed to be unknown but bounded without giving any other information, whose effect can be approximated using these bounds. In the first part of this work, an interval state estimator for such systems is designed to deal with these component faults and uncertainties. The result is then extended to find an interval state estimator for a non-cooperative LTV discrete-time system subject to component faults and uncertainties by similarity transformation of coordinates. The proposed interval state estimator guaranteed bounds on the observed states that are consistent with the system states. The observer convergence is also ensured. The designed method is simple and easy to be implemented. Two numerical examples are given to show the effectiveness of the proposed method.

Key words: component faults, interval state estimator, linear time-varying systems, non-cooperative systems

1. Introduction

State estimation of the unmeasurable systems is the fundamental problem in many engineering applications specially for designing controllers and fault detection [1–4]. The problem of state estimation for linear and nonlinear systems

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This work was supported by YangFan Innovative and Entepreneurial Research Team Projectof Guangdong Province under the grant 2016YT03G125, National Nature Science Foundation of China under the grant 61803161, Natural Science Foundation of Guangdong Province under the grant 2017A030313385 & 2018A030310371, Science and Technology Planning Project of Guangdong Province under the grant 2017B090914001, 2017A040403064, 2017B090901040, 2017B030306017 & 2017A040405023, Innovative Research Team Project of Jiangmen under the grant 2017TD03 and the scholarship from China Scholarship Council (CSC) under the grant CSC No. 2016GXZ344.

Received 25.10.2018. Revised: 5.12.2018 and 29.12.2018.

has been widely studied in the literature and numerous solutions already exist for such systems. Since the practical systems are generally affected by the uncertainties and disturbances, classical observers, for instance, Kalman/H_∞ filtering or Luenberger [5–7] are usually used to estimate the state of the systems. These observers are designed to guarantee asymptotic stability. However, in general, only local convergence is expected, i.e. to find the residual error [6]. To solve this problem, an alternative technique is recently developed to deal with these uncertainties and disturbances by determining certain upper and lower estimates of the systems at each time instant, which is known as set-membership or interval state estimators. There are many important contributions available for designing such estimators [8–15]. It is assumed that the disturbances and uncertainties are unknown but bounded. This technique is quite different from the classical observers which converges to the actual state of the system asymptotically. The interval state estimators can achieve both asymptotic convergence property and the state estimation of the system at each time instant by providing certain lower and upper bounds. Thus, an error bound is provided at any time instant. Interval observers are very popular these days because they make it possible to deal with large uncertainties, which is very significant for example when we consider large biological models. They are also successfully applied to many real time life problems [16, 17].

During the past few years, different kinds of interval observers have been presented for both continuous-time [16–18] and discrete-time [10, 19, 20] (linear and nonlinear) systems based on monotone system theory. It is known that positivity of the interval estimation error dynamics is one of the most restrictive assumption for the observers design [21]. This assumption was relaxed in [22–25] for LTI systems by using a time-varying change of coordinates. Moreover, to design a closed-loop observer for LTI systems, a time-invariant transformation of coordinates is proposed in [26] for a class of nonlinear system, where the observer gain and the transition matrix meet the Sylvester equation. The result is then extended to design a control strategy for nonlinear and LPV systems [27]. A particular class of periodic time-varying discrete-time system was investigated by using a time-varying transformation of coordinates in [28]. A static linear transformation of coordinates was proposed to ensure the stability and positivity of the observation error in [29] for a class of nonlinear systems. The results obtained in [29] were later used to design an interval observer for time-varying systems in [14].

Within this paper, an interval state estimator is proposed for linear time-varying discrete-time systems subject to component faults and uncertainties. The first contribution of this paper lies in how to design an interval state estimator for such systems based on monotone system theory. The second contribution of this work is related to the design of an interval state estimator for a non-cooperative LTV discrete-time systems subject to component faults and uncertainties based on similarity transformation of coordinates. It is shown that it is usually possible to design an interval state estimator for such type of systems by means of transformation of coordinates even though the given system is not cooperative. The uncertainties and component fault parameter vector are assumed to be unknown but bounded. This kind of interval observer is very useful in handling the effects of component faults in systems.

This paper is structured as follows. In Section 2, some preliminaries are given. The problem statement is formulated in Section 3. Section 4 is devoted to the design of the interval state estimator for LTV discrete-time systems subject to component faults and uncertainties. Two detailed examples are given to show the efficiency of the proposed design in Section 5. Finally, Section 6 concludes the paper.

2. Preliminaries

The set of real numbers, integers, nonnegative real numbers and nonnegative integers are denoted by R, Z, R_+ and Z_+ respectively where $R_+ = \{\tau \in R : \tau \geqslant 0\}$ and $Z_+ = Z \cap R_+$. A matrix $A(k) \in R^{n\times n}$ is called Metzler if and only if all the elements except the main diagonal are nonnegative, if all its elements are positive then it is a nonnegative matrix. It is Schur stable when the norm of all its eigenvalues is less than one. If $\upsilon^T P\upsilon > 0$ ($\upsilon^T P\upsilon < 0$) for all real vector $\upsilon \in R^n$, then $P \in R^{n\times n}$ is said to be positive (negative) definite denoted by $P \succ 0$ ($P \prec 0$). Similarly, $P \succeq 0$ ($P \preceq 0$) means semi-positive (semi-negative) definite matrix.

For a matrix $A \in R^{m\times n}$ we have $A^+ = \max\{A,\ 0\}$, $A^- = \max\{-A,\ 0\}$ and $A = A^+ - A^-$. The relationship $x_1 \leqslant x_2$ and $A_1 \leqslant A_2$ is understood elementwise for two vectors x_1, $x_2 \in R^n$ or matrices A_1, $A_2 \in R^{n\times n}$ respectively. I_n and E_p represents an identity matrix of dimension $n \times n$ and a matrix having all elements equal to 1 with dimension $p \times 1$ respectively. For a vector $x \in R^n$, the Euclidean norm is denoted by $|x|$ while $\|u\|_{[k_0,\ k_1]}$ denotes the L_∞ norm for a locally essential bounded and measurable input $u : Z \to R$. The L_∞ norm is given by $\|u\|_{[k_0,\ k_1]} = \sup\{|u(k)|,\ k \in [k_0,\ k_1]\}$ and if $k_1 = +\infty$ then the L_∞ norm is simply denoted by $\|u\|$.

The eigenvector for a matrix $A \in R^{n\times n}$ is denoted by $\lambda(A)$, the elementwise maximum norm is given by $\|A\|_{\max} = \max\left|A_{i,j}\right|$ for $i,\ j = 1\cdots n$ and the L_2 induced norm is given by $\|A\|_2 = \sqrt{\max_{i=1\cdots n}\lambda_i(A^T A)}$. The following equation shows the relationship between these norms [30]:

$$\|A\|_{\max} \leqslant \|A\|_2 \leqslant n\|A\|_{\max}.$$

Lemma 1 [21] *Given a non-autonomous system described by* $\dot{x}(t) = Ax(t) + B(t)$, *where* A *is a Metzler matrix and* $B(t) \geqslant 0$, *then, we have* $x(t) \geqslant 0$, $\forall t > 0$ *provided that* $x(0) \geqslant 0$.

Such type of systems are known as a cooperative systems or monotone. This lemma is also valid for time-varying discrete-time systems.

3. Problem statement

Consider the following discrete-time LTV system with component faults and uncertainties

$$\begin{cases} x(k+1) = A(\delta, k)x(k) + b(k), \\ \quad y(k) = C(k)x(k) + v(k), \end{cases} \tag{1}$$

where $x(k) \in R^n$, $y(k) \in R^p$, $b(k) : Z_+ \to R^n$ and $v(k) : Z_+ \to R^p$ represent the state vector, output signal, unknown but bounded input and the measurement noise respectively. $A : Z_+ \to R^{n\times n}$, $C : \ Z_+ \to R^{p\times n}$ are the matrix functions of appropriate dimensions and $\delta \in R^{n\times n}$ is the component fault parameter vector which is unknown but bounded and is considered to be in the set of admissible values Π. $A(\delta, k)$ in the system equation is supposed to be dependent on k and δ as:

$$A(\delta, k) = A(k) + \delta_1 A_1(k) + \delta_2 A_2(k) + \cdots + \delta_n A_s(k), \tag{2}$$

where $A(k)$, $A_1(k)$, $A_2(k)$, $\cdots$, $A_s(k)$ and δ_1, δ_2, δ_3, $\cdots$, δ_s are affine matrices and the elements of components faults parameter vector respectively. Then,

$$A(\delta, k) = A(k) + \sum_{i=1}^{s} \delta_i A_i(k). \tag{3}$$

Using (3), the system (1) can be written as:

$$\begin{cases} x(k+1) = A(k)x(k) + \sum\limits_{i=1}^{s} \delta_i A_i(k)x(k) + b(k), \\ \quad y(k) = C(k)x(k) + v(k). \end{cases} \tag{4}$$

The objective of this paper is to develop an interval state estimator for system described by (4). In this work, the following assumptions will be used.

Assumption 1 *Let $x(k) \in L_\infty^n$ and $v(k) \in L_\infty^n$, such that $\|v(k)\| \leqslant V$, where V is a positive constant.*

Assumption 2 *For known $\overline{\delta}_i$, $\underline{\delta}_i \in R^{n\times n}$, we have $\underline{\delta}_i \leqslant \delta_i \leqslant \overline{\delta}_i$, $\forall \delta \in \Pi$.*

Assumption 3 $\underline{b}(k) \leqslant b(k) \leqslant \overline{b}(k)$ *for all Z_+, where $\underline{b}$, $\overline{b} : Z_+ \to R^n$, $\underline{b}$, $\overline{b} \in L_\infty^n$.*

Assumption 4 *There exists a bounded matrix function $L_{obs} : R \rightarrow R^{n\times p}$ such that the matrix $D(k) = A(k) - L_{obs}C(k)$ is Schur stable and nonnegative (cooperative).*

Assumption 1 is common in the literature of estimation theory. Assumption 2 and Assumption 3 states that the magnitude of the component faults parameter vector and the external input are unknown but bounded and belongs to an interval $\left[\underline{\delta}, \overline{\delta}\right]$ and $\left[\underline{b}(k), \overline{b}(k)\right]$, respectively.

In the sequel derivation, we will need the following lemmas.

Lemma 2 [29] *Let $x \in R^n$ be a vector variable, $\underline{x} \leqslant x \leqslant \overline{x}$ for some $\underline{x}, \overline{x} \in R^n$, and $A \in R^{m\times n}$ be a constant matrix, then*

$$A^+\underline{x} - A^-\overline{x} \leqslant Ax \leqslant A^+\overline{x} - A^-\underline{x}. \tag{5}$$

Lemma 3 [29] *Let $\underline{A} \leqslant A \leqslant \overline{A}$ for some $\underline{A}$, $\overline{A}$, $A \in R^{n\times n}$ and $\underline{x} \leqslant x \leqslant \overline{x}$ for some $\underline{x}$, $\overline{x}$, $x \in R^n$, then*

$$\underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- \leqslant Ax \leqslant \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^-. \tag{6}$$

4. Interval observer design

A Luenberger type observer design for the system (4) can be described as

$$\begin{cases} x(k+1) = D(k)x(k) + \sum\limits_{i=1}^{k} \Upsilon_i(x(k)) + b(k) + L_{obs}(y(k) - v(k)), \\ \qquad y(k) = C(k)x(k) + v(k), \end{cases} \tag{7}$$

where $D(k) = A(k) - L_{obs}C(k)$ and $\Upsilon_i(x(k)) = \delta_i A_i(k)x(k)$.

From the Luenberger-based observer design described by (7), the upper and lower bounds for the system (4) can be estimated as

$$\begin{cases} \overline{x}(k+1) = D(k)\overline{x}(k) + \sum\limits_{i=1}^{k} \overline{\Upsilon}_i(\overline{x}(k), \underline{x}(k)) + \overline{b}(k) + L_{obs}y(k) + |L_{obs}|\, VE_p, \\ \underline{x}(k+1) = D(k)\underline{x}(k) + \sum\limits_{i=1}^{k} \underline{\Upsilon}_i(\overline{x}(k), \underline{x}(k)) + \underline{b}(k) + L_{obs}y(k) - |L_{obs}|\, VE_p, \end{cases} \tag{8}$$

where $\overline{x}(k)$, $\underline{x}(k)$ are the given lower and upper bounds of the state $x(k)$ and $|L_{obs}| = {L_{obs}}^+ + {L_{obs}}^-$.

By Lemma 3 and Assumption 2, we have

$$\begin{cases} \overline{\Upsilon}_i(\overline{x}(k),\underline{x}(k)) = \overline{\delta}_i^{\,+} A_i(k)\overline{x}^{+}(k) - \underline{\delta}_i^{\,+} A_i(k)\overline{x}^{-} \\ \qquad\qquad\qquad - \overline{\delta}_i^{\,-} A_i(k)\underline{x}^{+} + \underline{\delta}_i^{\,-} A_i(k)\underline{x}^{-}, \\ \underline{\Upsilon}_i(\overline{x}(k),\underline{x}(k)) = \underline{\delta}_i^{\,+} A_i(k)\underline{x}^{+}(k) - \overline{\delta}_i^{\,+} A_i(k)\underline{x}^{-} \\ \qquad\qquad\qquad - \underline{\delta}_i^{\,-} A_i(k)\overline{x}^{+} + \overline{\delta}_i^{\,-} A_i(k)\overline{x}^{-}, \end{cases} \tag{9}$$

$\underline{b}(k) \leqslant b(k) \leqslant \overline{b}(k)$.

Theorem 1 *Let Assumptions 1–3 be satisfied and the matrix* $D(k) = A(k) - L_{obs}C(k)$ *is Schur stable and Metzler for some* $L_{obs} \in R^{n\times p}$*, then the estimates* $\underline{x}(k)$, $\overline{x}(k)$ *for all* $k \in Z_+$ *are bounded and we obtained the following relationship between the solution of (4) and (8)*

$$\underline{x}(k) \leqslant x(k) \leqslant \overline{x}(k), \quad \forall k \in Z_+$$

provided that initial condition $\underline{x}(0) \leqslant x(0) \leqslant \overline{x}(0)$ *is satisfied.*

Proof. Consider the interval estimation error dynamics

$$\begin{cases} \overline{e}(k) = \overline{x}(k) - x(k), \\ \underline{e}(k) = x(k) - \underline{x}(k) \end{cases}$$

$$\begin{cases} \overline{e}(k+1) = D(k)\overline{e}(k) + \overline{\Phi}(k) + \overline{\wp}(k), \\ \underline{e}(k+1) = D(k)\underline{e}(k) + \underline{\Phi}(k) + \underline{\wp}(k), \end{cases}$$

where

$$\begin{cases} \overline{\Phi}(k) = \sum_{i=1}^{k} \overline{\Upsilon}_i(\overline{x}(k),\underline{x}(k)) - \sum_{i=1}^{k} \Upsilon_i(x(k)), \\ \underline{\Phi}(k) = \sum_{i=1}^{k} \Upsilon_i(x(k)) - \sum_{i=1}^{k} \underline{\Upsilon}_i(\overline{x}(k),\underline{x}(k)), \end{cases} \tag{10}$$

$$\begin{cases} \overline{\wp}(k) = \overline{b}(k) - b(k) + |L_{obs}|\,VE_p + L_{obs}v(k), \\ \underline{\wp}(k) = b(k) - \underline{b}(k) + |L_{obs}|\,VE_p - L_{obs}v(k). \end{cases} \tag{11}$$

$D(k)$ is Schur stable and Metzler according to Assumption 4, $\overline{\Phi}(k)$, $\underline{\Phi}(k) \geqslant 0$ for all $k \in Z_+$ by (6) and Assumption 2, $\overline{\wp}(k)$, $\underline{\wp}(k) \geqslant 0$ for all $k \in Z_+$ by Assumptions 1, 3 and $L_{obs} \leqslant |L_{obs}|$. $\overline{e}(0) = \overline{x}(0) - x(0) \geqslant 0$ and $\underline{e}(0) = x(0) - \underline{x}(0) \geqslant 0$ by construction. Thus, according to monotone system theory

[21], the variables $\overline{e}(k) \geqslant 0$, $\underline{e}(k) \geqslant 0$ for all $k \in Z_+$. Since all the inputs of $\overline{e}(k)$ and $\underline{e}(k)$ are bounded, $D(k)$ is Schur stable by Assumption 4, thus $\overline{e}(k)$, $\underline{e}(k) \in L_\infty^n$ and from boundedness of $x(k)$, we get $\overline{x}(k)$, $\underline{x}(k) \in L_\infty^n$. That implies,

$$\overline{e}(k) = \overline{x}(k) - x(k) \geqslant 0,$$
$$\underline{e}(k) = x(k) - \underline{x}(k) \geqslant 0,$$

for all $k \in Z_+$.

Therefore, we can write that the bounds of state $x(k)$ in (8) are bounded

$$\underline{x}(k) \leqslant x(k) \leqslant \overline{x}(k),$$

This completes the proof. □

Note that the matrix of interval estimation error dynamics must be positive in order to design interval observer. This means that the matrix $D(k) = A(k) - L_{obs}C(k)$ must satisfy Assumption 4, which is not always possible. This drawback can be overcome by means of similarity transformation of coordinates, $z(k) = \aleph x(k)$, where $\aleph$ is a non-singular transformation matrix. If there exists a matrix $L_{obs} \in R^{n\times p}$ such that $D(k) = A(k) - L_{obs}C(k)$ is Schur stable, then it is usually possible to find a non-singular transformation matrix $\aleph$, such that the matrix $\aleph(A(k) - L_{obs}C(k))\aleph^{-1}$ is Schur stable and Metzler (non-negative). The transformation matrix $\aleph$ can be found using *Procedure 1* of [31]. Then, Assumption 4 can be relaxed as:

Assumption 5 *There exists a matrix function $L_{obs} : R \to R^{n\times p}$ such that the matrix $D(k) = A(k) - L_{obs}C(k)$ is Schur stable.*

Using similarity transformation of coordinates Assumption 5 relaxes Assumption 4, thus the only requirement for the matrix $D(k) = A(k) - L_{obs}C(k)$ is to be Schur stable. So if $D(k) = A(k) - L_{obs}C(k)$ is Schur stable then $\aleph(A(k) - L_{obs}C(k))\aleph^{-1}$ is Schur stable as well as Metzler.

After similarity transformation of coordinates, the system (4) takes the form

$$\begin{cases} z(k+1) = \aleph A(k)\aleph^{-1}z(k) + \sum\limits_{i=1}^{k} \aleph\delta_i A_i(k)\aleph^{-1}z(k) + \aleph b(k), \\ y(k) = C(k)\aleph^{-1}z(k) + v(k) \end{cases} \tag{12}$$

which is a positive representation of (4) provided that $\sum\limits_{i=1}^{k} \aleph\delta_i A_i(k)\aleph^{-1} \geqslant 0$ and $\aleph b(k) \geqslant 0$. The observer takes the form similar to (7) in the new coordinates as:

$$\begin{cases} z(k+1) = M(k)z(k) + \sum\limits_{i=1}^{k} \Upsilon_i(z(k)) + \beta(k) + G(k)(y(k) - v(k)), \\ y(k) = C(k)\aleph^{-1}z(k) + v(k), \end{cases} \tag{13}$$

where

$$M(k) = \aleph D(k)\aleph^{-1},$$
$$\sum_{i=1}^{k} \Upsilon_i(z(k)) = \sum_{i=0}^{k} \aleph\delta_i A_i(k)\aleph^{-1} z(k),$$
$$\beta(k) = \aleph b(k), \qquad G(k) = \aleph L_{obs}.$$

The interval state estimator for the system (12) in new coordinates similar to (8) is

$$\begin{cases} \overline{z}(k+1) = M(k)\overline{z}(k) + \sum_{i=1}^{k} \overline{\Upsilon}_i(\overline{z}(k), \underline{z}(k)) + \overline{\beta}(k) + G(k)y(k) + \overline{G}(k)V, \\ \underline{z}(k+1) = M(k)\underline{z}(k) + \sum_{i=1}^{k} \underline{\Upsilon}_i(\overline{z}(k), \underline{z}(k)) + \underline{\beta}(k) + G(k)y(k) - \overline{G}(k)V, \end{cases} \tag{14}$$

where

$$\begin{cases} \overline{\beta}(k) = \aleph^{+}\overline{b}(k) - \aleph^{-}\underline{b}(k), \\ \underline{\beta}(k) = \aleph^{+}\underline{b}(k) - \aleph^{-}\overline{b}(k), \end{cases} \tag{15}$$

$$\overline{G}(k) = (G^{+}(k) + G^{-}(k))E_{p\times 1}, \tag{16}$$

$$\begin{cases} \overline{\Upsilon}_i(\overline{z}(k), \underline{z}(k)) = \overline{\Theta}^{+}{}_i(k)\overline{z}^{+}(k) - \underline{\Theta}^{+}{}_i(k)\overline{z}^{-}(k) - \overline{\Theta}^{-}{}_i(k)\underline{z}^{+}(k) \\ \qquad\qquad + \underline{\Theta}^{-}{}_i(k)\underline{z}^{-}(k), \\ \underline{\Upsilon}_i(\overline{z}(k), \underline{z}(k)) = \underline{\Theta}^{+}{}_i(k)\underline{z}^{+}(k) - \overline{\Theta}^{+}{}_i(k)\underline{z}^{-}(k) - \underline{\Theta}^{-}{}_i(k)\overline{z}^{+}(k) \\ \qquad\qquad + \overline{\Theta}^{-}{}_i(k)\overline{z}^{-}(k), \end{cases} \tag{17}$$

$$\begin{cases} \overline{\Theta}_i = (\aleph^{-1})^{+}(\aleph^{+}\overline{\delta}_i A_i(k) - \aleph^{-}\underline{\delta}_i A_i(k)) - (\aleph^{-1})^{-}(\aleph^{+}\underline{\delta}_i A_i(k) \\ \qquad - \aleph^{-}\overline{\delta}_i A_i(k)), \\ \underline{\Theta}_i = (\aleph^{-1})^{+}(\aleph^{+}\underline{\delta}_i A_i(k) - \aleph^{-}\overline{\delta}_i A_i(k)) - (\aleph^{-1})^{-}(\aleph^{+}\overline{\delta}_i A_i(k) \\ \qquad - \aleph^{-}\underline{\delta}_i A_i(k)), \end{cases} \tag{18}$$

According to Lemmas 1 and 2,

$$\aleph^{+}\underline{b}(k) - \aleph^{-}\overline{b}(k) = \underline{\beta}(k) \leqslant \beta(k) \leqslant \overline{\beta}(k) = \aleph^{+}\overline{b}(k) - \aleph^{-}\underline{b}(k), \tag{19}$$

$$\underline{\Upsilon}_i(\overline{z}(k), \underline{z}(k)) \leqslant \Upsilon_i(z(k)) \leqslant \overline{\Upsilon}_i(\overline{z}(k), \underline{z}(k)). \tag{20}$$

Theorem 2 *Let Assumptions 1–3, 5 be satisfied and there exists an observer gain L_{obs} and a non-singular transformation matrix $\aleph$ such that the matrix $M(k) = \aleph(A(k) - L_{obs}C(k))\aleph^{-1}$ is Schur stable and Metzler with the initial condition satisfying*

$$\underline{z}(0) \leqslant z(0) \leqslant \overline{z}(0),$$

and

$$\overline{z}(0) = \aleph^{+}\overline{x}(0) - \aleph^{-}\underline{x}(0),$$
$$\underline{z}(0) = \aleph^{+}\underline{x}(0) - \aleph^{-}\overline{x}(0),$$

Then the relationships between solutions of the systems (12) and (14) are obtained as

$$\underline{z}(k) \leqslant z(k) \leqslant \overline{z}(k),$$

for all $k \geqslant 0$ and (14) is an interval state estimator for the system (4) with new variable $z(k) = \aleph x(k)$ and

$$\underline{x}(k) \leqslant x(k) \leqslant \overline{x}(k).$$

Proof. Consider the upper bound of the estimation error dynamics

$$\begin{cases} \overline{e}(k+1) = \overline{z}(k+1) - z(k+1), \\ \overline{e}(k+1) = M(k)\overline{e}(k) + \overline{\psi}(k) + \overline{\vartheta}(k), \end{cases} \tag{21}$$

where

$$\begin{cases} \overline{\psi}(k) = \sum_{i=1}^{k} (\overline{\Upsilon}_i(\overline{z}(k), \underline{z}(k)) - \Upsilon_i(z(k))), \\ \overline{\vartheta}(k) = \overline{\beta}(k) - \beta(k) + \overline{G}(k)V + G(k)v(k), \end{cases} \tag{22}$$

$M(k)$ is Metzler by construction, $\overline{\psi}(k) \geqslant 0$ for all $k \in Z_+$ using (20) and $\overline{\vartheta}(k) \geqslant 0$ for all $k \in Z_+$ by (16), (19) and Assumption 1.

Then, according to monotone system theory [21], we obtained the following results

$$\overline{e}(k) = \overline{z}(k) - z(k) \geqslant 0. \tag{23}$$

Since $M(k)$ is Schur stable and Metzler by construction and all the inputs of $\overline{e}(k)$ are bounded, so $\overline{e}(k) \in L_\infty^n$ and then from $x(k) \in L_\infty^n$, we have $z(k) \in L_\infty^n$, and thus boundedness of $\overline{z}(k)$ is verified.

The lower bound of the estimation error dynamics can be proved by similar arguments.

$$\underline{e}(k) = z(k) - \underline{z}(k) \geqslant 0. \tag{24}$$

Then from (23) and (24), we deduce that $z(k)$ is bounded

$$\underline{z}(k) \leqslant z(k) \leqslant \overline{z}(k), \tag{25}$$

for all $k \in Z_+$.

If Assumption 5 is satisfied and $M(k)$ is Metzler, then the system (14) is stable and bounded interval state estimator for (4). Thus, by means of transformation of coordinates

$$x(k) = \aleph^{-1} z(k), \tag{26}$$

using (5) and (25), we get

$$(\aleph^{-1})^{+}\underline{z}(k) - (\aleph^{-1})^{-}\overline{z}(k) \leqslant \aleph^{-1} z(k) \leqslant (\aleph^{-1})^{+}\overline{z}(k) - (\aleph^{-1})^{-}\underline{z}(k),$$

which implies

$$\begin{cases} \overline{x}(k) = (\aleph^{-1})^{+}\overline{z}(k) - (\aleph^{-1})^{-}\underline{z}(k), \\ \underline{x}(k) = (\aleph^{-1})^{+}\underline{z}(k) - (\aleph^{-1})^{-}\overline{z}(k). \end{cases} \tag{27}$$

From (26) and (27), we deduce that

$$\underline{x}(k) \leqslant x(k) \leqslant \overline{x}(k),$$

for all $k \geqslant 0$.

This completes the proof for Theorem 2. □

5. Numerical examples

To illustrate the proposed interval state estimator technique two examples are given below.

5.1. Time-varying discrete-time system with components faults and uncertainties

Consider a second-order linear time-varying discrete-time system (4) with the following system matrices

$$A(\delta, k) = \begin{bmatrix} 0.4 - 0.3\sin(0.1k) + 0.5\sin(0.6k)\delta_1 & 0.6 + 0.3\cos(0.3k)\delta_2 \\ 0.1 & 0.6 - 0.2\cos(0.1k) + 0.1\delta_1 \end{bmatrix},$$

using (2),

$$A(k) = \begin{bmatrix} 0.4 - 0.3\sin(0.1k) & 0.6 \\ 0.1 & 0.6 - 0.2\cos(0.1k) \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.5\sin(0.6k) & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0.3\cos(0.3k) \\ 0 & 0 \end{bmatrix}.$$

$C = [0 \quad 1]$ and the unknown but bounded external input $b(k) = \gamma(k) + \lambda(k)$, such that

$$\gamma(k) = \begin{bmatrix} 0.5\cos(0.1k) \\ 0.9 \end{bmatrix}, \qquad \lambda(k) = d \times \begin{bmatrix} 0.9 \\ 0.6\sin(0.1k) \end{bmatrix}, \qquad d = 0.3,$$

$\overline{b}(k) = b(k) + d$, $\underline{b}(k) = b(k) - d$ and $v(k) = V \times \cos(0.5k)$ where $V = 0.05$ represents the unknown but bounded input and noise respectively.

The unknown but bounded component fault parameter vector $\underline{\delta}_i \leqslant \delta_i \leqslant \overline{\delta}_i$ is

$$\begin{bmatrix} 0 & -0.2 \\ 0.5 & -0.1 \end{bmatrix} \leqslant \begin{bmatrix} 0 & 0.2\sin(0.5k) \\ 0.5 & 0.1\cos(0.1k) \end{bmatrix} \leqslant \begin{bmatrix} 0 & 0.2 \\ 0.5 & 0.1 \end{bmatrix}.$$

For $L_{obs} = [0.5 \quad 0.3]^T$, the matrix $D(k) = A(k) - L_{obs}C$ is Schur stable and nonnegative and all the conditions of Theorem 1 are valid. The results for the interval state estimator (8) are shown in Fig. 1 and Fig. 2, which indicate that the states of the system are always between the lower and upper bound of the interval estimator. Fig. 1 shows the interval state estimation for the given system without the component faults while Fig. 2 shows the interval state estimation for the same system with component faults and uncertainty, occurs at $100s$.

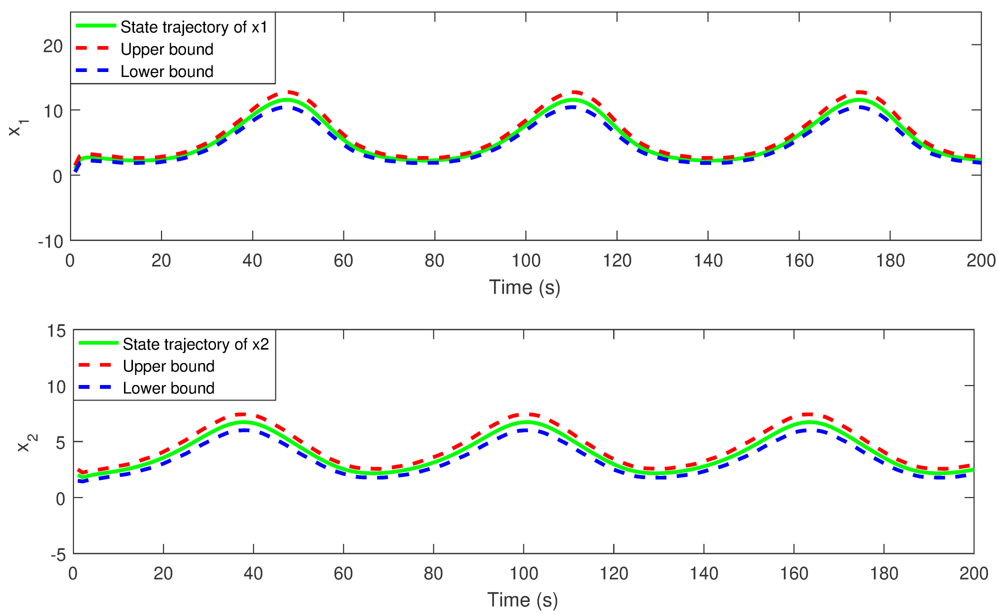

Figure 1: Simulation results of the interval estimator for first example without component faults

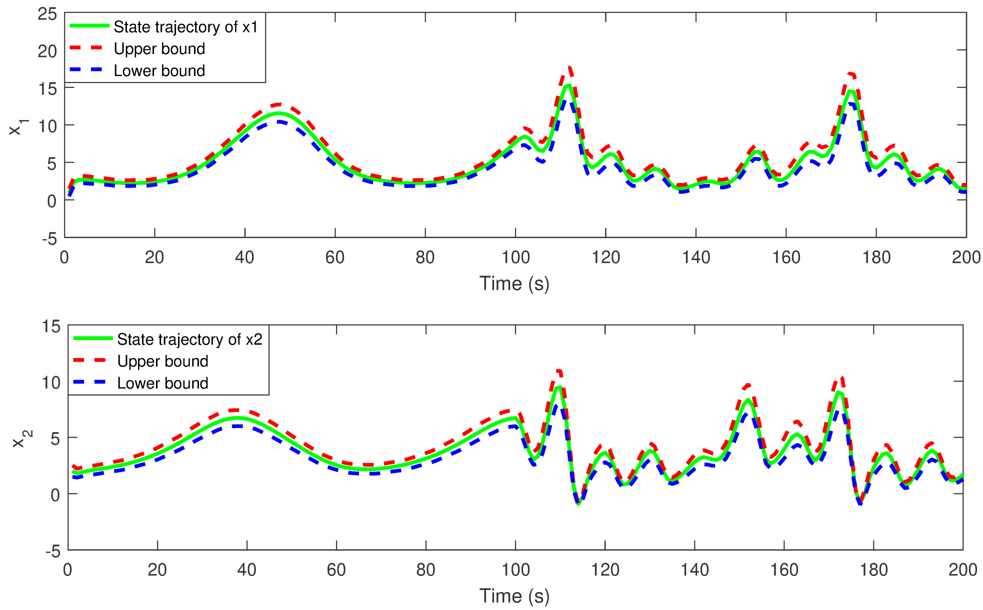

Figure 2: Simulation results of the interval estimator for first example with component faults

5.2. Transformation of coordinates

Consider an LTV discrete-time system:

$$\begin{cases} x(k+1) = A(k)x(k) + \sum_{i=1}^{2} \delta_i A_i(k)x(k) + b(k), \\ y(k) = C(k)x(k) + v(k). \end{cases}$$

For which we have,

$$A(\delta, k) = \begin{bmatrix} -0.5+0.1\sin(0.1k)+0.9\delta_1 & 0.7\delta_2 \\ -0.1-0.1\sin(0.1k)-0.1\cos(0.1k)\delta_1 & 0.2\cos(0.5k)+0.2\sin(0.3k)\delta_2 \end{bmatrix},$$

$$A(k) = \begin{bmatrix} -0.5+0.1\sin(0.1k) & 0 \\ -0.1-0.1\sin(0.1k) & 0.2\cos(0.5k) \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.9 & 0 \\ -0.1\cos(0.1k) & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0.7 \\ 0 & 0.2\sin(0.3k) \end{bmatrix}, \qquad C = [0 \;\; 1].$$

It is clear that there is no L_{obs} such that the matrix $D(k) = A(k) - L_{obs}C$ becomes Schur stable as well as nonnegative. For $L_{obs} = [-0.50 \;\; 0.35]^T$ the matrix $D(k) = A(k) - L_{obs}C$ is Schur Stable, thus Assumption 5 is satisfied. Using *Procedure 1* of [31], the non-singular transformation matrix is obtained as

$$\aleph = \begin{bmatrix} 1.25 & -0.25 \\ -0.25 & 0.8 \end{bmatrix},$$

such that the matrix $M(k) = \aleph D(k)\aleph^{-1}$ becomes nonnegative.

The unknown input is

$$b(k) = \gamma(k) + \lambda(k),$$

$$\gamma(k) = \begin{bmatrix} 0.5 \\ 0.9 \end{bmatrix}, \qquad \lambda(k) = d \times \begin{bmatrix} 0.9 \\ 0.6\sin(0.1k) \end{bmatrix}, \qquad d = 0.05,$$

where $\overline{b}(k) = b(k) + d$ and $\underline{b}(k) = b(k) - d$ represents the bounds of the external input, while the unknown and bounded noise is $v(k) = V \times \sin(k)$, such that $V = 0.1$, the bounded fault parameter vector is

$$\begin{bmatrix} 0 & -0.05 \\ 0.3 & -0.1 \end{bmatrix} \leqslant \delta = \begin{bmatrix} 0 & 0.05 - 0.1rand(1) \\ 0.3 & 0.1 - 0.2rand(1) \end{bmatrix} \leqslant \begin{bmatrix} 0 & 0.05 \\ 0.3 & 0.1 \end{bmatrix}.$$

Thus all the conditions of the Theorem 2 are satisfied. The results of simulations for the interval state estimator obtained (14) are given in Fig. 3 and Fig. 4.

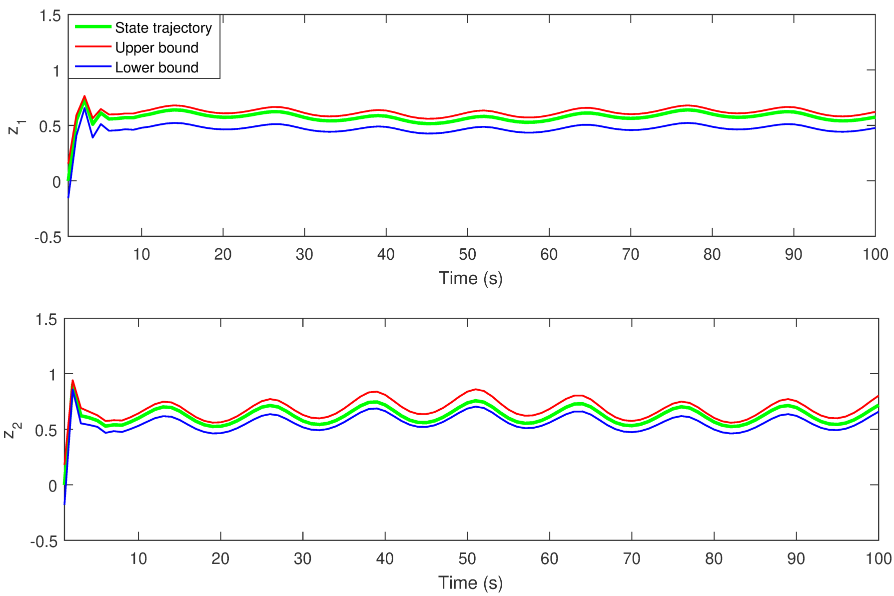

Figure 3: Simulation results of second example after similarity transformation of coordinates

The results show that it is usually possible to design an interval state estimator for any system even though the matrix $D(k) = A(k) - L_{obs}C(k)$ is not Metzler using similarity transformation of coordinates. The only requirement for designing interval observer is that the matrix $D(k) = A(k) - L_{obs}C(k)$ must be Schur stable. The states of the system are always within the estimated bounds. Fig. 3 and Fig. 4 show the simulation results after transformation of coordinates for a system without and with component faults and uncertainty respectively. The component faults and uncertainty occurs in the system at $50s$.

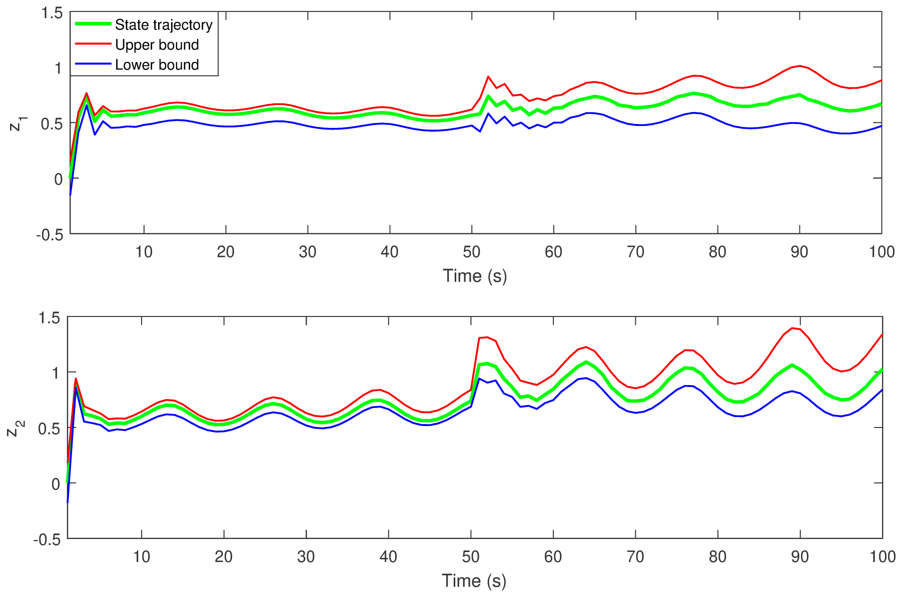

Figure 4: Simulation results of second example after similarity transformation of coordinates with component faults

6. Conclusion

In this paper, an interval state estimator has been developed for LTV discrete-time systems subject to component faults and uncertainties. The aim is to find out two bounds for the real value of the state vector at each time instant that are consistent with the error bounds and also ensured the observer convergence. As compared to the previous design techniques for the LTV discrete-time systems, this paper proposes a simple and easy technique for such systems having component faults and uncertainty. First, with a gain satisfying the positivity of the observation error, an interval state estimator is proposed and then using similarity transformation of coordinates a stable LTV discrete-time system is transformed to a stable and cooperative LTV discrete-time system to make this technique applicable to a large class of LTV systems, and can be used in handling the effects of component faults in such systems. Two numerical examples show the efficiency of the proposed technique.

References

[1] *G. Besançon* (Ed.): *Nonlinear observers and applications*. Lecture Notes in Control and Information Sciences, Berlin: Springer Verlag, 2007.

[2] T. Meurer, K. Graichen, and E.D. Gilles (Eds.): *Control and Observer Design for Nonlinear Finite and Infinite Dimensional Systems*. Lecture Notes in Control and Information Sciences, Springer, 2005.

[3] H. Nijmeijer and T.I. Fossen: *New Directions in Nonlinear Observer Design*. London, U.K.: Springer-Verlag, 1999.

[4] J. Tsinias: Time-varying observers for a class of nonlinear systems. *System and Control Letters*, **57** (2008), 1037–1047

[5] F. Yang, Z. Wang, and Y.S. Hung: Robust Kalman filtering for discrete-time-varying uncertain systems with multiplicative noises. *IEEE Transactions on Automatic Control*, **47**(7) (2002), 1179–1183.

[6] R. Reif and R. Unbehauen: The extended Kalman filter as an exponential observer for nonlinear systems. *IEEE Transactions on Signal Processing*, **47**(8) (1999), 2324–2328.

[7] H.F. Grip, A. Saberi, and T.A. Johansen: State and parameter estimation for nonlinearly parameterized systems: an H_∞-based approach. *In IFAC world congress*, (2011), 2997–3002.

[8] F. Mazenc, T.N . Dinh, and S.I. Niculescu: Interval observers for discrete-time systems. *IEEE 51st Annual Conference on Decision and Control*, Maui, Hawaii, (2012), 6755–6760.

[9] R. Lamouchi, M. Amairi, T. Raïssi, and M. Aoun: Interval observer design for Linear Parameter-Varying systems subject to components faults. *24th Mediterranean Conference on Control and Automation (MED)*, Athens, Greece, 2016.

[10] D. Efimov, T. Raïssi, W. Perruquetti, and A. Zolghadri: Estimation and control of discrete-time LPV systems using interval observers. *IEEE 52nd Annual Conference on Decision and Control*, (2013), 5036–5041.

[11] T. Raïssi, G. Videau, and A. Zolghadri: Interval observer design for consistency checks of nonlinear continuous-time systems. *Automatica*, **46**(3) (2010), 518–527.

[12] S. Chebotarev, D. Efimov, T. Raďssi and A. Zolghadri: Interval observers for continuous-time LPV systems with L_2/L_∞ performance. *Automatica*, **58** (2015), 82–89.

[13] F. Mazenc, N.T. Dinh, and S.N. Niculescu: Robust interval observers and stabilization design for discrete-time systems with input and output. *Automatica*, **49**(11) (2013), 3490–3497.

[14] E.H.R. Thabet, T. Raïssi, C. Combastel, D. Efimov, and A. Zolghadri: An effective method to interval observer design for time-varying systems. *Automatica*, **50** (2014), 2677–2684.

[15] F. Cacace, A. Germani and C. Manes: A New Approach to Design Interval Observers for Linear Systems. *IEEE Transactions on Automatic Control*, **60**(6) (2015), 1665–1670.

[16] M. Moisan, O. Bernard, and J.L. Gouzé: Near optimal interval observers bundle for uncertain bio-reactors. *Automatica*, **45**(1) (2009), 291–295.

[17] O. Bernard and J.L. Gouzé: Closed loop observers bundle for uncertain biotechnological models. *Journal of Process Control*, **14** (2004), 765–774.

[18] S. Chebotarev, D. Efimov, T. Raïssi, and A. Zolghadri: On interval observer design for a class of continuous-time LPV systems. *IFAC Proceedings*, **46**(23) (2013), 68–73.

[19] F. Mazenc, S. I. Niculescu, and O. Bernard: Exponentially Stable Interval Observers for Linear Systems with Delay. *SIAM Journal on Control and Optimization*, **50** (2012), 286–305.

[20] F. Mazenc, T.N. Dinh, and S.I. Niculescu: Continuous – Discrete Interval Observers for Monitoring Microalgae Cultures. *Biotechnology Progress*, **25**(3) (2009), 667–675.

[21] H.L. Smith: *Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems*. American Mathematical Society, Mathematical surveys and monographs, 1995.

[22] G. Goffaux, A.V. Wouwer, and O. Bernard: Asymptotically Stable Interval Observers for Planar Systems with Complex Poles. *IEEE Transactions on Automatic Control*, **55**(2) (2010), 523–527.

[23] C. Combastel and S.A. Raka: A stable interval observer for LTI systems with no multiple poles. *IFAC Proceedings*, **44**(1) (2011), 14335–14341.

[24] C. Combastel: Stable interval observers in C for linear systems with time-varying input bounds. *IEEE Transactions on Automatic Control*, **58**(2) (2013), 481–487.

[25] F. Mazenc and O. Bernard: Interval observers for linear time-invariant systems with disturbances. *Automatica*, **47**(1) (2011), 140–147.

[26] T. Raďssi, D. Efimov, and A. Zolghadri: Interval state estimation for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, **57**(1) (2012), 260–265.

[27] T. Raďssi, D. Efimov, and A. Zolghadri: Control of nonlinear and LPV systems: interval observer-based framework. *IEEE Transactions on Automatic Control*, **58**(3) (2013), 773–782.

[28] D. Efimov, T. Raïssi, T. Perruquetti, and A. Zolghadri: Interval observers for time-varying discrete-time systems. *IEEE Transactions on Automatic Control*, **58**(12) (2013), 3218–3224.

[29] D. Efimov, T. Raïssi, S. Chebotarev, and A. Zolghadri: On set-membership observer design for a class of periodical time-varying systems. *Conference on decision and control*, Maui, HI, USA, 2012.

[30] D. Efimov, T. Raïssi, S. Chebotarev, and A. Zolghadri: Interval state observer for nonlinear time-varying systems. *Automatica*, **49**(1) (2013), 200–205.

[31] Z. He and X. Wei: Control of nonlinear systems based on interval observer design. *IET Control Theory and Applications*, **12**(4) (2018), 543–548.