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Asymptotic properties of discrete linear fractional equations

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Abstract. In this paper we study the dynamical behavior of linear discrete-time fractional systems. The first main result is that the norm of the difference of two different solutions of a time-varying discrete-time Caputo equation tends to zero not faster than polynomially. The second main result is a complete description of the decay to zero of the trajectories of one-dimensional time-invariant stable Caputo and Riemann-Liouville equations. Moreover, we present Volterra convolution equations, that are equivalent to Caputo and Riemann-Liouvile equations and we also show an explicit formula for the solution of systems of time-invariant Caputo equations.

Key words: linear discrete-time fractional systems, Caputo equation, Riemann-Liouville equation, Volterra convolution equation, stability.

1. Introduction

Recently, the theory of fractional calculus became very popular and its development is still very fast (see e.g. [9, 37] and the references therein). In the literature, one can find results on theoretical problems as well as practical applications (see for example [24, 27, 32, 36, 40]). In control theory the stability is one of the most important properties of control systems and it is widely studied for standard order dynamical systems. In the theory of fractional systems the stability problem was well investigated for linear continuous-time systems and meanwhile it is completely solved for time-invariant systems [12, 33–35, 39].

This paper is devoted to study linear discrete-time fractional systems. In the discrete-time framework four types of fractional differences are considered: forward/backward Caputo and forward/backward Riemann-Liouville operators [1, 5, 8]. For linear discrete time-invariant fractional systems the stability problem is studied in [3, 6, 13–15, 25, 30]. In the historically first paper [6], the authors show that all solutions $x:\mathbb{N}_1\to\mathbb{R}$ of the one-dimensional fractional difference equation

$$\left(\nabla_0^\alpha x\right)\left(n\right)=\lambda x\left(n\right),\ n\in\mathbb{N}_1,$$

with positive initial value $x(0)$, where ∇_0^α denotes the backward Riemann-Liouville (or nabla) operator, tend to infinity, if the order α of the fractional equation is in the interval [0.5, 1] and the coefficient λ of the equation is in the interval (0, 1). A much deeper result about the same type of equation is obtained in [14], where it is shown that if the order α of the system is in the interval $(0,1)$, the coefficient λ of the equation is in the interval $(0,1)$ and the initial value $x(0)$ is positive, then all solutions tend to infinity, but if

$$\lambda\in(-\infty,0]\cup(2^\alpha,\infty)$$

then the equation is asymptotically stable. Moreover, in [14] the authors obtain the exact rate of divergence for $\alpha,\lambda\in(0,1)$ and $x(0)>0$, and the exact rate of convergence for $\alpha\in(0,1)$ and

$$\lambda\in(-\infty,0]\cup(2,\infty).$$

In both cases it is shown that the rates are polynomial in these subregions of stability and instability. This is in contrast to integer-order systems, where it is well-known that the growth rates of solutions are exponential. Higher-dimensional discrete linear time-invariant fractional equations

$$(\Delta^\alpha x)(n+1-\alpha)=Ax(n),\quad n\in\mathbb{N}_0,$$

with coefficient matrix $A\in\mathbb{R}^{d\times d}$, are considered in [15] in the context of discretizations of linear continuous time-invariant Riemann-Liouville equations with discretization step size $h>0$. From the results of [15], with $h=1$, one may obtain both necessary and sufficient conditions for stability of higher-dimensional linear discrete time-invariant fractional equations with backward Riemann-Liouville operator. In certain subregions of the stability region, solutions x converge with the rate $n^{-1-\alpha}$, i.e. the limit $\lim_{n\to\infty}x(n)n^{1+\alpha}$ exists and is different from zero. The relation of stability of solutions to properties of the discrete Mittag-Leffler function, as well as results about scalar fractional backward equations, are presented in [30]. The stability of higher-dimensional equations with forward Caputo operator is investigated in [3]. The main result [3, Thm. 3.2] provides

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a necessary and sufficient condition for the asymptotic stability, which is that solutions $z \in \mathbb{C}$ of the equation

$$\det\left(I - z^{-1}(1-z^{-1})^{-\alpha}A\right) = 0,$$

where A is the coefficient matrix, lie inside the unit circle. Further investigation of linear equation with forward Caputo operator can be found in [13], where the authors present explicit necessary and sufficient conditions for stability of linear time-invariant fractional systems. They also prove that $O(n^{-\alpha})$ is an upper bound for the decay rate as $n \to \infty$. In [13] a deeper analysis of one-dimensional time-invariant Riemann-Liouville and Caputo equations with coefficient λ is performed and it is shown [13, Cor. 4.1] that scalar Riemann-Liouville equations are asymptotically stable if and only if $\lambda \in (-2^{\alpha}, 0]$, and scalar Caputo equations, if and only if $\lambda \in (-2^{\alpha}, 0)$. Moreover, they show that for $\lambda \in (-2\alpha, 0)$, the exact decay rates of solutions of scalar Caputo and Riemann-Liouville equations are $n^{-\alpha}$ and $n^{-\alpha-1}$, respectively. Finally, the paper [25] continues investigations of systems of backward Riemann-Liouville equations from [15] and answers on open question from [15] about stability on the boundary curve separating asymptotic stability and unstability regions.

Backward and forward Caputo and Riemann-Liouville fractional differences can be expressed as a convolution of sequences with generalized binomial coefficients, a summary of these facts is presented e.g. in [1]. Using these relations, a discrete-time fractional equation can be expressed as a convolution Volterra difference equation. Then its dynamical behavior may be studied using results about asymptotic behavior of convolutions of sequences (see e.g. [11, 17, 18, 28, 31] and the references therein). This approach is considered e.g. in [4, 10, 38] for general Volterra equations to obtain the exact decay rates of solutions. In [15] and [13], results from [4] are used to determine the exact decay rates of solutions. However, the required assumptions allow to determine decay rates only in certain subregions of the stability region.

The main objective of this paper is to investigate decay and growth rates of solutions to linear discrete-time forward Caputo and Riemann-Liouville equations. It is organized as follows: Section 2 collects basic notation and facts. A partial analysis of decay rates of one-dimensional time-invariant forward Caputo and Riemann-Liouville equations is presented in Section 3. Section 4 contains a polynomial separation result for solutions to linear systems of time-varying Caputo equations. The last section concludes with some remarks.

2. Basic notions

We introduce some necessary notions concerning fractional summation and fractional differences. Denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{Z}$ the set of integers, by $\mathbb{N} := \mathbb{Z}_{\geq 0}$ the set of natural numbers $\{0, 1, 2, \dots\}$ including 0, and by $\mathbb{Z}_{\leq 0} := \{0, -1, -2, \dots\}$ the set of non-positive integers. For $a \in \mathbb{R}$ we denote by $\mathbb{N}_a := a + \mathbb{N}$ the set $\{a, a+1, \dots\}$. By $\Gamma\colon \mathbb{R} \setminus \mathbb{Z}_{\leq 0} \to \mathbb{R}$ we denote the Euler Gamma function defined by

$$\Gamma(\alpha) := \lim_{n\to\infty} \frac{n^{\alpha} n!}{\alpha(\alpha+1)\cdots(\alpha+n)} \qquad (\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}), \tag{1}$$

which is well-defined, since the limit exists (see e.g. [26, p. 156]), and

$$\Gamma(\alpha) = \begin{cases} \int\limits_0^{\infty} x^{\alpha-1} e^{-x}\,\mathrm{d}x & \text{if } \alpha > 0, \\ \dfrac{\Gamma(\alpha+1)}{\alpha} & \text{if } \alpha < 0 \text{ and } \alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}. \end{cases} \tag{2}$$

Note that $\Gamma(\alpha) > 0$ for all $\alpha > 0$.

For $s \in \mathbb{R}$ with $s+1, s+1-\alpha \notin \mathbb{Z}_{\leq 0}$ the falling factorial power $(s)^{(\alpha)}$ is defined by

$$(s)^{(\alpha)} := \frac{\Gamma(s+1)}{\Gamma(s+1-\alpha)} \qquad \left(s \in (\mathbb{R} \setminus \mathbb{Z}_{\leq -1}) \cap (\mathbb{R} \setminus \alpha + \mathbb{Z}_{\leq -1})\right). \tag{3}$$

By $\lceil x \rceil := \min\{k \in \mathbb{Z}\colon k \geq x\}$ we denote the least integer greater or equal to x and by $\lfloor x \rfloor := \max\{k \in \mathbb{Z}\colon k \leq x\}$ the greatest integer less or equal to x.

For $a \in \mathbb{R}$, $\nu \in \mathbb{R}_{>0}$ and a function $f : \mathbb{N}_a \to \mathbb{R}$, the ν-th delta fractional sum $\Delta_a^{-\nu} f\colon \mathbb{N}_{a+\nu} \to \mathbb{R}$ of f is defined as

$$(\Delta_a^{-\nu} f)(t) := \frac{1}{\Gamma(\nu)} \sum_{k=a}^{t-\nu} (t-k-1)^{(\nu-1)} f(k) \qquad (t \in \mathbb{N}_{a+\nu}).$$

In the majority of our further considerations we will assume that $a = 0$ and then we will write simply $\Delta^{-\nu} f$ instead of $\Delta_0^{-\nu} f$.

Let $\alpha \in (0,1)$, $a \in \mathbb{R}$ and $f\colon \mathbb{N}_a \to \mathbb{R}$. The Caputo forward difference ${}_{\mathrm{C}}\Delta_a^{\alpha} f\colon \mathbb{N}_{a+1-\alpha} \to \mathbb{R}$ of f of order α is defined as the composition ${}_{\mathrm{C}}\Delta_a^{\alpha} := \Delta_a^{-(1-\alpha)} \circ \Delta$ of the $(1-\alpha)$-th delta fractional sum with the classical difference operator $t \mapsto \Delta f(t) := f(t+1) - f(t)$, i.e.

$$({}_{\mathrm{C}}\Delta_a^{\alpha} f)(t) := (\Delta_a^{-(1-\alpha)} \Delta f)(t) \qquad (t \in \mathbb{N}_{a+1-\alpha}).$$

The Riemann-Liouville forward difference ${}_{\text{R-L}}\Delta_a^{\alpha} f\colon \mathbb{N}_{a+1-\alpha} \to \mathbb{R}$ of f of order α is defined as ${}_{\text{R-L}}\Delta_a^{\alpha} := \Delta \circ \Delta_a^{-(1-\alpha)}$, i.e.

$$({}_{\text{R-L}}\Delta_a^{\alpha} f)(t) := (\Delta \Delta_a^{-(1-\alpha)} f)(t) \qquad (t \in \mathbb{N}_{a+1-\alpha}).$$

Similarly, as for the fractional sum, if $a = 0$ we simply write ${}_{\mathrm{C}}\Delta^{\alpha} f$ and ${}_{\text{R-L}}\Delta^{\alpha} f$.

Finally, we list several results for the binomial coefficient

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \qquad (\alpha \in \mathbb{R},\ k \in \mathbb{N}),$$

with the convention that for $k = 0$ the empty product

$$\prod_{\ell=1}^{k} (\alpha - \ell + 1) = \alpha(\alpha-1)\cdots(\alpha-k+1)$$

equals 1, i.e. $\binom{\alpha}{0} := 1$. These results can be found e.g. in [13, p. 656] and [22, pp. 165].

$$\binom{\alpha+k-1}{k} = (-1)^k \binom{-\alpha}{k} \quad (\alpha \in \mathbb{R}, k \in \mathbb{N}), \tag{4}$$

$$\sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} = (-1)^n \binom{\alpha-1}{n} \quad (\alpha \in \mathbb{R}, n \in \mathbb{N}), \tag{5}$$

and

$$\lim_{n\to\infty} \binom{\alpha}{n+1} \frac{\Gamma(-\alpha) n^{1+\alpha}}{(-1)^{n+1}} = 1 \quad (\alpha \in \mathbb{R}\setminus\mathbb{Z}), \tag{6}$$

since

$$1 = \lim_{n\to\infty} \Gamma(-\alpha)(-1)^{n+1} n^{1+\alpha} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!(n+1)} \frac{n+1}{n}$$
$$= \lim_{n\to\infty} \binom{\alpha}{n+1} \frac{\Gamma(-\alpha) n^{1+\alpha}}{(-1)^{n+1}}$$

by (1).

In our further consideration we need the following technical lemma.

Lemma 1. Let $\alpha > 0$ and the sequence $(u_{-\alpha}(k))_{k\in\mathbb{N}}$ be defined by

$$u_{-\alpha}(k) := (-1)^k \binom{-\alpha}{k} \quad (k \in \mathbb{N}). \tag{7}$$

Then the following statements hold:

(i) $u_{-\alpha}(k) > 0$ for $k \in \mathbb{N}$.

(ii) If $0 < \alpha < 1$, then $(u_{-\alpha}(k))_{k\in\mathbb{N}}$ is a decreasing sequence.

(iii) $\sum_{k=0}^{n} u_{-\alpha}(k) = u_{-\alpha-1}(n)$ for $n \in \mathbb{N}$.

(iv) There exist $\overline{m}$, $\overline{M} > 0$ such that

$$\frac{\overline{m}}{n^{1-\alpha}} < u_{-\alpha}(n) < \frac{\overline{M}}{n^{1-\alpha}} \quad (n \in \mathbb{N}\setminus\{0\}).$$

(v) For $n \in \mathbb{N}$,

$$u_{-(i+1)\alpha-1}(n-i) = \sum_{k=i}^{n} u_{-\alpha}(n-k) u_{-i\alpha-1}(k-i) \quad (i \in \{0,\dots,n\}), \tag{8}$$

i.e. for $0 \le i \le n$,

$$(-1)^{n-i} \binom{-(i+1)\alpha-1}{n-i} = \sum_{k=i}^{n} (-1)^{n-k} \binom{-\alpha}{n-k} (-1)^{k-i} \binom{-i\alpha-1}{k-i}. \tag{9}$$

(vi) For $i \in \mathbb{N}$, $(u_{-i\alpha-1}(n))_{n\in\mathbb{N}}$ is a positive increasing sequence and $u_{-i\alpha-1}(n) \ge 1$ for $i, n \in \mathbb{N}$.

Proof. Let $\alpha > 0$.

(i) $u_{-\alpha}(0) = 1$ and for $k \in \mathbb{N}\setminus\{0\}$,

$$u_{-\alpha}(k) = (-1)^k \frac{(-\alpha)(-\alpha-1)\cdots(-\alpha-k+1)}{k!}$$
$$= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}.$$

Thus, $u_{-\alpha}(k) > 0$ for $k \in \mathbb{N}$.

(ii) If $\alpha \in (0,1)$ then

$$u_{-\alpha}(k+1) = \frac{\alpha(\alpha+1)\cdots(\alpha+k)}{(k+1)!}$$
$$= \frac{\alpha+k}{k+1} u_{-\alpha}(k) < u_{-\alpha}(k) \quad (k \in \mathbb{N}),$$

i.e. $(u_{-\alpha}(k))_{k\in\mathbb{N}}$ is a decreasing sequence.

(iii) This follows directly from (5).

(iv) Using (1),

$$\Gamma(\alpha) = \lim_{n\to\infty} \frac{n^\alpha n!}{\alpha(\alpha+1)\cdots(\alpha+n)}$$
$$= \lim_{n\to\infty} \frac{1}{(-1)^n n^{-\alpha}(\alpha+n) \frac{(-\alpha)(-\alpha-1)\cdots(-\alpha-n+1)}{n!}}$$
$$= \lim_{n\to\infty} \frac{1}{n^{1-\alpha}\left(\frac{\alpha}{n}+1\right)(-1)^n \binom{-\alpha}{n}}$$
$$= \lim_{n\to\infty} \frac{1}{n^{1-\alpha}\left(\frac{\alpha}{n}+1\right) u_{-\alpha}(n)}.$$

Hence,

$$\lim_{n\to\infty} \Gamma(\alpha)\left(\frac{\alpha}{n}+1\right) n^{1-\alpha} u_{-\alpha}(n) = 1.$$

Using (i) and the fact that $\Gamma(\alpha) > 0$, there exist $m, M > 0$ such that

$$m \le \Gamma(\alpha)\left(\frac{\alpha}{n}+1\right) n^{1-\alpha} u_{-\alpha}(n) \le M \quad (n \in \mathbb{N}).$$

Hence

$$\frac{\overline{m}}{n^{1-\alpha}} \le \frac{m}{\Gamma(\alpha)\left(\frac{\alpha}{n}+1\right) n^{1-\alpha}} \le u_{-\alpha}(n)$$
$$\le \frac{M}{\Gamma(\alpha)\left(\frac{\alpha}{n}+1\right) n^{1-\alpha}} \le \frac{\overline{M}}{n^{1-\alpha}} \quad (n \in \mathbb{N}\setminus\{0\})$$

with

$$\overline{m} := \frac{m}{\Gamma(\alpha)(\alpha+1)}$$

and

$$\overline{M} := \frac{M}{\Gamma(\alpha)}.$$

(v) Let $n \in \mathbb{N}$, $i \in \{0,\dots,n\}$. Expanding the equality

$$(1-x)^{-(i+1)\alpha-1} = (1-x)^{-\alpha}(1-x)^{-i\alpha-1} \quad (x \in \mathbb{R}\setminus\{-1\})$$

with the binomial series, yields

$$\sum_{n=0}^{\infty}(-1)^n\binom{-(i+1)\alpha-1}{n}x^n = \sum_{k=0}^{\infty}(-1)^k\binom{-\alpha}{k}x^k\sum_{j=0}^{\infty}(-1)^j\binom{-i\alpha-1}{j}x^j$$

for $|x|<1$

Comparing the coefficients of x^{n-i}, we receive the claim (9)

$$(-1)^{n-i}\binom{-(i+1)\alpha-1}{n-i} = \sum_{k=i}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}(-1)^{k-i}\binom{-i\alpha-1}{k-i}.$$

(vi) Let $i \in \mathbb{N}$. From

$$u_{-i\alpha-1}(n) = (-1)^n\binom{-i\alpha-1}{n} = \frac{(i\alpha+1)(i\alpha+2)\cdots(i\alpha+n)}{1\cdot 2\cdots n} \geq 1 \quad (n\in\mathbb{N}),$$

it follows that $n \mapsto u_{-i\alpha-1}(n)$ is increasing. □

For two sequences $f,g:\mathbb{N}\to\mathbb{R}$ the convolution $f*g:\mathbb{N}\to\mathbb{R}$ is defined as

$$(f*g)(n) = \sum_{i=0}^{n} f(n-i)g(i) \quad (n\in\mathbb{N}).$$

In this paper we consider linear bounded non-autonomous fractional difference systems of the form

$$(\Delta^{\alpha}x)(n+1-\alpha) = A(n)x(n) \quad (n\in\mathbb{N}), \tag{10}$$

where $x\colon \mathbb{N}\to\mathbb{R}^d$, Δ^{α} is either the Caputo ${}_{\mathrm{C}}\Delta^{\alpha}$ or Riemann-Liouville ${}_{\mathrm{R\text{-}L}}\Delta^{\alpha}$ forward difference operator of a real order $\alpha \in (0,1)$ and $A\colon \mathbb{N}\to\mathbb{R}^{d\times d}$ is bounded, i.e.

$$M := \sup_{n\in\mathbb{N}}\|A(n)\| < \infty. \tag{11}$$

For an initial value $x_0 \in \mathbb{R}^n$, (10) has a unique solution $x\colon \mathbb{N}\to\mathbb{R}^d$ which satisfies the initial condition $x(0)=x_0$. We denote x by $\varphi_{\mathrm{C}}(\cdot,x_0)$ or $\varphi_{\mathrm{R\text{-}L}}(\cdot,x_0)$, respectively. In Theorem 1 below, we characterize solutions of (10) as solutions of an associated Volterra convolution equation. For the proof of this characterization we use the following Lemma which provides an alternative representation of fractional differences [1,2] and fractional sums [23].

Lemma 2. Let $\alpha>0$ and $f\colon\mathbb{N}\to\mathbb{R}$. Then

$$({}_{\mathrm{R\text{-}L}}\Delta^{\alpha}f)(n) = \sum_{k=0}^{\alpha+n}(-1)^k\binom{\alpha}{k}f(n-k+\alpha) \quad (n\in\mathbb{N}_{1-\alpha}), \tag{12}$$

$$({}_{\mathrm{C}}\Delta^{\alpha}f)(n) = \sum_{k=0}^{\alpha+n}(-1)^k\binom{\alpha}{k}f(n-k+\alpha) - \frac{(n)^{(-\alpha)}}{\Gamma(1-\alpha)}f(0) \quad (n\in\mathbb{N}_{1-\alpha}), \tag{13}$$

$$(\Delta^{-\alpha}f)(n) = \sum_{k=0}^{-\alpha+n}(-1)^k\binom{-\alpha}{k}f(n-k-\alpha) \quad (n\in\mathbb{N}_{\alpha}). \tag{14}$$

Theorem 1. (Equivalent Volterra difference equation)
Let $\alpha\in(0,1)$.

(a) $x\colon\mathbb{N}\to\mathbb{R}^d$ is a solution of (10) with Caputo forward difference operator if and only if

$$x(n+1) = x(0) + \sum_{k=0}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}A(k)x(k) \quad (n\in\mathbb{N}). \tag{15}$$

and this equation is equivalent to

$$x(n+1) = A(n)x(n) + \sum_{k=0}^{n}(-1)^{n-k}\binom{\alpha}{n-k+1}x(k) + (-1)^{n+1}\binom{\alpha-1}{n+1}x(0) \quad (n\in\mathbb{N}). \tag{16}$$

(b) $x\colon\mathbb{N}\to\mathbb{R}^d$ is a solution of (10) with Riemann-Liouville forward difference operator if and only if

$$x(n+1) = \sum_{k=0}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}A(k)x(k) + (-1)^{n+1}\binom{-\alpha}{n+1}x(0) \quad (n\in\mathbb{N}) \tag{17}$$

and this equation is equivalent to

$$x(n+1) = A(n)x(n) + \sum_{k=0}^{n}(-1)^{n-k}\binom{\alpha}{n-k+1}x(k) \quad (n\in\mathbb{N}). \tag{18}$$

Proof. (a) In order to prove (15), we recall the following identity from [3, Thm. 2.4] (it should be noticed that there is a misprint in [3, Thm. 2.4] and the correct formula is stated some lines above this theorem). Let $a\in\mathbb{R}$, $x\colon\mathbb{N}_a\to\mathbb{R}$, $\alpha\in\mathbb{R}_{>0}\setminus\mathbb{N}$, $m:=\lceil\alpha\rceil$.

Then

$$x(t)=\sum_{k=0}^{m-1}\frac{(t-a)^{(k)}}{k!}\Delta^k x(a) + \frac{1}{\Gamma(\alpha)}\sum_{s=a+m-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}({}_{C}\Delta_a^{\alpha}x)(s) \quad (t\in\mathbb{N}_{a+m}). \tag{19}$$

Using (19) for $a=0$ and $\alpha\in(0,1)$, we have

$$x(t)=x(0)+\frac{1}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}({}_{C}\Delta^{\alpha}x)(s) \quad (t\in\mathbb{N}\setminus\{0\}).$$

For $n\in\mathbb{N}$ and $t:=n+1$, the substitution $s=k+(1-\alpha)$ yields

$$x(n+1)=x(0)+\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n}(n-k+\alpha-1)^{(\alpha-1)}({}_{C}\Delta^{\alpha}x)(k+1-\alpha) \quad (n\in\mathbb{N}).$$

From the definition (3) of falling factorial power,

$$\frac{1}{\Gamma(\alpha)}(n-k+\alpha-1)^{(\alpha-1)}=\frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)\Gamma(n-k+1)} = \binom{n-k+\alpha-1}{n-k}=\binom{\alpha+(n-k)-1}{n-k} \quad (k\in\{0,\dots,n\}).$$

Using (4),

$$\frac{1}{\Gamma(\alpha)}(n-k+\alpha-1)^{(\alpha-1)}=(-1)^{n-k}\binom{-\alpha}{n-k} \quad (k\in\{0,\dots,n\})$$

and therefore

$$x(n+1)=x(0)+\sum_{k=0}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}({}_{C}\Delta^{\alpha}x)(k+1-\alpha) \quad (n\in\mathbb{N}).$$

With (10) for the Caputo forward difference operator

$$({}_{C}\Delta^{\alpha}x)(k+1-\alpha)=A(k)x(k) \quad (k\in\mathbb{N}),$$

we receive equation (15). In order to prove (16), use (10) and apply (13) to $x(\cdot+1-\alpha)\colon \mathbb{N}_{1-\alpha}\to\mathbb{R}$ to get

$$\begin{aligned}A(n)x(n)&=({}_{C}\Delta^{\alpha}x)(n+1-\alpha)\\ &=\sum_{k=0}^{n+1}(-1)^k\binom{\alpha}{k}x(n+1-k)-\frac{(n+1-\alpha)^{(-\alpha)}}{\Gamma(1-\alpha)}x(0)\\ &=x(n+1)+\sum_{k=1}^{n+1}(-1)^k\binom{\alpha}{k}x(n+1-k)-\frac{(n+1-\alpha)^{(-\alpha)}}{\Gamma(1-\alpha)}x(0)\\ &=x(n+1)+\sum_{k=0}^{n}(-1)^{n-k+1}\binom{\alpha}{n-k+1}x(k)\\ &\quad-\frac{(n+1-\alpha)^{(-\alpha)}}{\Gamma(1-\alpha)}x(0).\end{aligned}$$

Using (4),

$$(-1)^{n+1}\binom{\alpha-1}{n+1}=\binom{n+1-\alpha}{n+1} =\frac{\Gamma(n+2-\alpha)}{\Gamma(n+2)\Gamma(1-\alpha)}=\frac{(n+1-\alpha)^{(-\alpha)}}{\Gamma(1-\alpha)},$$

and (16) follows. Similarly one can show that (16) implies (15), and (15) implies (10).

(b) In order to prove (17), we recall the following identity from [7, Theorem 4.10]. Let $a\in\mathbb{R}$, $\alpha\in(0,1)$, $f\colon\mathbb{N}_a\to\mathbb{R}$. Then

$$(\Delta_{a-\alpha+1}^{-\alpha}\,{}_{\text{R-L}}\Delta_a^{\alpha}f)(b) = f(b)-(-1)^{b-a}\binom{-\alpha}{b-a}f(a) \quad (b\in\mathbb{N}_a\setminus\{a\}).$$

For $n\in\mathbb{N}$, $a=0$, $b=n+1$ and $x\colon\mathbb{N}\to\mathbb{R}$, we get

$$(\Delta_{-\alpha+1}^{-\alpha}\,{}_{\text{R-L}}\Delta^{\alpha}x)(n+1-\alpha) =x(n+1)-(-1)^{n+1}\binom{-\alpha}{n+1}x(0) \quad (n\in\mathbb{N})$$

and by (14) with $f(n)=A(n)x(n)$, we may show that

$$(\Delta^{-\alpha}f)(n+\alpha) =\sum_{k=0}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}A(k)x(k) \quad (n\in\mathbb{N}).$$

Combining the last two equalities and the fact that x is a solution of (10) with the Riemann-Liouville operator, we get (17). In the same way as we proved (16), but using (12) instead of (13), we may show (18). □

The following theorem is a special case of [19, Proposition 1].

Theorem 2. (Variation of constants formula for Volterra difference equations) Let $A,B\colon\mathbb{N}\to\mathbb{R}^{d\times d}$ and $g\colon\mathbb{N}\to\mathbb{R}^d$. Then, if $R\colon\mathbb{N}\to\mathbb{R}^{d\times d}$ satisfies the equation

$$R(n+1)=A(n)R(n)+\sum_{k=0}^{n}B(n-k)R(k) \quad (n\in\mathbb{N})$$

with the initial condition $R(0)=I$, then the unique solution $x\colon\mathbb{N}\to\mathbb{R}^d$ of the equation

$$x(n+1)=A(n)x(n)+\sum_{k=0}^{n}B(n-k)x(k)+g(n) \quad (n\in\mathbb{N})$$

with the initial condition $x(0)=x_0\in\mathbb{R}^d$, is given by

$$x(n)=R(n)x_0+\sum_{k=0}^{n-1}R(n-k-1)g(k) \quad (n\in\mathbb{N}).$$

Combining Theorem 2 with the Volterra representations (16) and (18) of Theorem 1 in the one-dimensional and time-invariant case, we obtain the following lemma.

Lemma 3. Suppose that $r\colon \mathbb{N}\to\mathbb{R}$ is the solution of the scalar Riemann-Liouville equation

$$r(n+1)=\lambda r(n)+\sum_{k=0}^{n}(-1)^{n-k}\binom{\alpha}{n-k+1}r(k) \qquad (n\in\mathbb{N})$$

with initial condition $r(0)=1$, then the solution $x\colon \mathbb{N}\to\mathbb{R}$ of the Caputo equation

$$x(n+1)=\lambda x(n)+\sum_{k=0}^{n}(-1)^{n-k}\binom{\alpha}{n-k+1}x(k) \\ +(-1)^{n+1}\binom{\alpha-1}{n+1}x(0) \quad (n\in\mathbb{N})$$

with initial condition $x(0)=x_0$, is given by the convolution $x=r*g$, i.e.

$$x(n)=(r*g)(n) \quad (n\in\mathbb{N})$$

of r with $g\colon \mathbb{N}\to\mathbb{R}$, defined by

$$g(n)=(-1)^{n+1}\binom{\alpha-1}{n+1}x_0 \quad (n\in\mathbb{N}).$$

In the autonomous case, we have the following formula for the solution of (10).

Theorem 3. Let $A\in\mathbb{R}^{d\times d}$, $x_0\in\mathbb{R}^d$. Then the solution $x\colon \mathbb{N}\to\mathbb{R}^d$ of the autonomous Caputo forward difference equation (10)

$$({}_{\mathrm{C}}\Delta^{\alpha}x)(n+1-\alpha)=Ax(n) \qquad (n\in\mathbb{N}),$$

with initial condition $x(0)=x_0$, is given by

$$\varphi_{\mathrm{C}}(n,x_0)=\sum_{i=0}^{n}\frac{(n+i(\alpha-1))^{(i\alpha)}}{\Gamma(i\alpha+1)}A^i x_0 \quad (n\in\mathbb{N}). \tag{20}$$

Proof. Let $A\in\mathbb{R}^{d\times d}$, $x_0\in\mathbb{R}^d$. For $n=0$, (20) obviously holds. To prove (20) by induction over $n\in\mathbb{N}$, note that by (15),

$$\varphi_{\mathrm{C}}(n+1,x_0)=x_0+\sum_{k=0}^{n}u_{-\alpha}(n-k)A\varphi_{\mathrm{C}}(k,x_0) \quad (n\in\mathbb{N}), \tag{21}$$

with $u_{-\alpha}(n)=(-1)^n\binom{-\alpha}{n}$ defined in (7). Using (21), the induction hypothesis (20) for a fixed $n\in\mathbb{N}$, and formula (8),

$$\begin{aligned}\varphi_{\mathrm{C}}(n+1,x_0)&=x_0+\sum_{k=0}^{n}u_{-\alpha}(n-k)A\sum_{i=0}^{k}u_{-i\alpha-1}(k-i)A^i x_0\\&=x_0+\sum_{i=0}^{n}\sum_{k=i}^{n}u_{-\alpha}(n-k)u_{-i\alpha-1}(k-i)A^{i+1}x_0\\&=x_0+\sum_{i=0}^{n}u_{-(i+1)\alpha-1}(n-i)A^{i+1}x_0\\&=\sum_{i=0}^{n+1}u_{-i\alpha-1}(n+1-i)A^i x_0,\end{aligned}$$

proving (20) for $n+1$, and the induction step is complete. □

3. Asymptotic behavior of scalar autonomous equations

Consider the one-dimensional and time-invariant equation (10)

$$(\Delta^{\alpha}x)(n+1-\alpha)=\lambda x(n) \quad (n\in\mathbb{N}), \tag{22}$$

with $\lambda\in\mathbb{R}$ and Caputo or Riemann-Liouville forward difference operator, respectively. It has been shown in [13, Corollary 4.1] that the Caputo equation is asymptotically stable if and only if $-2^{\alpha}<\lambda<0$ and that the Riemann-Liouville equation is asymptotically stable if and only if $-2^{\alpha}<\lambda\le 0$. In [13, Theorem 1.4 and Corollary 4.2] the following decay rates have been shown

$$\varphi_{\mathrm{C}}(n,x_0)=O(n^{-\alpha}) \quad \text{as } n\to\infty, \tag{23}$$

for $-2^{\alpha}<\lambda<0$ and

$$\lim_{n\to\infty}\varphi_{\text{R-L}}(n,y_0)n^{1+\alpha}=\frac{-y_0}{\lambda^2\Gamma(-\alpha)} \tag{24}$$

for $-2\alpha<\lambda<0$. The main result of this section is the exact representation of the asymptotic relation (23) when $\lambda\in(-2\alpha,0)$.

Theorem 4. (Decay rate of asymptotically stable solutions; the Caputo case) Let $\lambda\in(-2\alpha,0)$, $x_0\in\mathbb{R}$. Then

$$\lim_{n\to\infty}\varphi_{\mathrm{C}}(n,x_0)n^{\alpha}=\frac{-x_0}{\lambda\Gamma(1-\alpha)}. \tag{25}$$

In order to show Theorem 4, we prove the following preparatory lemmas.

Lemma 4. Let $\lambda\in(-2^{\alpha},0)$, $r_0\in\mathbb{R}$. Then the series $\sum_{n=0}^{\infty}\varphi_{\text{R-L}}(n,r_0)$ is absolutely convergent and

$$\sum_{n=0}^{\infty}\varphi_{\text{R-L}}(n,r_0)=\frac{-r_0}{\lambda}. \tag{26}$$

Proof. Let $\lambda\in(-2^{\alpha},0)$, $r_0\in\mathbb{R}$. Using equation (18) of Theorem 1(b), (22) with the Riemann-Liouville operator is equivalent to

$$r(n+1)=\sum_{i=0}^{n}a(n-i)r(i), \quad r(0)=r_0, \tag{27}$$

where $\varphi_{\text{R-L}}(n,r_0)=r(n)$, $n\in\mathbb{N}$, and

$$a(n)=\begin{cases}\lambda+\alpha & \text{for } n=0,\\ (-1)^n\binom{\alpha}{n+1} & \text{for } n\in\mathbb{N}\setminus\{0\}.\end{cases}$$

Using the fact that for $\lambda\in(-2^{\alpha},0)$ equation (27) is uniformly asymptotically stable, [21, Theorem 2] implies that the series $S:=\sum_{n=0}^{\infty}r(n)$ is absolutely convergent. By (27), we get

$$S-r(0)=\sum_{n=0}^{\infty}\sum_{i=0}^{n}a(n-i)r(i). \tag{28}$$

Since the series $\sum_{i=0}^{\infty} a(i)$ and $\sum_{i=0}^{\infty} r(i)$ are absolutely convergent, Mertens's theorem [29] implies that their Cauchy product

$$\sum_{n=0}^{\infty}\sum_{i=0}^{n} a(n-i)r(i)$$

is also convergent and

$$\sum_{n=0}^{\infty}\sum_{i=0}^{n} a(n-i)r(i) = \sum_{n=0}^{\infty} a(n) \sum_{n=0}^{\infty} r(n).$$

Using the fact that for $\alpha > 0$ and $x \in [-1,1]$ the series

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = (1+x)^{\alpha}$$

converges absolutely, we get by setting $x = -1$,

$$\sum_{n=0}^{\infty} a(n) = 1+\lambda.$$

By (28), we have $S - r_0 = (1+\lambda)S$ and finally

$$\sum_{n=0}^{\infty} r(n) = -\frac{r_0}{\lambda},$$

proving (26). □

Lemma 5. Let $\alpha \in (0,1)$. Then

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=m+1}^{n-m}\frac{(n-i)^{-1-\alpha}i^{-\alpha}}{n^{-\alpha}} = 0.$$

Proof. Let $m,n \in \mathbb{N}\setminus\{0\}$, $m < 2n$, and denote

$$\begin{aligned} I(n,m) &:= \sum_{i=m+1}^{n-m}\frac{(n-i)^{-1-\alpha}i^{-\alpha}}{n^{-\alpha}} \\ &= \sum_{i=m+1}^{\left[\frac{n}{2}\right]-m}\frac{(n-i)^{-1-\alpha}i^{-\alpha}}{n^{-\alpha}} + \sum_{i=\left[\frac{n}{2}\right]-m+1}^{n-m}\frac{(n-i)^{-1-\alpha}i^{-\alpha}}{n^{-\alpha}}. \end{aligned}$$

For

$$m+1 \le i \le \left[\frac{n}{2}\right]-m$$

we have

$$(n-i)^{-1-\alpha} \le \left(\left[\frac{n}{2}\right]+m\right)^{-1-\alpha}$$

and therefore

$$\begin{aligned} I(n,m) \le{} & \frac{\left(\left[\frac{n}{2}\right]+m\right)^{-1-\alpha}}{n^{-\alpha}}\sum_{i=m+1}^{\left[\frac{n}{2}\right]-m} i^{-\alpha} \\ &+ \frac{\left(\left[\frac{n}{2}\right]-m+1\right)^{-\alpha}}{n^{-\alpha}}\sum_{i=\left[\frac{n}{2}\right]-m+1}^{n-m}(n-i)^{-1-\alpha}. \end{aligned} \tag{29}$$

Observe that for $\alpha \in (0,1)$ we have

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k} i^{-\alpha} = 0. \tag{30}$$

This follows from the inequality

$$\sum_{i=1}^{k} i^{-\alpha} \le \int_{1}^{k+1} x^{-\alpha}\,\mathrm{d}x = \frac{(k+1)^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1}.$$

Using (30), the limit of the first term in (29) for $n\to\infty$ satisfies

$$\begin{aligned} &\lim_{n\to\infty}\frac{\left(\left[\frac{n}{2}\right]+m\right)^{-1-\alpha}}{n^{-\alpha}}\sum_{i=m+1}^{\left[\frac{n}{2}\right]-m} i^{-\alpha} = \\ &= \lim_{n\to\infty}\frac{\left(\left[\frac{n}{2}\right]+m\right)^{-\alpha}}{n^{-\alpha}}\frac{1}{\left[\frac{n}{2}\right]+m}\sum_{i=m+1}^{\left[\frac{n}{2}\right]-m} i^{-\alpha} \\ &= \lim_{n\to\infty}\frac{\left(\left[\frac{n}{2}\right]+m\right)^{-\alpha}}{n^{-\alpha}}\lim_{n\to\infty}\frac{1}{\left[\frac{n}{2}\right]+m}\sum_{i=m+1}^{\left[\frac{n}{2}\right]-m} i^{-\alpha} \\ &= \left(\frac{1}{2}\right)^{-\alpha}\lim_{n\to\infty}\frac{1}{\left[\frac{n}{2}\right]+m}\sum_{i=m+1}^{\left[\frac{n}{2}\right]-m} i^{-\alpha} = 0. \end{aligned} \tag{31}$$

Estimating the limit superior of the second term in (29), we get

$$\begin{aligned} &\limsup_{n\to\infty}\frac{\left(\left[\frac{n}{2}\right]-m+1\right)^{-\alpha}}{n^{-\alpha}}\sum_{i=\left[\frac{n}{2}\right]-m+1}^{n-m}(n-i)^{-1-\alpha} \\ &\le \lim_{n\to\infty}\frac{\left(\left[\frac{n}{2}\right]-m+1\right)^{-\alpha}}{n^{-\alpha}} \\ &\quad\cdot\limsup_{n\to\infty}\sum_{i=\left[\frac{n}{2}\right]-m+1}^{n-m}(n-i)^{-1-\alpha} \\ &\le \left(\frac{1}{2}\right)^{-\alpha}\sum_{i=m}^{\infty} i^{-1-\alpha}. \end{aligned} \tag{32}$$

From (31), (32) and the fact that

$$\lim_{m\to\infty}\sum_{i=m}^{\infty} i^{-1-\alpha} = 0,$$

we get the claim. □

Lemma 6. Let $g, r\colon \mathbb{N}_0 \to \mathbb{R}$ satisfy the conditions

$$\lim_{n\to\infty}\frac{g(n)}{n^{-\alpha}} = d \tag{33}$$

and there exists a constant $c \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that

$$\left| \frac{r(n)}{n^{-\alpha-1}} \right| < c \tag{34}$$

for all $n > n_0$. Then

$$\lim_{n\to\infty} \frac{(r*g)(n)}{n^{-\alpha}} = d \sum_{i=0}^{\infty} r(i).$$

Proof. Let $m \in \mathbb{N}$. For $n > 2m$, we have

$$\frac{(r*g)(n)}{n^{-\alpha}} = E(n) + F(n) + G(n),$$

where

$$E(n) = \sum_{i=0}^{m} \frac{r(n-i)g(i)}{n^{-\alpha}},$$

$$F(n) = \sum_{i=m+1}^{n-m} \frac{(n-i)^{-1-\alpha} i^{-\alpha}}{n^{-\alpha}} \frac{r(n-i)}{(n-i)^{-1-\alpha}} \frac{g(i)}{i^{-\alpha}},$$

$$G(n) = \sum_{j=0}^{m-1} \frac{r(m-1-j)g(n-m+j+1)}{n^{-\alpha}}.$$

Condition (34) implies that

$$\lim_{n\to\infty} \frac{r(n-k)}{n^{-\alpha}} = 0$$

for each $k \in \mathbb{N}$ and therefore

$$\lim_{n\to\infty} E(n) = 0. \tag{35}$$

From condition (33) we have

$$\lim_{n\to\infty} \frac{g(n-k)}{n^{-\alpha}} = d$$

for each $k \in \mathbb{N}$, and therefore

$$\lim_{n\to\infty} G(n) = d \sum_{i=0}^{m-1} r(i). \tag{36}$$

Conditions (34) and (33) imply that $\theta < \infty$, where

$$\theta = \sup_{j\in\mathbb{N}\setminus\{0\}} \frac{r(j)}{j^{-1-\alpha}} \sup_{j\in\mathbb{N}\setminus\{0\}} \frac{g(j)}{j^{-\alpha}},$$

and therefore

$$|F(n)| \le \theta \sum_{i=m+1}^{n-m} \frac{(n-i)^{-1-\alpha} i^{-\alpha}}{n^{-\alpha}}. \tag{37}$$

Using (35), (36) and (37), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \left| \frac{1}{n^{-\alpha}} \sum_{i=0}^{n} r(n-i)g(i) - d\sum_{i=0}^{\infty} r(i) \right| \\
&= \limsup_{n\to\infty} \left| E(n) + F(n) + G(n) - d\sum_{i=0}^{\infty} r(i) \right| \\
&\le \limsup_{n\to\infty} |E(n)| + \limsup_{n\to\infty} \left| F(n) - d\sum_{i=m}^{\infty} r(i) \right| \\
&\quad + \limsup_{n\to\infty} \left| G(n) - d\sum_{i=0}^{m-1} r(i) \right| \\
&\le \theta \limsup_{n\to\infty} \sum_{i=m+1}^{n-m} \frac{(n-i)^{-1-\alpha} i^{-\alpha}}{n^{-\alpha}} + d\sum_{i=m}^{\infty} |r(i)|.
\end{aligned}$$

Taking the limit superior for $m \to \infty$, using Lemma 5 and the absolute convergence of the series $\sum_{i=0}^{\infty} r(i)$, we get the conclusion of the lemma. □

Proof of Theorem 4. The proof of (25) follows now from (24) with $y_0 = 1$, by using Lemmas 3 and 6 and equality (26), since for

$$g(n) = (-1)^{n+1} \binom{\alpha-1}{n+1} x_0,$$

by (6), we have

$$\lim_{n\to\infty} \frac{g(n)}{n^{-\alpha}} = \frac{x_0}{\Gamma(1-\alpha)}.$$

□

4. Separation results

Exponential rates of convergence or divergence of trajectories of dynamical systems can be described by Lyapunov exponents. The Lyapunov exponent of a sequence of real numbers $a(n)$ is defined as

$$\limsup_{n\to\infty} \frac{1}{n} \ln |a(n)|. \tag{38}$$

Theorem 4 shows that the Lyapunov exponent

$$\limsup_{n\to\infty} \frac{1}{n} \ln |\varphi_C(n, x_0)|$$

of a non-zero solution $\varphi_C(\cdot, x_0)$ of a one-dimensional Caputo system (10) with $x_0 \in \mathbb{R} \setminus \{0\}$ equals 0. In other words, solutions of (10) do not decay exponentially. The next theorem shows that this is also true for in higher dimensions. In what follows, we establish a lower bound on the exponential separation between two solutions of (10). As a consequence, we show that the Lyapunov exponent of an arbitrary non-trivial solution of (10) is always non-negative.

Theorem 5. Consider equation (10) with Caputo operator. Let $\lambda > \frac{\alpha}{1-\alpha}$, $x, y \in \mathbb{R}^d$ and $x \neq y$. Then

$$\limsup_{n\to\infty} n^{\lambda} \|\varphi_C(n,x) - \varphi_C(n,y)\| = \infty. \tag{39}$$

Consequently, for any $x_0 \in \mathbb{R}^d \setminus \{0\}$ we have

$$\limsup_{n\to\infty} \frac{1}{n} \ln \|\varphi_C(n,x_0)\| \geq 0. \tag{40}$$

Proof. Let $x, y \in \mathbb{R}^d$ with $x \neq y$ and $\lambda > \frac{\alpha}{1-\alpha}$. Suppose the contrary, i.e. there exists $K \in \mathbb{R}$ such that

$$\limsup_{n\to\infty} n^{\lambda} \|\varphi_C(n,x) - \varphi_C(n,y)\| < K,$$

which implies that

$$\lim_{n\to\infty} \|\varphi_C(n,x) - \varphi_C(n,y)\| = 0 \tag{41}$$

and therefore

$$L := \sup_{n\in\mathbb{N}} \|\varphi_C(n,x) - \varphi_C(n,y)\| < \infty. \tag{42}$$

Furthermore, there exists $N \in \mathbb{N}$ such that

$$\|\varphi_C(n,x) - \varphi_C(n,y)\| \leq K n^{-\lambda} \qquad (n \geq N). \tag{43}$$

Considering the Caputo equation in the form given by (15), we have

$$\begin{aligned}
&\varphi_C(n,x) - \varphi_C(n,y)\\
&\quad = x - y + \sum_{k=0}^{n} u_{-\alpha}(n-k) A(k)(\varphi_C(k,x) - \varphi_C(k,y))\\
&\quad = x - y + \sum_{k=0}^{n} B(n,k)(\varphi_C(k,x) - \varphi_C(k,y)),
\end{aligned}$$

where

$$B(n,k) := u_{-\alpha}(n-k) A(k),$$

with $u_{-\alpha}(\cdot)$ given by (7). Thus,

$$\begin{aligned}
\|x-y\| &\leq \|\varphi_C(n,x) - \varphi_C(n,y)\|\\
&\quad + \left\| \sum_{k=0}^{n} B(n,k)(\varphi_C(k,x) - \varphi_C(k,y)) \right\|.
\end{aligned}$$

Letting $n \to \infty$ and using (41), we obtain that

$$\limsup_{n\to\infty} \left\| \sum_{k=0}^{n} B(n,k)(\varphi_C(k,x) - \varphi_C(k,y)) \right\| > 0. \tag{44}$$

Since $\lambda > \frac{\alpha}{1-\alpha}$, there exists $\delta \in \left(\frac{\alpha}{\lambda}, 1-\alpha\right)$. To gain a contradiction to inequality (44), it is sufficient to show that

$$\limsup_{n\to\infty} \sum_{k=0}^{\lceil n^{\delta}\rceil - 1} B(n,k)(\varphi_C(k,x) - \varphi_C(k,y)) = 0 \tag{45}$$

and

$$\limsup_{n\to\infty} \sum_{k=\lceil n^{\delta}\rceil}^{n} B(n,k)(\varphi_C(k,x) - \varphi_C(k,y)) = 0. \tag{46}$$

By definition of $B(n,k)$ and non-negativity of the sequence $u_{-\alpha}(n)$ as in Lemma 1, we have

$$\begin{aligned}
&\left\| \sum_{k=0}^{\lceil n^{\delta}\rceil - 1} B(n,k)(\varphi_C(k,x) - \varphi_C(k,y)) \right\|\\
&\quad \leq \sum_{k=0}^{\lceil n^{\delta}\rceil - 1} \|B(n,k)\| \|\varphi_C(k,x) - \varphi_C(k,y)\|\\
&\quad \leq \sum_{k=0}^{\lceil n^{\delta}\rceil - 1} M u_{-\alpha}(n-k) \|\varphi_C(k,x) - \varphi_C(k,y)\|\\
&\quad \leq ML \sum_{k=0}^{\lceil n^{\delta}\rceil - 1} u_{-\alpha}(n-k),
\end{aligned}$$

where we used (42) to obtain the last inequality. By Lemma 1(i), the sequence $(u_{-\alpha}(n))_{n\in\mathbb{N}}$ is decreasing. Thus,

$$\left\| \sum_{k=0}^{\lceil n^{\delta}\rceil - 1} B(n,k)(\varphi_C(k,x) - \varphi_C(k,y)) \right\| \leq ML \lceil n^{\delta}\rceil u_{-\alpha}(n - \lceil n^{\delta}\rceil).$$

Using Lemma 1(iv), we obtain that

$$\begin{aligned}
&\left\| \sum_{k=0}^{\lceil n^{\delta}\rceil - 1} B(n,k)(\varphi_C(k,x) - \varphi_C(k,y)) \right\|\\
&\quad \leq ML(n^{\delta} + 1) \frac{\overline{M}}{(n - \lceil n^{\delta}\rceil)^{1-\alpha}},
\end{aligned}$$

which, together with the fact that $\delta < 1-\alpha$, proves (45). To conclude the proof we show (46). For this purpose, we use the estimate

$$\begin{aligned}
&\left\| \sum_{k=\lceil n^{\delta}\rceil}^{n} B(n,k)(\varphi_C(k,x) - \varphi_C(k,y)) \right\|\\
&\quad \leq \sum_{k=\lceil n^{\delta}\rceil}^{n} \|B(n,k)\| \|\varphi_C(k,x) - \varphi_C(k,y)\|\\
&\quad \leq M \sum_{k=\lceil n^{\delta}\rceil}^{n} u_{-\alpha}(n-k) \|\varphi_C(k,x) - \varphi_C(k,y)\|.
\end{aligned}$$

Let $n \in \mathbb{N}$ satisfy that $n^\delta \geq N$. Using (43), we obtain that

$$\left\| \sum_{k=\lceil n^\delta \rceil}^{n} B(n,k)(\varphi_C(k,x) - \varphi_C(k,y)) \right\| \leq MK\lceil n^\delta \rceil^{-\lambda} \sum_{k=\lceil n^\delta \rceil}^{n} u_{-\alpha}(n-k)$$

By Lemma 1(i) and (5), we have

$$\sum_{k=\lceil n^\delta \rceil}^{n} u_{-\alpha}(n-k) \leq \sum_{k=0}^{n} u_{-\alpha}(n-k) = u_{-(\alpha+1)}(n).$$

Thus,

$$\left\| \sum_{k=\lceil n^\delta \rceil}^{n} B(n,k)(\varphi_C(k,x) - \varphi_C(k,y)) \right\| \leq MK\lceil n^\delta \rceil^{-\lambda} u_{-(\alpha+1)}(n).$$

In light of Lemma 1(iv) for $\alpha + 1$, we have

$$\left\| \sum_{k=\lceil n^\delta \rceil}^{n} B(n,k)(\varphi_C(k,x) - \varphi_C(k,y)) \right\| \leq MKn^{-\delta\lambda} \frac{\overline{M}}{n^{-\alpha}}.$$

Note that $\delta\lambda > \alpha$, (46) is proved and the proof is complete. □

It is still an open question whether an analog result also holds for Riemann-Liouville equations.

Whereas the results in Section 3 show that an asymptotically stable solution of a scalar linear Caputo differential equations decays to zero with a polynomial rate of convergence, the next theorem shows that under certain assumptions a solution of Riemann-Liouville equation, which tends to infinity, grows exponentially fast.

Theorem 6. Consider the time-invariant version of the Riemann-Liouville difference equation (10)

$$({}_{\text{R-L}}\Delta^\alpha x)(n+1-\alpha) = Ax(n) \qquad (n \in \mathbb{N}),$$

with $A \in \mathbb{R}^{d \times d}$ such that

$$\det\left(I - z^{-1}(1-z^{-1})^{-\alpha} A\right) = 0 \tag{47}$$

does not have a complex solution z satisfying $|z| = 1$. If there exists an $x^* \in \mathbb{R}^d$ such that

$$\limsup_{n\to\infty} \|\varphi_{\text{R-L}}(n,x^*)\| = \infty,$$

then for every $x_0 \in \mathbb{R}^d$ there exist $q > 1$ and $M > 0$ such that

$$\|\varphi_{\text{R-L}}(n,x_0)\| \geq Mq^n \qquad (\text{for infinitely many} n \in \mathbb{N}).$$

Proof. Consider the Riemann-Liouville equation (10) in the form given by (18). By [21, Theorem 2(i) & (iii)] there exists a solution $z_0 \in \mathbb{C}$ of the equation (47) with $|z_0| > 1$. Using (16), the $\mathscr{Z}$-transform $\overline{\varphi_{\text{R-L}}}(z,x_0)$ of the solution $\varphi_{\text{R-L}}(n,x_0)$ takes the form

$$\overline{\varphi_{\text{R-L}}}(z,x_0) = \left(z\left(1-\frac{1}{z}\right)^\alpha I - A\right)^{-1} z\left(1-\frac{1}{z}\right)^{\alpha-1} x_0.$$

A zero of (47) represents a singular point of $\overline{\varphi_{\text{R-L}}}(\cdot,x_0)$. It is known that if $\overline{f}(z)$ is the $\mathscr{Z}$-transform of a sequence $f\colon \mathbb{N} \to \mathbb{R}$, then its radius of convergence r is given by the distance of the origin to an outermost (non-removable) singular point (see e.g. [20, Chapter 6]). Hence, if there is a zero z_0 with $|z_0| > 1$, then also the radius r of convergence of at least one component $\overline{\varphi}_i(\cdot,x_0)$ of $\overline{\varphi}(\cdot,x_0)$ satisfies $r > 1$. Using the Cauchy-Hadamard theorem we have

$$r = \limsup_{n\to\infty} \sqrt[n]{|\varphi_i(n,x_0)|} > 1$$

and the conclusion follows. □

5. Conclusions

In this paper we investigated dynamic properties of discrete fractional Caputo and Riemann-Liouville linear equations. For multidimensional time-varying equations we presented equivalent Volterra convolution equations for the Caputo and the Riemann Liouville case (Theorem 1). Moreover, for Caputo time-invariant equations we provide an explicit formula for the solution (Theorem 3). The result given by Theorem 4 describes the exact decay rate of a scalar time-invariant Caputo equation, namely $(n^{-\alpha})$. Although the decay rate of linear fractional equations is polynomial, the rate of divergence is exponential as it is shown in Theorem 6. From a dynamical systems point of view Theorem 5 is surprising as it provides a polynomial lower bound for the norm of differences between two different solutions of a Caputo time-varying linear equation. In particular, this theorem implies that the classical Lyapunov exponent defined by (38) is not an appropriate tool for stability analysis of fractional equations. An appropriate modification of the definition of Lyapunov exponents for discrete fractional equations is an important challenge of the theory of Lyapunov exponents. Such a proposition was presented for the continous-time case in [16]. Finally, we conjecture that Theorem 5 is also true for Riemann-Liouville equations.

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