10.24425/acs.2019.130198

Archives of Control Sciences
Volume 29(LXV), 2019
No. 3, *pages 413–422*

The LQG homing problem for a Wiener process with random infinitesimal parameters

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The problem of optimally controlling a Wiener process until it leaves an interval (a, b) for the first time is considered in the case when the infinitesimal parameters of the process are random. When $a = -\infty$, the exact optimal control is derived by solving the appropriate system of differential equations, whereas a very precise approximate solution in the form of a polynomial is obtained in the two-barrier case.

Key words: stochastic optimal control, first-passage time, dynamic programming, Brownian motion

1. Introduction

Let $\{X(t), t \geqslant 0\}$ be a controlled Wiener process defined by the stochastic differential equation

$$\mathrm{d}X(t) = \mu \mathrm{d}t + b_0 u[X(t)]\mathrm{d}t + \sigma \mathrm{d}W(t), \tag{1}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $b_0 > 0$ are constants, and $\{W(t), t \geqslant 0\}$ is a standard Brownian motion.

We assume that $X(0) = x \in [a, b]$, and we define the first-passage time

$$T(x) = \inf\{t \geqslant 0 : X(t) = a \text{ or } b \mid X(0) = x\}. \tag{2}$$

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This work was supported by the Natural Sciences and Engineering Research Council of Canada.
Received 11.01.2019.

The problem of finding the control u^* that minimizes the expected value of the cost function

$$J(x) := \int_0^{T(x)} \left\{ \frac{1}{2} q_0 u^2[X(t)] + \lambda \right\} \mathrm{d}t + K[X(T(x))], \tag{3}$$

where q_0 and λ are positive constants and K is a general termination cost function, is a particular case of what Whittle [4] has called *LQG homing* problems. As will be seen, it is possible here to determine the value of u^* by considering the uncontrolled process obtained by setting $u[X(t)] \equiv 0$ in Eq. (1).

The above cost function could be generalized by using a cost criterion that takes the risk sensitivity of the optimizer into account; see Kuhn [1] or Whittle [5].

Solving explictly an LQG homing problem is usually difficult, especially in more than one dimension. Makasu [3] was able to obtain an explicit solution to a two-dimensional problem.

In this paper, we are interested in finding the optimal control in the case when the infinitesimal parameters μ and σ are random, and vary according to a continuous-time Markov chain. This problem has applications in finance, in particular, where there are regime switches.

In a previous work (Lefebvre and Moutassim [2]), the authors considered the case when σ is constant, and μ switches between $+1$ and -1. They were able to obtain relatively good approximate solutions. However, they could not find an exact solution to any particular problem.

It turns out that we can indeed find an exact optimal control in the case when $a = -\infty$, that is, in the one-barrier case. In the two-barrier case, and with μ constant, we will see that a solution in the form of a polynomial is very good, when compared with a precise numerical solution, at least for the particular problem considered.

In the next section, we will use dynamic programming to find the system of differential equations that must be solved to determine the value function, which depends on the state of the Markov chain and from which the optimal control is deduced at once. Then, the one-barrier case will be solved in Section 3. A particular two-barrier problem will be treated in Section 4, and we will end with a few remarks in Section 5.

2. Dynamic programming

Let $\{Y(t), t \geqslant 0\}$ be a continuous-time Markov chain with state space $E = \{1, 2\}$. When the process enters state i, it remains there for a random time τ_i having an exponential distribution with parameter ν_i. Then, it will move to the other state.

The controlled process $\{X(t), t \geqslant 0\}$ defined in (1) is replaced by the two-dimensional process $\{(X(t), Y(t)), t \geqslant 0\}$, in which $X(t)$ satisfies

$$\mathrm{d}X(t) = \mu_{Y(t)}\,\mathrm{d}t + b_0 u[X(t), Y(t)]\,\mathrm{d}t + \sigma_{Y(t)}\,\mathrm{d}W(t). \tag{4}$$

We are now looking for the control that minimizes the expected value of the cost function

$$J(x,i) := \int_0^{T(x,i)} \left\{\frac{1}{2} q_{0,i} u^2[X(t), i] + \lambda_i\right\} \mathrm{d}t + K_i[X(T(x,i))], \tag{5}$$

where $q_{0,i} > 0$ and

$$T(x,i) = \inf\left\{t \geqslant 0 : X(t) = a \text{ or } b \mid X(0) = x \in [a,b], Y(0) = i\right\}, \tag{6}$$

for $i = 1, 2$. Furthermore, we assume that the final cost is constant:

$$K_i[X(T(x,i))] \equiv k_i \in \mathbb{R}, \quad \text{for } i = 1, 2.$$

To determine the value of the optimal control, we will make use of dynamic programming. We define the value function

$$F(x,i) = \inf_{u[X(t),i],\ 0 \leqslant t \leqslant T(x,i)} E[J(x,i)], \tag{7}$$

for $i = 1, 2$. We can show (see Lefebvre and Moutassim [2]) that the function $F(x,i)$ satisfies the dynamic programming equation

$$0 = \inf_{u(x,i)} \left\{\frac{1}{2} q_{0,i} u^2(x,i) + \lambda_i + [\mu_i + b_0 u(x,i)] F'(x,i) \right.$$
$$\left. + \frac{1}{2}\sigma_i^2 F''(x,i) + \nu_i\,[F(x,j) - F(x,i)]\right\}, \tag{8}$$

in which $j \neq i$.

From the above equation, we deduce at once that the optimal control $u^*(x,i)$ is given by

$$u^*(x,i) = -\frac{b_0}{q_{0,i}} F'(x,i), \quad \text{for } i = 1, 2. \tag{9}$$

Finally, substituting the expression for $u^*(x,i)$ into the dynamic programming equation, we obtain the following system of non-linear second-order differential-difference equations:

$$0 = \lambda_1 + \mu_1 F'(x,1) - \frac{1}{2}\frac{b_0^2}{q_{0,1}}[F'(x,1)]^2 + \frac{1}{2}\sigma_1^2 F''(x,1)$$
$$+ \nu_1\,[F(x,2) - F(x,1)], \tag{10}$$

$$0 = \lambda_2 + \mu_2 F'(x,2) - \frac{1}{2}\frac{b_0^2}{q_{0,2}}\left[F'(x,2)\right]^2 + \frac{1}{2}\sigma_2^2 F''(x,2)$$
$$+ \nu_2\left[F(x,1) - F(x,2)\right]. \tag{11}$$

The boundary conditions are

$$F(a,i) = F(b,i) = k_i, \quad \text{for } i = 1,2. \tag{12}$$

In the next section, we will find an exact solution to the above system in the case when the interval $[a,b]$ becomes $(-\infty, b]$. Notice that we then have only one boundary condition, namely $F(b,i) = k_i$. Moreover, because λ_i is assumed to be positive, we can write (if k_i is finite) that

$$\lim_{x\to-\infty} F(x,i) = \infty. \tag{13}$$

3. The one-barrier case

First, let us assume that $\lambda_1 = \lambda_2 = \lambda > 0$, $\mu_1 = \mu_2 = \mu$, $\sigma_1 = \sigma_2 = \sigma$, $q_{0,1} = q_{0,2} = q_0$, $\nu_1 = \nu_2 = \nu$ and $k_1 = k_2 = 0$. Then, we can write that

$$F(x,1) \equiv F(x,2) := F(x), \tag{14}$$

and we must solve the non-linear differential equation

$$0 = \lambda + \mu F'(x) - \frac{1}{2}\frac{b_0^2}{q_0}\left[F'(x)\right]^2 + \frac{1}{2}\sigma^2 F''(x). \tag{15}$$

In the one-barrier case, we have the boundary condition $F(b) = 0$, and we must also have

$$\lim_{x\to-\infty} F(x) = \infty. \tag{16}$$

Let

$$\alpha := \frac{q_0\sigma^2}{b_0^2} \tag{17}$$

and define

$$\Phi(x) = e^{-F(x)/\alpha}. \tag{18}$$

We find that the function $\Phi(x)$ satisfies the second-order *linear* differential equation

$$\frac{1}{2}\sigma^2\Phi''(x) + \mu\Phi'(x) = \theta\Phi(x), \tag{19}$$

where

$$\theta := \frac{\lambda}{\alpha}, \tag{20}$$

and is such that

$$\Phi(b) = 1. \tag{21}$$

Furthermore, we can write that

$$\lim_{x \to -\infty} \Phi(x) = 0. \tag{22}$$

Remark 1 *Equation (19) is the Kolmogorov backward equation satisfied by the mathematical expectation*

$$L(x) := e^{-\theta T_0(x)}, \tag{23}$$

where $T_0(x)$ is the same as the random variable $T(x)$ defined in (2), but for the uncontrolled Wiener process $\{X_0(t), t \geqslant 0\}$ with infinitesimal parameters μ and σ^2. Moreover, the conditions in (21) and (22) are the appropriate ones. By uniqueness, we can assert that $\Phi(x) \equiv L(x)$. Therefore in this particular problem, as mentioned in the Introduction, we are able to derive the optimal control by computing a mathematical expectation for the uncontrolled process. Notice that $L(x)$ is the moment-generating function of the first-passage time T_0, or the Laplace transform of its probability density function.

Proposition 1 *In the one-barrier case considered above, the optimal control $u^*(x)$ is given by the constant*

$$u^*(x) \equiv -\frac{1}{b_0}\left\{\mu - \left[\mu^2 + 2\lambda\frac{b_0^2}{q_0}\right]^{1/2}\right\}. \tag{24}$$

Proof. Using the fact that, when $\lambda > 0$, the condition in (22) must be fulfilled, we deduce from the general solution of Eq. (19) and the boundary condition $\Phi(b) = 1$ that

$$\Phi(x) = \exp\left\{\frac{1}{\sigma^2}\left[\mu - \left(\mu^2 + 2\theta\sigma^2\right)^{1/2}\right](b - x)\right\}, \tag{25}$$

from which we easily obtain the optimal control. □

Remark 2 *Notice that the optimal control does not depend on σ. Hence, if only the infinitesimal variance of the process is random in the optimal control problem, and the various parameters (like λ_i) do not depend on i, we can conclude that*

$$F(x, 1) \equiv F(x, 2). \tag{26}$$

This result can be generalized as follows.

Proposition 2 *Assume that $\lambda_i > 0$, for $i = 1, 2$. Then, in the one-barrier case, we have*

$$F(x, i) = \gamma (b - x) + k_i, \tag{27}$$

for $i = 1, 2$, where γ is the positive solution of the polynomial equation

$$\frac{1}{2} b_0^2 \left\{ \frac{1}{\nu_1 q_{0,1}} + \frac{1}{\nu_2 q_{0,2}} \right\} \gamma^2 + \left\{ \frac{\mu_1}{\nu_1} + \frac{\mu_2}{\nu_2} \right\} \gamma = \frac{\lambda_1}{\nu_1} + \frac{\lambda_2}{\nu_2}. \tag{28}$$

Furthermore, we must have

$$\begin{aligned} k_1 - k_2 &= \frac{1}{\nu_1} \left(-\lambda_1 + \mu_1 \gamma + \frac{1}{2} \frac{b_0^2}{q_{0,1}} \gamma^2 \right) \\ &= -\frac{1}{\nu_2} \left(-\lambda_2 + \mu_2 \gamma + \frac{1}{2} \frac{b_0^2}{q_{0,2}} \gamma^2 \right). \end{aligned} \tag{29}$$

Proof. These results are obtained by substituting the expressions for the functions $F(x, i)$ into the system (10), (11). Notice that the constant γ must indeed be positive, because the value function must be non-negative if $k_i \geqslant 0$. □

From the above proposition, we can easily calculate the (constant) optimal control $u^*(x, i)$ (see Eq. (9)):

$$u^*(x, i) = \frac{b_0 \gamma}{q_{0,i}}, \quad \text{for } i = 1, 2. \tag{30}$$

Remark 3 *In the case when $\lambda_1 = \lambda_2 = \lambda > 0$, $\mu_1 = \mu_2 = \mu$, $\sigma_1 = \sigma_2 = \sigma$, $q_{0,1} = q_{0,2} = q_0$, $\nu_1 = \nu_2 = \nu$ and $k_1 = k_2 = 0$, one can check that we retrieve the solution given in Proposition 1. Actually, in that case, we deduce from (29) that we must have $k_1 - k_2 = 0$.*

In the next section, we will turn to the two-barrier case. We will present an approximate solution that is very precise.

4. The two-barrier case

In the two-barrier case, the function $F(x, i)$ cannot be affine. Let

$$c_i^2 := \frac{b_0^2}{2 q_{0,i}}, \quad \text{for } i = 1, 2. \tag{31}$$

The system that we want to solve is

$$0 = \lambda_1 + \mu_1 F'(x,1) - c_1^2 \left[F'(x,1)\right]^2 + \frac{1}{2}\sigma_1^2 F''(x,1) + \nu_1 \left[F(x,2) - F(x,1)\right], \tag{32}$$

$$0 = \lambda_2 + \mu_2 F'(x,2) - c_2^2 \left[F'(x,2)\right]^2 + \frac{1}{2}\sigma_2^2 F''(x,2) + \nu_2 \left[F(x,1) - F(x,2)\right]. \tag{33}$$

We will try to find approximate solutions to the above system in the form of a polynomial. We choose the values shown in the following table for the various parameters in the system:

Parameter	Value
Infinitesimal means	$\mu_1 = \mu_2 = 0$
Infinitesimal variances	$\sigma_1^2 = 1$ and $\sigma_2^2 = 4$
Coefficient c_i^2	$c_1^2 = 1/2$ and $c_2^2 = 2$
Parameters ν_i and λ_i	1, for $i = 1, 2$
Final costs	$k_1 = k_2 = 0$
Interval $[a, b]$	$[-1, 1]$

The system (32), (33) reduces to

$$0 = 1 - \frac{1}{2}\left[F'(x,1)\right]^2 + \frac{1}{2}F''(x,1) + F(x,2) - F(x,1), \tag{34}$$

$$0 = 1 - 2\left[F'(x,2)\right]^2 + 2F''(x,2) + F(x,1) - F(x,2). \tag{35}$$

We tried polynomials of degree 6 as approximations $F_{\text{appr}}(x,i)$ to the value functions defined as follows:

$$F_{\text{appr}}(x,1) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6, \tag{36}$$

$$F_{\text{appr}}(x,2) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6. \tag{37}$$

Substituting these expressions into (34), (35), we obtain the following system of non-linear equations for the constants $(a_i)_{0\leqslant i\leqslant 6}$ and $(b_i)_{0\leqslant i\leqslant 6}$:

$$\left.\begin{aligned}
a_2 - a_0 + b_0 - \frac{1}{2}a_1^2 + 1 = 0,\\
-a_1 + 3\,a_3 + b_1 - 2\,a_1a_2 = 0,\\
-a_2 + 6\,a_4 + b_2 - 2\,a_2^2 - 3\,a_1a_3 = 0,\\
-a_3 + 10\,a_5 + b_3 - 4\,a_1a_4 - 6\,a_2a_3 = 0,\\
-a_4 + 15\,a_6 + b_4 - \frac{9}{2}a_3^2 - 5\,a_1a_5 - 8\,a_2a_4 = 0,\\
-2\,b_1^2 + 4\,b_2 + a_0 - b_0 + 1 = 0,\\
-8\,b_1b_2 - b_1 + 12\,b_3 + a_1 = 0,\\
-12\,b_1b_3 - 8\,b_2^2 - b_2 + 24\,b_4 + a_2 = 0,\\
-16\,b_1b_4 - 24\,b_2b_3 - b_3 + 40\,b_5 + a_3 = 0,\\
-20\,b_1b_5 - 32\,b_2b_4 - 18\,b_3^2 - b_4 + 60\,b_6 + a_4 = 0.
\end{aligned}\right\} \tag{38}$$

We must add to the above system the following conditions that are deduced from the boundary conditions $F(-1,i) = F(1,i) = 0$, for $i = 1,2$:

$$\left.\begin{aligned}
a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 0,\\
a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 = 0,\\
b_0 + b_1 + b_2 + b_3 + b_4 + b_5 + b_6 = 0,\\
b_0 - b_1 + b_2 - b_3 + b_4 - b_5 + b_6 = 0.
\end{aligned}\right\} \tag{39}$$

We can determine the coefficients $(a_i)_{0\leqslant i\leqslant 6}$ and $(b_i)_{0\leqslant i\leqslant 6}$ by making use of a MATLAB routine for non-linear systems. We find the following coefficients:

a_0	a_1	a_2	a_3	a_4	a_5	a_6
0.613	0.000	−0.6767	0.000	0.095	−0.000	−0.0313
b_0	b_1	b_2	b_3	b_4	b_5	b_6
0.2897	0.000	−0.3308	−0.000	0.0509	0.000	−0.0097

Thus, the approximate solutions are given by

$$F_{\text{appr}}(x,1) = 0.613 - 0.6767\,x^2 + 0.095\,x^4 - 0.0313\,x^6 \tag{40}$$

and

$$F_{\text{appr}}(x,2) = 0.2897 - 0.3308\,x^2 + 0.0509\,x^4 - 0.0097\,x^6. \tag{41}$$

Notice that $F_{\text{appr}}(x,1)$ and $F_{\text{appr}}(x,2)$ are even functions.

In Figure 1, we show these two approximate solutions, as well as solutions obtained by using precise numerical methods. We see that our approximate solutions are very close to the curves that correspond to precise numerical solutions. These polynomial solutions are actually more precise than the ones computed in Lefebvre and Moutassim [2] in the case when $\mu_1 = 1$, $\mu_2 = -1$ and $\sigma_1 = \sigma_2 = 1$. Although there is no guarantee that the polynomial solutions will always be really good, they lead to a suboptimal control that is very easy to compute and to implement, which is important in practice.

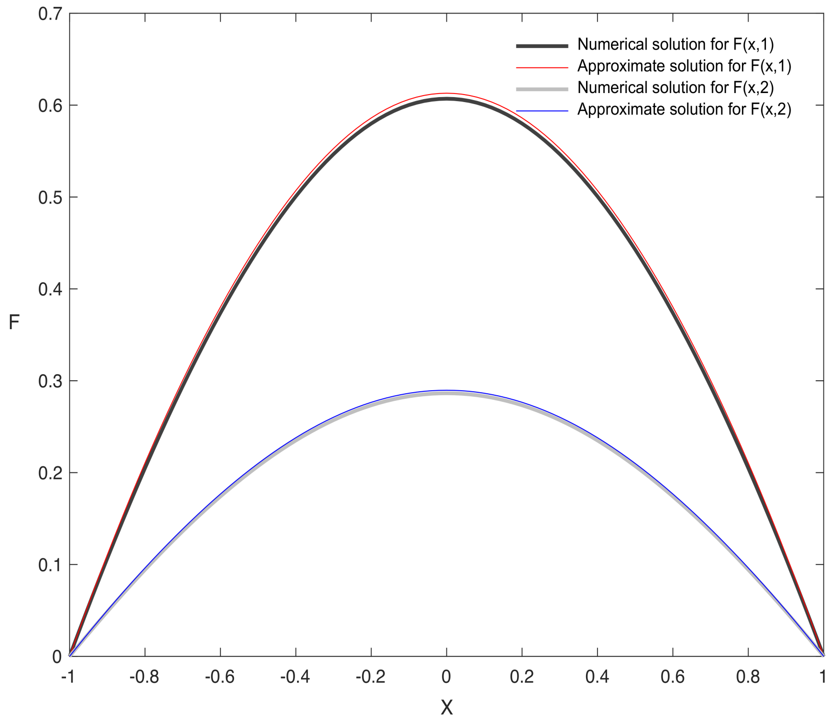

Figure 1: Approximate solutions and numerical solutions.

5. Concluding remarks

In this paper, we obtained exact and very precise approximate solutions to LQG homing problems for a Wiener process having random infinitesimal parameters. This work complements the one presented in Lefebvre and Moutassim [2].

As mentioned in the Introduction, it is generally very difficult to solve explicitly LQG homing problems. Here, we were able to derive the exact solution for a very important diffusion process, namely the Wiener process, in a more complicated case.

In theory, we could try to generalize the results presented in this paper by assuming that the state space of the Markov chain is the set $\{1, 2, \ldots, k\}$. But having two possible states is already an important improvement over the deterministic case and is sufficient in many applications. For example, in financial mathematics, there are random switches from a *bull* to a *bear* market (or vice versa).

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