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Non integer order, discrete, state space model of heat transfer process using Grünwald-Letnikov operator

K. OPRZĘDKIEWICZ*, K. DZIEDZIC, and Ł. WIĘCKOWSKI

AGH University of Science and Technology, al. A. Mickiewicza 30, 30-059 Kraków, Poland

Abstract. The paper is intented to show a new, state space, discrete, non integer order model of a one-dimensional heat transfer process. The proposed model derives directly from time continuous, state space model and it uses the discrete Grünwald-Letnikov operator to express the fractional order difference with respect to time. Stability and spectrum decomposition for the proposed model are recalled, the accuracy and convergence are analyzed too. The convergence of the proposed model does not depend on parameters of heater and measuring sensors. The dimension of the model assuring stability and predefined rate of convergence and stability is estimated. Analytical results are confirmed by experiments.

Key words: non integer order systems, heat transfer, fractional order state equation, fractional order backward difference, Grünwald-Letnikov operator, convergence.

1. Introduction

Mathematical models of distributed parameter systems based on partial differential equations can be described in an infinite-dimensional state space, usually in a Hilbert space, but Sobolev space can also be applied. This problem has been analyzed by many Authors. Fundamentals has been drawn by [27], they are given also in [13], analysis of a hyperbolic system in Hilbert space was presented by [2]. This paper gives also a broad overview of literature.

The modeling of processes and phenomena hard to analyse with the use of other tools is one of main areas where non integer order calculus is applied. Non integer models of different physical phenomena were presented by many Authors. The amount of FO models of various processes is collected in the book [5]. The book [4] presents fractional order models of chaotic systems and Ionic Polymer Metal Composites (IPMC). FO models of ultracapacitor are given for example by [8]. The use of fractional calculus to modeling diffusion processes is considered in [9, 29, 31]. A collection of recent results employing new Atangana-Baleanu operator can be found in [10]. In this book i.e. the FO blood alcohol model, the Christov diffusion equation and fractional advection-dispersion equation for groundwater transport process are presented.

Heat transfer processes can also be described using non integer order approach. For example a temperature–heat flux relationship for heat flow in semi-infinite conductor is presented in [5], the beam heating problem is given in [8], the FO transfer function temperature models in the room are presented by [6], the temperature models in three dimensional solid body are given in [15]. The use of fractional order approach to the modeling and control of heat systems is also presented in [30].

The paper is organized as follows: at the beginning some elementary ideas and definitions are presented. Next we propose discrete state space model using discrete PSE approximation. Elementary properties: spectrum decomposition, stability, accuracy and convergence of the model are discussed. Finally the experimental verification of the proposed results is presented.

2. Preliminaries

A presentation of elementary ideas is started with a definition of a non integer order, integro-differential operator. It was given for example by [5, 11, 12, 28].

Definition 1. The elementary non integer order operator. The non integer order integro-differential operator is defined as follows:

$$ {}_aD_t^\alpha f(t) = \begin{cases} \dfrac{d^\alpha(t)}{dt^\alpha} & \alpha > 0 \\ f(t) & \alpha = 0\,. \\ \int\limits_a^t f(\tau)(d\tau)^\alpha & \alpha < 0 \end{cases} \tag{1}$$

where a and t denote time limits for operator calculation, $\alpha \in \mathbb{R}$ denotes the non integer order of the operation.

The fractional order, integro-differential operator is described by different definitions, given by Grünvald and Letnikov (GL definition), Riemann and Liouville (RL definition) and Caputo (C definition). Relations between Caputo

*e-mail: kop@agh.edu.pl

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and Riemann-Liouville, between Riemann-Liouville and Grünwald-Letnikov operators are given in [1, 5]. Discrete versions of these operators are analysed with details in [7]. In the further consideration the GL definition is used ([4, 25]):

Definition 2. The Grünwald-Letnikov definition of the FO operator.

$$ {}_{0}^{GL}D_t^{\alpha} f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{l=0}^{\left[\frac{t}{h}\right]} (-1)^l \binom{\alpha}{j} f(t-lh). \tag{2} $$

In (2) $\binom{\alpha}{l}$ is the binomial coefficient:

$$ \binom{\alpha}{l} = \begin{Bmatrix} 1, & l=0 \\ \dfrac{\alpha(\alpha-1)\dots(\alpha-l+1)}{l!}, & l>0 \end{Bmatrix}. \tag{3} $$

A fractional order linear state space system is considered for example in [1, 5]. Using GL operator and for homogenous initial condition it takes the following form:

$$ \begin{aligned} {}_{0}^{GL}D_t^{\alpha} x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t). \end{aligned} \tag{4} $$

where $\alpha \in (0, 1)$ denotes the fractional order of the state equation, $x(t) \in \mathbb{R}^N$, $u(t) \in \mathbb{R}^L$, $y(t) \in \mathbb{R}^P$ are the state, control and output vectors respectively. A, B, C are the state, control and output matrices, respectively.

The GL definition is limit case for $h \to 0$ of the Fractional Order Backward Difference (FOBD), commonly employed in discrete FO calculations (see for example [26], p. 68):

Definition 3. The Fractional Order Backward Difference-FOBD.

$$ (\Delta^{\alpha} x) = \frac{1}{h^{\alpha}} \sum_{l=0}^{L} (-1)^l \binom{\alpha}{l} x(t-lh). \tag{5} $$

Denote coefficients $(-1)^l \binom{\alpha}{l}$ by d_l:

$$ d_l = (-1)^l \binom{\alpha}{l}. \tag{6} $$

The coefficients (6) can be also calculated with the use of the following, equivalent recursive formula (see for example [4], p. 12), useful in numerical calculations:

$$ \begin{aligned} d_0 &= 1 \\ d_l &= \left(1 - \frac{1+\alpha}{l}\right) d_{l-1}, \quad l = 1, \dots, L. \end{aligned} \tag{7} $$

It is proven in [3] that:

$$ \sum_{l=1}^{\infty} d_l = 1 - \alpha. \tag{8} $$

From (7) and (8) we obtain at once that:

$$ \sum_{l=2}^{\infty} d_l = 1. \tag{9} $$

In (5) L denotes a memory length necessary to correct approximation of a non integer order operator. Unfortunately good accuacy of approximation requires to use a long memory L what can make difficulties during implementation.

The discrete, fractional order state equation using definition (5) is written as follows (see for example [7, 14]):

$$ \begin{cases} (\Delta_L^{\alpha} x)(t+h) = A^+ x(t) + B^+ u(t) \\ y(t) = C^+ x(t). \end{cases} \tag{10} $$

where $x(t) \in \mathbb{R}^N$ is the state vector, $u(t) \in \mathbb{R}^P$ is the control, $y(t) \in \mathbb{R}^M$ is the output. A^+, B^+ and C^+ are state, control and output matrices respectively. If we shortly denote k-th time instant: hk by k, then equation (10) turns to:

$$ \begin{cases} (\Delta_L^{\alpha} x)(k+1) = A^+ x(k) + B^+ u(k) \\ y(k) = C^+ x(k). \end{cases} \tag{11} $$

where:

$$ A^+ = h^{\alpha} A. \tag{12} $$

$$ B^+ = h^{\alpha} B. \tag{13} $$

$$ C^+ = C. \tag{14} $$

The solution of state equation (11) takes the form:

$$ x(k+1) = P^+ x(k) - \sum_{l=2}^{L} A_l^+ x(k-l) + h^{\alpha} B^+ u(k) \tag{15} $$

where:

$$ P^+ = A^+ + \alpha I. \tag{16} $$

$$ P^+ = d_l I_{N\times N}. \tag{17} $$

Finally the Final Value Thoerem (FVT) should be recalled. It allows to calculate the steady-state value of a time function described by the Laplace transform or the "z" transform. It is formulated as follows:

Theorem 1. Final Value Theorem for continuous time. Let $f(t)$ is a function of time t and $F(s)$ is its Laplace transform. Assume that $F(s)$:

1. has no poles in the right part of the complex plane,
2. has maximally one pole on the imaginary axis: $s = 0$.

then:

$$ \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s). \tag{18} $$

Theorem 2. Final Value Theorem for discrete time. Let $f^+(k)$ is a discrete function of time, defined in k time moments and $F^+(z)$ is its z-transform. Assume that $F^+(z)$:
1. has no poles outside the unit circle,
2. has maximally one pole on the unit circle: $z = 1$.
then:

$$\lim_{k\to\infty} f^+(k) = \lim_{z\to 1}(z-1)F^+(z). \tag{19}$$

3. The experimental system and its models

The simplified scheme of the considered heat plant is shown in Fig. 1. It has the form of a thin copper rod heated by an electric heater Δx_u long, localized at one end of rod. The output temperature is measured using Pt-100 RTD sensors Δx long located in points: 0.29, 0.50 and 0.73 of rod length. More details of the construction are given in the section "Experimental Results".

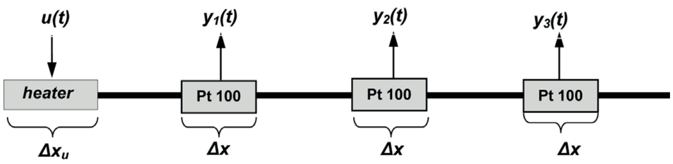

Fig. 1. The simplified scheme of the experimental system

The fundamental time continuous model describing the heat conduction in the rod is the partial differential equation of the parabolic type with the homogeneous Neumann boundary conditions at the ends, the homogeneous initial condition, the heat exchange along the length of rod and distributed control and observation. This equation with integer orders of both differentiations has been considered in many papers, for example in [16–18].

3.1. The time-continuous model. The time-continuous, non integer order model with respect to time using Caputo operator is given in [19]. The model considering the both time and space coordinates employing Caputo and Riesz operators is presented with details in papers [21, 22]. Here its finite dimensional approximaton of size N, using GL operator is recalled. It takes the following form:

$$\begin{cases} {}_0^{GL}D_t^{\alpha} Q(t) = AQ(t) + Bu(t) \\ Q(0) = 0 \\ y(t) = k_0 CQ(t). \end{cases} \tag{20}$$

In (20) $\alpha > 0$ denotes non integer order of the system with respect to time, $Q(t) \in \mathbb{R}^{N+1}$ is the state vector, $u(t) \in \mathbb{R}$ is the control signal, $y(t) \in \mathbb{R}^3$ is the output signal measured by RTD sensors, k_0 is the coefficient necessary to fit the response of the model to experimental response.

The state matrix A is the diagonal matrix:

$$A = diag\{\lambda_{\beta_0}, \lambda_{\beta_1}, \ldots, \lambda_{\beta_N}\} \tag{21}$$

where:

$$\lambda_{\beta_n} = -a_w \pi^{\beta} n^{\beta} - R_a, \quad n = 0, 1, \ldots, N. \tag{22}$$

In (22) β is the fractional order of the system with respect to length, a_w and R_a denote coefficients of heat conduction and heat exchange. The spectrum σ of the state operator A is expressed as underneath:

$$\sigma(A) = \{\lambda_{\beta_0}, \lambda_{\beta_1}, \ldots, \lambda_{\beta_N}\}. \tag{23}$$

From (22) it follows at once that $\lambda_{\beta_0} > \lambda_{\beta_1} > \ldots > \lambda_{\beta_N}$.

The input operator B has the following form:

$$B = [b_0, b_1, \ldots, b_N]^T \tag{24}$$

where:

$$b_n = \begin{cases} x_u, \ n = 0 \\ \dfrac{\sqrt{2}\sin(n\pi x_u)}{n\pi}, \ n = 1, \ldots, N. \end{cases} \tag{25}$$

The output operator C is expressed as follows:

$$C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}. \tag{26}$$

Each row of the output operator C is associated to one RTD sensor:

$$C_j = [c_{j0}, c_{j1}, \ldots, c_{jN}] \quad j = 1, 2, 3 \tag{27}$$

where:

$$c_{jn} = \begin{cases} x_{j2} - x_{j1}, \ n = 0, \\ \dfrac{\sqrt{2}\left(\sin(n\pi x_{j2}) - \sin(n\pi x_{j1})\right)}{n\pi}, \ n = 1, \ldots, N, \\ \qquad j = 1, 2, 3. \end{cases} \tag{28}$$

Coordinates x_1 and x_2 depend on sensor location on the rod and they are equal:

$$\begin{cases} x = 0.29: \quad x_1 = 0.26, \ x_2 = 0.32 \\ x = 0.50: \quad x_1 = 0.47, \ x_2 = 0.53 \\ x = 0.73: \quad x_1 = 0.70, \ x_2 = 0.76 \end{cases}$$

From (25) and (26) it turns out, that the control function $b(x)$ and output function $c(x)$ are the interval constant functions.

4. The proposed discrete model

The discrete time model follows directly from continuous model (20) after use FOBD (5). Then solution (15) takes the following form:

$$\begin{cases} Q^+(k+1) = P^+Q^+(k) - \sum_{l=1}^{L} A_l^+ Q^+(k-l) + B^+u(k) \\ y^+(k) = C^+Q^+(k). \end{cases} \tag{29}$$

In (29) A_l^+ is expressed by (17), P^+, B^+ and C^+ take the following form:

$$\begin{cases} P^+ = diag\{\lambda_{\beta_0}^+, \lambda_{\beta_1}^+, \ldots, \lambda_{\beta_N}^+\} \\ B^+ = h^\alpha B \\ C^+ = C \end{cases} \tag{30}$$

where:

$$\lambda_{\beta_n}^+ = \alpha + h^\alpha \lambda_{\beta_n} = \alpha - h^\alpha (a_w \pi^\beta n^\beta + R_a). \tag{31}$$

In (22) is proven that the spectrum of the time-continuous system can be decomposed into single, seperated eigenvalues (analogically as in the integer order case). This property is mapped to discrete time system. Particularly the solution (29) can be decomposed to separated "subsolutions" associated to the single eigenvalues (31).

4.1. Decomposition of the system. The state vector $Q^+(k)$ of the discrete model (29) can be expressed as:

$$Q^+(k) = \begin{bmatrix} q_1^+(k) \\ \ldots \\ q_N^+(k) \end{bmatrix}. \tag{32}$$

The matrices P^+ and A_l^+ describing the solution of the discrete system (29) are diagonal matrices. Consequently the solution (29) can be decomposed into N independent modes, associated with n-th state variable $Q_n^+(k)$ and described by n-th eigenvalue. The n-th mode of solution for fixed memory length L takes the form as follows:

$$q_n^{+L}(k+1) = \lambda_{\beta_n}^+ q_n^+(k) - \sum_{l=2}^{L} d_l q_n^+(k-l) + \\ + b_n^+ u^+(k), \quad n = 0, \ldots, N. \tag{33}$$

For each memory length the solution takes the following form:

$$q_n^{+\infty}(k+1) = \lambda_{\beta_n}^+ q_n^+(k) - \sum_{l=2}^{\infty} d_l q_n^+(k-l) + \\ + b_n^+ u^+(k), \quad n = 0, \ldots, N. \tag{34}$$

Between input of the system and the j-th output the discrete transfer function $G_j^+(z^{-1})$ can be defined:

$$G_j^{+L\infty}(z^{-1}) = \sum_{n=0}^{N} G_{nj}^{+L,\infty}(z^{-1}) \quad j = 1, 2, 3. \tag{35}$$

The upper index "L" denotes the fixed memory length, index "∞" denotes each memory length. The transfer function $G_{nj}^{+L}(z^{-1})$ associated to n-th mode of solution for fixed memory length L is as follows:

$$G_{nj}^{+L}(z^{-1}) = \frac{c_{jn}^+ b_n^+ z^{-1}}{1 - z^{-1}\lambda_{\beta n}^+ + \sum_{l=2}^{L} d_l z^{-l-1}} \quad j = 1, 2, 3. \tag{36}$$

and analogically $G_{nj}^{+\infty}(z^{-1})$ can be defined:

$$G_{nj}^{+\infty}(z^{-1}) = \frac{c_{jn}^+ b_n^+ z^{-1}}{1 - z^{-1}\lambda_{\beta n}^+ + \sum_{l=2}^{\infty} d_l z^{-l-1}} \quad j = 1, 2, 3. \tag{37}$$

4.2. Stability. The stability conditions for the model (29–31) are proven in the paper [20]. The fundamental result is that too high order N of the considered discrete model can cause its instability. Propositions describing the maximal permissible order N assuring the preservation of the stability are recalled here.

Proposition 1. Maximum size of model N_{sL} assuring the stability of the discrete model for fixed memory length L. Let us consider the discrete model of heat transfer process described by (29). The size N_{sL} of finite-dimensional approximation assuring the stability of the discrete model (29) meets the following inequality:

$$N_{sL} \leq Int\left(\left(\frac{1 + \alpha - h^\alpha R_a + \sum_{l=2}^{L} dl}{h^\alpha a_w \pi^\beta}\right)^{\frac{1}{\beta}}\right). \tag{38}$$

Proposition 2. Maximum size of model $N_{s\infty}$ assuring the stability of the discrete model for each memory length. Let us consider the discrete model of heat transfer process described by (29). The size N_{sL} of finite-dimensional approximation assuring the stability of the discrete model (29) meets the following inequality:

$$N_{s\infty} \leq Int\left(\left(\frac{2 + \alpha - h^\alpha R_a}{h^\alpha a_w \pi^\beta}\right)^{\frac{1}{\beta}}\right). \tag{39}$$

In (38) and (39) $Int(x)$ denotes an integer number nearest to x.

From (8) it turns out that condition (39) is the limit case of (38) for $L \to \infty$. Results of numerical calculations show that the both propositions give practically the same result ([20]).

4.3. Accuracy. The accuraccy of the considered model can be described using approach given in papers [23] and [24]. The steady-state error of the considered model is defined as follows:

$$\varepsilon = y^{ss} - y^{+ss}. \tag{40}$$

where y^{ss} and y^{+ss} are steady-state responses of continuous and discrete model respectively. They are equal:

$$y^{ss} = \lim_{t\to\infty} y(t). \tag{41}$$

$$y^{ss+} = \lim_{k\to\infty} y^{+}(k). \tag{42}$$

Both above responses can be calculated using Final Value Theorem (FVT) (18) and (19) for continuous and discrete systems respectively.

For time continuous system and the control in the form of the Heaviside function: $u(k) = 1(k)$ the steady state response is equal:

$$y^{ss} = -CA^{-1}B. \tag{43}$$

With respect to (21–26) it is as follows:

$$y^{ss} = \left[y_1^{ss}, y_2^{ss}, y_3^{ss}\right]^T \tag{44}$$

where:

$$y_j^{ss} = \sum_{n=0}^{N} y_{nj}^{ss}, \quad j = 1, 2, 3. \tag{45}$$

$$y_{nj}^{ss} = \frac{c_{jn} b_n}{\lambda_{\beta_n}}, \quad j = 1, 2, 3. \tag{46}$$

Next the steady state response of the discrete system needs to be given. Fixed memory length L and each memory length need to be analysed separately. Using FVT Theorem (19) and discrete transfer functions (36, 37) and with respect to (6–8) and (9) we obtain the steady-state response of system $y^{+ss} = \left[y_1^{+ssL,\infty}, y_2^{+ssL,\infty}, y_3^{+ssL,\infty}\right]^T$. Upper index "$L$" denotes the fixed memory length, index "∞" denotes each memory length. For fixed memory length the steady state response is equal:

$$y_j^{+ssL} = \sum_{n=0}^{N} y_{nj}^{ssL}, \quad j = 1, 2, 3 \tag{47}$$

where:

$$y_{nj}^{+ssL} = \frac{c_{jn}^{+} b_n^{+}}{1 - \lambda_{\beta n}^{+} + \sum_{l=2}^{L} dl} \quad j = 1, 2, 3. \tag{48}$$

Next the steady state response for each memory length equals to:

$$y_j^{+ss\infty} = \sum_{n=0}^{N} y_{nj}^{ss\infty} \quad j = 1, 2, 3 \tag{49}$$

where:

$$y_{nj}^{+ss\infty} = \frac{c_{jn}^{+} b_n^{+}}{2 - \lambda_{\beta n}^{+}} \quad j = 1, 2, 3. \tag{50}$$

With respect to (30) and (31) y_{nj}^{ssL} and $y_{nj}^{ss\infty}$ take the form:

$$y_{nj}^{+ssL} = \frac{h^{\alpha} c_{jn} b_n}{1 - \alpha + \sum_{l=2}^{L} dl - h^{\alpha}\lambda_{\beta n}} \quad j = 1, 2, 3 \tag{51}$$

$$y_{nj}^{+ss\infty} = \frac{h^{\alpha} c_{jn} b_n}{2 - \alpha - h^{\alpha}\lambda_{\beta n}} \quad j = 1, 2, 3. \tag{52}$$

Finally the steady-state error with respect to (40) takes the following form:

$$\varepsilon^{ssL,\infty} = \left[\varepsilon_1^{ssL,\infty}, \varepsilon_2^{ssL,\infty}, \varepsilon_3^{ssL,\infty}\right]^T \tag{53}$$

where:

$$\varepsilon_j^{ssL,\infty} = \left|y_j^{ss} - y_j^{+ssL,\infty}\right|, \quad j = 1, 2, 3. \tag{54}$$

With respect to (51) and (52):

$$\varepsilon_j^{ssL,\infty} = \left|\sum_{n=0}^{N} \varepsilon_{jn}^{ssL,\infty}\right|. \tag{55}$$

Each component of (55) is as follows:

$$\varepsilon_{nj}^{ssL} = c_{jn} b_n \left(\frac{1 + \sum_{l=2}^{L} dl - \alpha - 2h^{\alpha}\lambda_{\beta n}}{\lambda_{\beta n}\left(1 + \sum_{l=2}^{L} dl - \alpha - h^{\alpha}\lambda_{\beta n}\right)}\right) \quad j = 1, 2, 3 \tag{56}$$

$$\varepsilon_{nj}^{ss\infty} = c_{jn} b_n \left(\frac{2 - \alpha - 2h^{\alpha}\lambda_{\beta n}}{\lambda_{\beta n}\left(2 - \alpha - h^{\alpha}\lambda_{\beta n}\right)}\right) \quad j = 1, 2, 3. \tag{57}$$

4.4. Convergence. The convergence can be analyzed with respect to order N or with respect to memory length L. Unfortunately the analysis with respect to L can be done numerically only. However, this analysis with respect to N can be done analytically using approach given in the paper [23]. It is presented below.

The rate of convergence ROC_N for the model of the size N is defined as the absolute value of the steady state value of the N-th mode of solution for the j-th output:

$$ROC_{Nj}^{L,\infty} = \left|y_{Nj}^{+ssL,\infty}\right|. \tag{58}$$

where upper indices L and ∞ denote fixed memory length and each memory length respectively, $y_{Nj}^{+ssL,\infty}$ is calculated using (51) and (52) with $n = N$. The size of model $N_{\Delta L}$ assuring the predefined value Δ_L of ROC for fixed memory length L can be estimated. It is described by the following proposition:

Proposition 3. Minimum size of model $N_{\Delta L}$ assuring the predefined Rate of Convergence Δ_L of the discrete model for fixed memory length L. Let us consider the discrete model of heat transfer process described by (29). The size $N_{\Delta L}$ of model assuring the predefined value Δ_L of ROC meets the following inequality:

$$N_{\Delta L} \geq Int\left(\frac{\sqrt{\sqrt{S_L^2 + \frac{8h^{2\alpha}a_w}{\Delta_L}} - S_L}}{\pi\sqrt{h^\alpha a_w}}\right) \tag{59}$$

where:

$$S_L = 1 - \alpha + \sum_{l=2}^{L} dl + h^\alpha R_a. \tag{60}$$

In (59) $Int(..)$ denotes the nearest integer value.

Proof. The condition $ROC_{Nj}^L \leq \Delta_L$ with respect to (25) and (28) is equivalent to:

$$\Delta_L \leq \left|\frac{h^\alpha}{S_L + h^\alpha a_w \pi^\beta N_{\Delta L}^\beta}\right| \cdot P \tag{61}$$

where:

$$P = \frac{2}{N_{\Delta L}^2\pi^2}\left|\sin\left(\frac{N\pi(x_{j2}-x_{j1})}{2}\right)\cos\left(\frac{N\pi(x_{j2}+x_{j1})}{2}\right)\cdot \sin\left(\frac{N\pi x_u}{2}\right)\right|. \tag{62}$$

The factor P expressed by (62) is not greater than $\frac{2}{N_{\Delta L}^2\pi^2}$, because the expression inside absolute value $|..|$ does not exceed one. It allows to assume that:

$$P \leq \frac{2}{N_{\Delta L}^2\pi^2}. \tag{63}$$

This gives the upper estimation of $N_{\Delta L}$, but (61) takes to simplier form:

$$\Delta_L \leq \left|\frac{2h^\alpha}{N_{\Delta L}^2\pi^2\left(S_L + h^\alpha a_w \pi^\beta N_{\Delta L}^\beta\right)}\right|. \tag{64}$$

The expression inside $|\ldots|$ is always positive for $S_L > 0$. This allows to ignore the absolute value. Next to simplify further calculations assume that $\beta = 2$. Then the (64) takes the form:

$$\begin{aligned}\Delta_L \leq \frac{2h^\alpha}{N_{\Delta L}^2\pi^2\left(S_L + h^\alpha a_w \pi^\beta N_{\Delta L}^\beta\right)} \iff \\ \iff \Delta_L\pi^4 h^\alpha a_w N_{\Delta L}^4 + \Delta_L \pi^2 S_L N_{\Delta L}^2 - 2h^\alpha \geq 0\end{aligned} \tag{65}$$

Solving the double quadratic inequality (65) we obtain directly the condition (59) and the proof is completed. □

The case of each memory length is the limit case for $L \to \infty$ and it is expressed as follows:

Proposition 4. Minimum size of model $N_{\Delta\infty}$ assuring the predefined Rate of Convergence Δ_∞ of the discrete model for each memory length. Let us consider the discrete model of heat transfer process described by (29). The size $N_{\Delta\infty}$ of model assuring the predefined value Δ_∞ of ROC meets the following inequality:

$$N_{\Delta\infty} \geq Int\left(\frac{\sqrt{\sqrt{S_\infty^2 + \frac{8h^{2\alpha}a_w}{\Delta_\infty}} - S_\infty}}{\pi\sqrt{h^\alpha a_w}}\right) \tag{66}$$

where:

$$S_\infty = 2 - \alpha + h^\alpha R_a. \tag{67}$$

In (66) $Int(..)$ denotes the nearest integer value.

Proof. The each memory length is a limit case: $L \to \infty$ of condition (59). This gives:

$$S_\infty = \lim_{L\to\infty} S_L \tag{68}$$

where S_L is expressed by (60). The rest of the proof is identical as (61–65). □

The use of both conditions (59) and (66) gives practically the same result. It can be explained by the fact that the factor S_∞ in (66) is the limit case of factor S_L in (59) and the sum (9) fastly goes to one. The simplification (63) gives the upper, "cautious" estimation of $N_{\Delta L,\infty}$. However it allows to analyze the convergence independently on size and location of heater and sensors.

Finally the size N reccommended for the model can be proposed as follows:

$$N_\Delta < N < N_s. \tag{69}$$

5. Experimental results

Experiments were led using the experimental system shown with details in Fig. 2. The length of rod equals to 260 [mm]. The control signal in the system is the standard current 0–20 [mA] given from analog output of the PLC. This signal is amplified to the range 0–1.5 [A] and it is the input signal for the heater. The temperature distribution along the rod is measured by the stan-

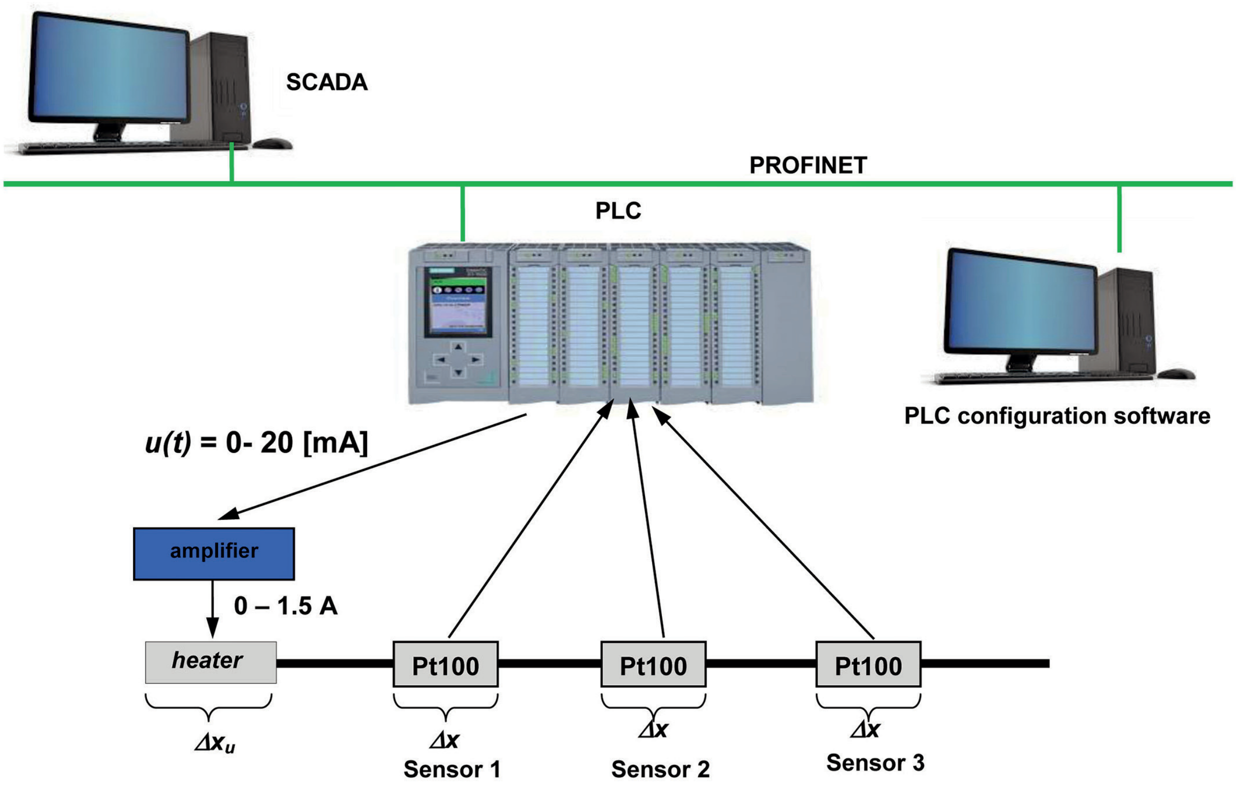

Fig. 2. The construction of the experimental system

dard RTD sensors Pt-100. Signals from the sensors are directly read by analog inputs of the PLC in Celsius degrees. Data from PLC are collected by SCADA. The whole system is connected via PROFINET industrial network. The temperature distribution with respect to time and length is shown in the Fig. 3. The step response of the model was tested in time range from 0 to $T_f = 300$ [s] with sample time $h = 1$ [s], parameters of the model (20–31) were calculated via the minimization of the MSE (Mean Square Error) cost function (70) using MATLAB *fminsearch* function. The parameters are given in the Table 1 (see [20]).

$$MSE = \frac{1}{3K_s}\sum_{j=1}^{3}\sum_{k=1}^{K_s}\left(y_{pj}^{+}(k) - y_j^{+}(k)\right)^2. \tag{70}$$

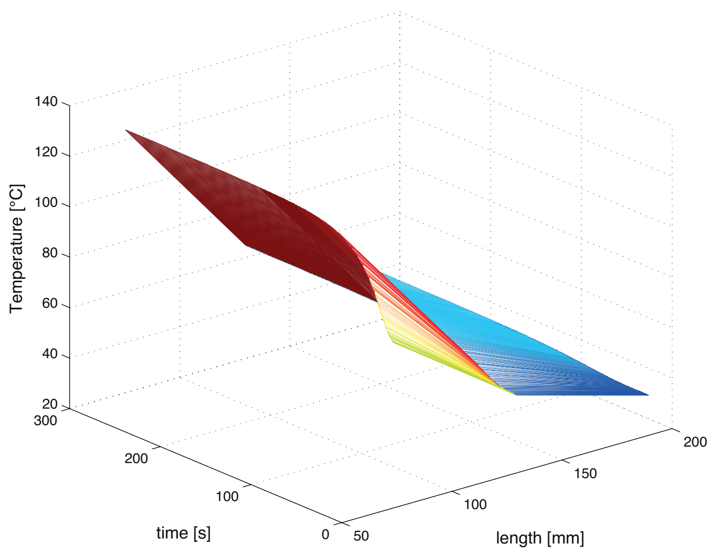

Fig. 3. The spatial-time temperature distribution in the plant

Table 1
Parameters of the heat plant, $n = 13$, $L = 150$

Parameter	α	β	a	R_a
value	0.9448	2.0336	0.0006	0.0531

In (70) K_s denotes the number of collected samples for one sensor, $y^{+}p_j(k)$ and $y_j^{+}(k)$ are responses of plant and model in k-th time moment and at j-th output respectively.

The stability of the considered model is analysed in [20]. The use of conditions (38) and (39) from Propositions 1 and 2 gives $N_{sL} \le 20$, $N_{s\infty} \le 20$.

The steady state error is analysed next. The error $\varepsilon_{N2}^{ss\infty}$ as a function of size N, calculated with respect to (54) for each memory length is shown in Fig. 4, the comparison steady state errors for each memory length vs fixed memory length is shown in the Fig. 5. Next the convergence of the considered model needs to be considered. In should be assumed that we need to find the size $N_{\Delta L}$ and $N_{\Delta\infty}$ of model assuring the value

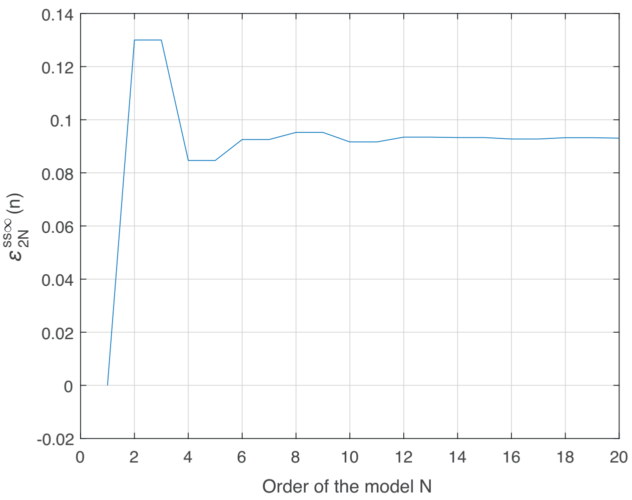

Fig. 4. The steady state error $\varepsilon_{N2}^{ss\infty}$ as a function of N for each memory length

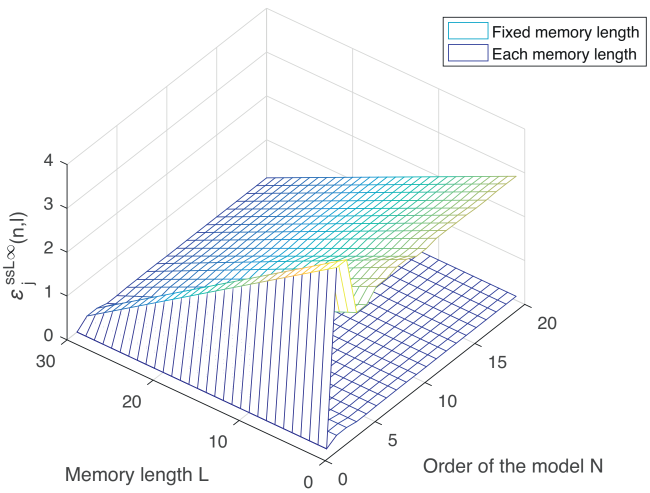

Fig. 5. The comparison of steady state error $\varepsilon_{N2}^{ss\infty}$ for each memory length to fixed memory length

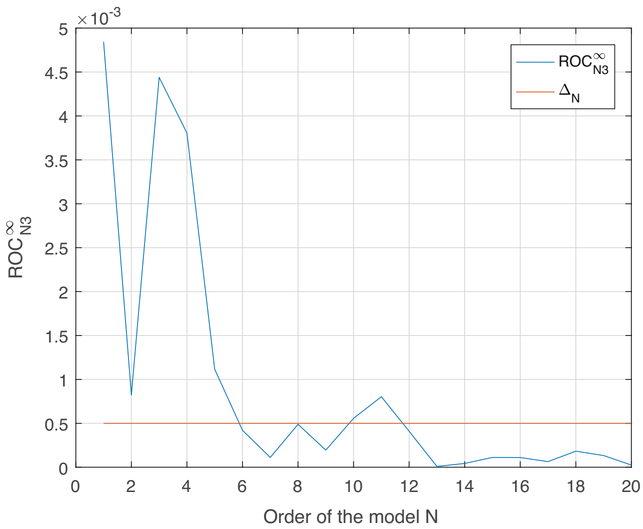

Fig. 6. The Rate of Convergence as a function of N for each memory length

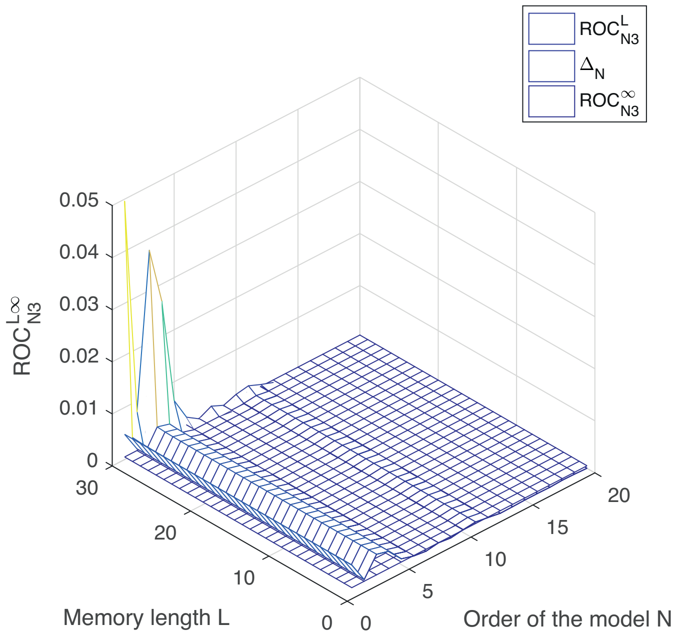

Fig. 7. The comparison of Rate of Convergence for each memory length to fixed memory length

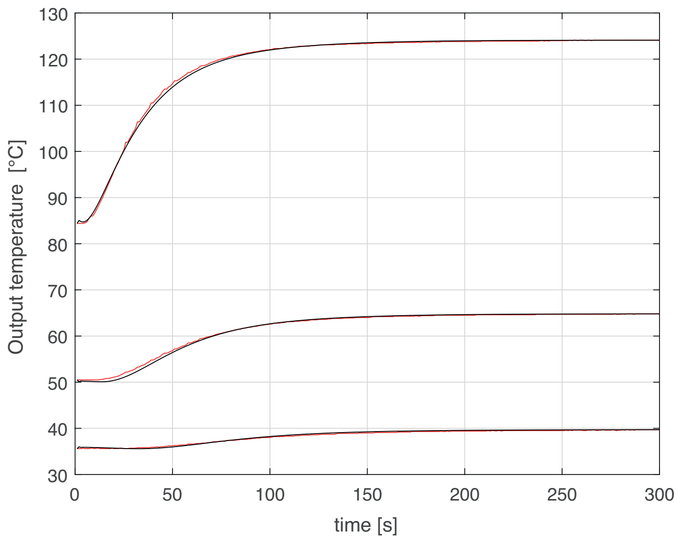

Fig. 8. Comparison of experimental step response to step response of the model. Red line is the experimental step response, black line denotes the step response of the model

$\Delta_N = 0.0005$. The use of conditions (59) and (66) from Propositions 3 and 4 gives the values: $N_L = N_\infty \geq 13$. This result is verified by diagrams shown in Figs 6 and 7.

Finally, the order N of the model assuring the keeping stability and predefined rate of convergence is as follows:

$$13 \leq N \leq 20.$$

The comparison experimental step response to step response of the model with parameters given in the Table 1 and order of the model $N = 13$ and the order of the discrete GL operator equal $L = 150$ is given in Fig. 8. For diagrams in this figure the cost function (70) equals to: $MSE = 0.0719$.

6. Final conclusions

The final conclusions are as follows:

- The most important result is the analytical estimation of accuracy and convergence of the proposed model. This allows to optimize its size during digital implementation. Additionally the convergence does not depend on size and localization of the heater and sensors.
- The presented results can be generalized into class of fractional order, linear systems with the diagonal state matrix and spectrum containing single, separated, real eigenvalues.
- Future studies of the presented issue will cover applying an implicit scheme , i.e. where in equation (10) we have $A^+x(t+h)$ on the right side. This scheme is expected to be more accurate and the choice of the step size h will not be as critical as in the explicit scheme considered in this paper.
- The numerical optimization of the presented model using biologically inspired optimization methods, for example Particle Swarm Optimization algorithm or Grey Wolf algorithm is also planned to do. It can be explained by the fact that the value of order N is the only one possible to analytical estimation in the form of interval $[N_s; N_\Delta]$. All other parameters of the model need to be numerically assigned only.

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