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CONTROL AND ROBOTICS

Assignability of numerical characteristics of time-varying systems

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Abstract. The main aim of this article is to survey and discuss the existing state of art concerning the assignability by a feedback of numerical characteristics of linear continuous and discrete time-varying systems. Most of the results present necessary or sufficient conditions for different formulation of the Lyapunov spectrum assignability problem. These conditions are expressed in terms of various controllability types and optimalizability of the controlled systems and certain properties of the free system such as: regularity, diagonalizability, boundness away, integral separation and reducibility.

Key words: assignability, controllability, Lyapunov spectrum, linear time-varying systems.

1. Introduction

From the point of view of the classical control theory the primary stabilization problem is a problem of stabilization of the plant described by linear time-invariant system given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ t \in \mathbb{R}, \tag{1}$$

where A and B are real matrices of appropriate sizes. By stabilization of this model we understand a problem of finding linear time-invariant feedback

$$u(t) = Ux(t)$$

such that the closed-loop system

$$\dot{x}(t) = (A + BU)x(t), \quad x \in \mathbb{R}^n, \ t \in \mathbb{R} \tag{2}$$

is assymptoticly stable. On the base on the well-known stability criterion (see [1]) the stabilization problem leads to the location in the open left half complex plane all eigenvalues $\lambda_i(A+BU)$, $i = 1, \ldots, n$ of the matrix $A + BU$ by an appropriate chosen feedback matrix U. It is clear that the solution of this problem, if it exists, may be not unique because the eigenvalues $\lambda_i(A+BU)$ may be located in the open left half complex plane in many ways. Whereas the minimal distance of the eigenvalues $\lambda_i(A+BU)$ from the imaginary axis, i.e. the number

$$\alpha = -\max Re\lambda_i(A+BU)$$

describes asymptotic decaying rate of the solution as $e^{-\alpha t}$, then depending on the particular placement of the eigenvalues we may change qualitative behaviour convegency to zero characterized, among others, by degree and amplitudes of oscillation [1]. The natural requirement of designing the stabilizing feedback in such a way that closed-loop system is fast and smooth led to problems of regulator synthesis with additional qualitative criteria [1]. One of the primary methods of designing such a control strategy for linear systems with time-invariant coefficients is the pole placement method, also known as the pole-shifting or the spectrum assignment method [1], which based on construction of the feedback in such a way that the eigenvalues of the matrix $A + BU$ have a priori given location.

It is worth to mention that this method is not limited only to stabilization problems, where the decaying to zero is the most importing feature. In the case of tracking systems the very important issue is to keep the transient states in given limits when the set point is changing. If system (1) is controllable (see [2–5] for definition), then introducing control in the form

$$u(t) = Ux(t) + v(t),$$

where U is a given matrix of size $m \times n$ we do not change this property i.e. system

$$\dot{x}(t) = (A+BU)x(t) + Bv(t), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ t \in \mathbb{R} \tag{3}$$

is controllable [1]. Changing by the feedback U location of eigenvalues of the matrix $A + BU$, we may ensure the required transient states constrains. A whole range of issues related to solving such problems, very closed to engineering practice [6], are usually referred to as the modal control theory, details of which can be found for example in the book [7].

A typical example of modal control task is a problem of assigning the spectrum of system (3) by the selection of the matrix U to ensure that

$$\lambda_i(A+BU) = \mu_i, \quad i = 1, \ldots, n$$

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for a priori given sequence of complex numbers $\mu_1, \mu_2, \ldots, \mu_n$. In the case of a single input systems ($m = 1$) the problem of assignability of the spectrum of the matrix $A+BU$ is solvable if and only the square matrix

$$S = \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix}$$

is invertible. A very important observation connected to this criterion is the fact that invertibility of the matrix S is also, in the considered case, the necessary and sufficient condition for controllability of system (1) (see eg., [2]). Basing on this observation Romanian mathematician W. M. Popov at the beginning of 1960's proved in [8] that in case of arbitrary m the necessary and sufficient condition for assignability of the spectrum of the matrix $A+BU$ is

$$rank \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} = n.$$

Slightly earlier R. Kalman in [9] showed that this condition is also a necessary and sufficient for controllability of system (1). Finally, W. M. Wonham in the paper [10] clarified that if the set $\mu_1, \mu_2, \ldots, \mu_n$ is symmetric with respect to the real axis, then the matrix U can be selected as a real matrix.

In order to cope with growing requirements formulated for control systems in the process of the model building we use linear time-varying systems. The most frequently used class of linear time-varying system is the one of periodic systems. They appear in the natural way in signal processing and communication as, for example, filters which incorporate modulators in the signal path (see [11]), as well as in control. We also obtain linear time-varying systems when we simplify certain complicated models as it is the case of the linearization of the nonlinear dynamics around of the given trajectory [12, 13]. Another application of linear time-varying models is presented in [14] where the authors analysis electrical circuits while in [15] it is shown how to design a simplified observer on the base of the linear time-varying model. Further industrial applications of linear time-varying models are presented in [16] to model current-mode control of a converter, highway vehicles with time-varying velocity (see also [17]) and servo system with moving operating point. It should be also noticed that switched and jump (see [18, 19]) linear systems belong to the class of time-varying systems. Finally, we may obtain a linear time-varying model when we apply a time-varying feedback to a time-invariant system in order to improved the control quality.

In the one of the first paper [20] about the modal control for time-varying systems, P. Brunovsky showed that for system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad t \in \mathbb{R} \tag{4}$$

with ω-periodic and continuously differentiable coefficients solvability of the assignability problem for their characteristic multipliers is equivalent to complete controllability of the system. Moreover, the feedback function $U(\cdot)$ may be selected as ω-periodic and continuously differentiable. If we would like to extend that result to general time-varying system then we have to give the answer to the following three questions.

1. How to define the spectrum of the closed-loop system

$$\dot{x}(t) = (A(t) + B(t)U(t))x(t), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}? \tag{5}$$

2. How to define the controllability?
3. What class of feedback functions U should be considered?

It seems to be natural to consider in time-varying case the Lyapunov spectrum (see [21] for the definition and basic properties) as a counterpart of spectrum of matrix A. However, we should remember that for time-invariant system the Lyapunov spectrum consists of real parts of the eigenvalues of matrix A.

For the time-varying systems we have many nonequivalent concepts of controllability [2]. In this paper we will consider three of them which are called: controllability, complete controllability and uniform complete controllability.

Regarding to the class of feedback function $U(\cdot)$ we will mainly consider a class of bounded piecewise continuous functions but we also present some results for the class of essentially bounded functions.

In the paper we will consider several formulations of the assignability of the Lyapunov exponents of both continuous and discrete linear time-varying systems and we will describe solvability of this problem in the context of controllability. Additionally, we will present some results for assignability for another numerical characteristics.

The paper consists of two parts. The first one is about continuous-time systems, whereas the second one contains results for discrete-time systems. Each part has the same structure and starts with basic notation, definitions of Lyapunov spectrum and concepts from theory of autonomous systems. Next we discuss relations between stabilizability, controllability and optimizability. The main section of each part is entitled Assignability and contains results about different formulations of assignability of the Lyapunov spectrum as well as regularity coefficients and central exponents.

2. Continuous-time systems

2.1. Basic notation. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with a fixed orthonormal basis and the standard norm $\|\cdot\|$. By $\mathbb{R}^{n\times m}$ we denote the space of all real $n\times m$-matrices with the spectral norm, i.e. with the operator norm generated in $\mathbb{R}^{n\times m}$ by Euclidean norms in $\mathbb{R}^n$ and $\mathbb{R}^m$.

For symmetric matrix $W \in \mathbb{R}^{n\times n}$ we will write $W > 0$ ($W \geq 0$) if the matrix W is positive (nonnegative) definite, i.e. $x^T W x > 0$, ($x^T W x \geq 0$) for each nonzero vector $x \in \mathbb{R}^n$ where superscript "T" denotes transposition operation. For two symmetric matrices $W, V \in \mathbb{R}^{n\times n}$ we will write $W > V$, ($W \geq V$) when $(W-V)$ is positive (nonnegative) definite matrix.

By I_n we will denote the identity matrix of order n when the dimension of the matrix follows from the context we will omit the sub-index n. By $\mathbb{R}_+$ we denote the interval $[0,\infty)$ and by $\mathbb{R}^n_{\leq}$ we denote the set of all nondecreasing sequences of n real numbers and for $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_{\leq}$ denote by $O_\delta(\mu)$ the set of all sequences $\nu = (\nu_1, \nu_2, \ldots, \nu_s) \in \mathbb{R}^n_{\leq}$ such that $\max_{j=1,2,\ldots,n} |\nu_j - \mu_j| < \delta$.

For an interval $\mathscr{I} \subset \mathbb{R}$ the set of all essentially bounded and measurable functions $f\colon \mathscr{I} \to \mathbb{R}^{n\times m}$ is denoted $L^\infty(\mathscr{I},\, \mathbb{R}^{n\times m})$, $L^\infty_{loc}(\mathbb{R}_+,\, \mathbb{R}^{n\times m})$ is the set of all functions $f\colon \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ such that for any compact interval $\mathscr{I} \subset \mathbb{R}_+$ the restriction of f to $\mathscr{I}$ belongs to $L^\infty(\mathscr{I},\, \mathbb{R}^{n\times m})$, and $L^2(\mathscr{I},\, \mathbb{R}^{n\times m})$ is the set of integrable functions $f\colon \mathscr{I} \to \mathbb{R}^{n\times m}$ such that $\int_{\mathscr{I}} \|f(s)\|^2\, \mathrm{d}s < \infty$. By $PC(\mathscr{I},\mathbb{R}^{n\times m})$ we will denote the set of all bounded piecewise continuous functions $f\colon \mathscr{I} \to \mathbb{R}^{n\times m}$. A function $B\colon \mathscr{I} \to \mathbb{R}^{n\times m}$ is called piecewise uniformly continuous on $\mathscr{I}$ if the following conditions are satisfied: $B \in PC(\mathscr{I},\mathbb{R}^{n\times m})$, there exists $\Delta_0 > 0$ such that the length of each continuity interval $\mathscr{I}_j (j \in J \subset \mathbb{N})$ of the function B satisfies the inequality $|\mathscr{I}_j| \geq \Delta_0$, and for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|B(t) - B(s)\| \leq \varepsilon$ for each $j \in J$ and for all $t,s \in \mathscr{I}_j$ satisfying the inequality $|t-s| \leq \delta$. By $PUC(\mathscr{I},\mathbb{R}^{n\times m})$ we will denote the set of all piecewise uniformly continuous functions $f\colon \mathscr{I} \to \mathbb{R}^{n\times m}$. If $m = 1$ then we will simple write $L^\infty(\mathscr{I},\, \mathbb{R}^n)$, $L^\infty_{loc}(\mathbb{R}_+,\, \mathbb{R}^n)$, $L^2(\mathscr{I},\mathbb{R}^n)$, $PC(\mathscr{I},\mathbb{R}^n)$ and $PUC(\mathscr{I},\mathbb{R}^n)$.

We consider the continuous time-varying linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{6}$$

where $A \in L^\infty(\mathbb{R}_+,\mathbb{R}^{n\times n})$ and $B \in PC(\mathbb{R}_+,\mathbb{R}^{n\times m})$ (occasionally we also consider system (6) with unbounded coefficients but then it will be clearly stated), x is the n-dimensional state vector and $u \in PC(\mathbb{R}_+,\mathbb{R}^m)$ is called a control. A solution corresponding to the initial condition $x(t_0) = x_0 \in \mathbb{R}^n$ at time $t_0 \in \mathbb{R}_+$ and the input function u is denoted by $x(\cdot,x_0,t_0,u)$.

For the homogeneous system

$$\dot{x}(t) = A(t)x(t) \tag{7}$$

the unique solution corresponding to the initial condition $x(t_0) = x_0 \in \mathbb{R}^n$ is denoted $x(\cdot,x_0,t_0)$ and its state transition matrix is denote by $\Phi_A(\cdot,\cdot)$. For a bounded function $U \in L^\infty(\mathbb{R}_+,\mathbb{R}^{m\times n})$ we will denote by $\|U\|_\infty$ the supremum norm defined by

$$\|U\|_\infty = \sup_{t\in[0,\infty)} \|U(t)\|.$$

For $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$ the Lyapunov exponent $\lambda(x_0)$ of (7) is defined as follows

$$\lambda(x_0) = \limsup_{t\to\infty} \frac{1}{t} \ln \|x(t,x_0,t_0)\|.$$

It easy to show that the value of $\lambda(x_0)$ does not depend on t_0. It is well known (see for example [22, 23]) that the set of Lyapunov exponents of all nontrivial solutions of system (7) contains at most n elements. Moreover, all Lyapunov exponents $\lambda_i'(A)$, $i \in \mathbb{N}$, $i \leq r \leq n$ are finite and fulfil the inequalities

$$-\infty < \lambda_1'(A) < \lambda_2'(A) < \ldots < \lambda_r'(A) < \infty.$$

For each $\lambda_i'(A)$ we consider the linear subspaces of $\mathbb{R}^n$

$$E_i = \left\{ v \in \mathbb{R}^n \colon \lambda(v) \leq \lambda_i'(A) \right\}$$

and we set $E_0 = \{0\}$. The multiplicity n_i of the Lyapunov exponent $\lambda_i'(A)$ is defined as

$$n_i = \dim E_i - \dim E_{i-1}, \quad i = 1,2,\ldots,r.$$

Observe that

$$\sum_{i=1}^r n_i = n.$$

The sequence

$$\lambda(A) = (\lambda_1(A),\, \lambda_2(A),\, \ldots,\, \lambda_n(A)),$$

where each Lyapunov exponent $\lambda_i'(A)$ appears n_i times, will be called the Lyapunov spectrum of (7). It is also well known [22] that

$$\lambda_n(A) = \limsup_{t\to\infty} \frac{1}{t} \ln \|\Phi_A(t,t_0)\|. \tag{8}$$

For $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$ the Bohl exponent $\beta(x_0)$ of solution $x(\cdot,x_0,t_0)$ of (7) is defined as follows

$$\beta(x_0) = \limsup_{t-\tau\to\infty} \frac{1}{t-\tau} \ln \frac{\|x(t,x_0,t_0)\|}{\|x(\tau,x_0,t_0)\|}$$

and the Bohl exponent of system (7) is defined by

$$\beta(A) = \limsup_{t-\tau\to\infty} \frac{1}{t-\tau} \ln \|\Phi_A(t,\tau)\|.$$

It is easy to show that the value of $\beta(x_0)$ does not depend on t_0. In contrast to the Lyapunov exponents the structure of the set $\{\beta(x_0)\colon x_0 \in \mathbb{R}^n \setminus \{0\}\}$ and the relation between $\beta(x_0)$ and $\beta(A)$ are much more complicated. The full description of the set of all Bohl exponents of a given system is presented in [24]. It has been also shown in [24] that for each number $p \leq P$ there exists a system (7) such that

$$\sup\{\beta(x_0)\colon x_0 \in \mathbb{R}^n \setminus \{0\}\} = p$$

and

$$\beta(A) = P.$$

Let the control u in system (6) be defined as a linear state feedback $u(t) = U(t)x(t)$, where $U\colon \mathbb{R}_+ \to \mathbb{R}^{m\times n}$. We assume that $U \in PC(\mathbb{R}_+,\mathbb{R}^{m\times n})$, unless otherwise is stated. Then the original system (6) becomes

$$\dot{x}(t) = (A(t) + B(t)U(t))x(t). \tag{9}$$

The spectrum of the closed-loop system (9) will be denoted by

$$\lambda(A+BU) = (\lambda_1(A+BU),\, \lambda_2(A+BU),\, \ldots,\, \lambda_n(A+BU)).$$

Now we will present certain definitions and concepts from the theory of linear time-varying differential equations which we will use in the further part of this paper (see [25]).

Definition 1. The number

$$\sigma_L(A) = \sum_{i=1}^{n} \lambda_i(A) - \liminf_{t\to\infty} \frac{1}{t} \int_0^t trA(s)\,\mathrm{d}s$$

is called the Lyapunov regularity coefficient of system (7). System (7) is called regular (in the Lyapunov sense) if $\sigma_L(A) = 0$.

Definition 2. Suppose that $L\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is piecewise continuously differentiable, $L(t)$ is invertible for all $t \in \mathbb{R}_+$ and

$$\sup_{t\geq 0} \left(\|L(t)\| + \|L^{-1}(t)\| + \|\dot{L}(t)\|\right) < \infty,$$

then the transformation

$$y = L(t)x$$

is called the Lyapunov transformation.

Definition 3. System (7) is called dynamically equivalent to system

$$\dot{y}(t) = G(t)y(t) \tag{10}$$

if there exists a Lyapunov transformation L such that

$$G(t) = L(t)A(t)L^{-1}(t) + \dot{L}(t)L^{-1}(t)$$

for all $t \in \mathbb{R}_+$. If there exists a Lyapunov transformation such that all the matrices $G(t)$, $t \in \mathbb{R}_+$ are diagonal then system (7) is called diagonalizable.

Definition 4. The Lyapunov spectrum of (7) is called stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $Q\colon PC(\mathbb{R}_+, \mathbb{R}^{n\times n})$ the inequality

$$\|Q\|_\infty < \delta$$

implies

$$\lambda(A+Q) \in O_\varepsilon(\lambda(A)),$$

where $\lambda(A+Q)$ is the Lyapunov spectrum of so-called disturbed system

$$\dot{x}(t) = (A(t) + Q(t))x(t).$$

Let us briefly describe the relationships between the introduced concepts. It is not difficult to give examples of system with the following properties: system is regular and not integrally separated; system is regular and integrally separated; system is irregular and not integrally separated; system is irregular and integrally separated.

2.2. Stability, stabilizability, controllability and optimalizability. In this subsection we will define the concepts of stability, stabilizability, controllability and optimizability of continuous time-varying linear systems.

Definition 5. [26] System (7) is called (uniformly exponentially) exponentially stable if there exist positive constants M, $\omega \in \mathbb{R}_+$ such that

$$\left(\|\Phi_A(t,t_0)\| \leq Me^{-\omega(t-t_0)}\right)$$

$$\|\Phi_A(t,0)\| \leq Me^{-\omega t}$$

for all $(t > t_0 \geq 0)$ $t \geq 0$.

The defined above concepts of stability are characterized by Lyapunov and Bohl exponents in the following ways.

Theorem 1. [27, 28] System (7) is (uniformly exponentially) exponentially stable if and only if

$$(\beta(A) < 0)$$

$$\lambda_n(A) < 0.$$

Among many different kinds of controllability we will consider the following ones [9].

Definition 6. System (6) is called controllable at $t_0 \in \mathbb{R}_+$ if and only if for each $x_0 \in \mathbb{R}^n$ there exists $t_1 \geq t_0$ and $u \in PC([t_0,t_1],\mathbb{R}^m)$ such that

$$x(t_1,x_0,t_0,u) = 0. \tag{11}$$

Definition 7. System (6) is called completely controllable if and only if for each $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ there exists $t_1 > t_0$ and $u \in PC([t_0,t_1],\mathbb{R}^m)$ such that

$$x(t_1,x_0,t_0,u) = 0. \tag{12}$$

Definition 8. System (6) is called uniformly completely controllable if there exist ℓ, $T \in (0,\infty)$ such that for each $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ there exists $u \in PC([t_0,t_0+T],\mathbb{R}^m)$ such that

$$x(t_1,x_0,t_0,u) = 0 \tag{13}$$

and

$$\|u\|_\infty \leq \ell\|x_0\|.$$

It can be shown that if we replace the zero vector by an arbitrary one on right hand sides of equations (11)–(13) then we obtain concepts which are equivalent to the original ones. Moreover, the definition of controllability at t_0 can be also formulated in the following seemingly stronger way [29].

Remark 1. System (6) is controllable at $t_0 \in \mathbb{R}_+$ if and only if there exists $t_1 \geq t_0$ such that for each $x_0 \in \mathbb{R}^n$ there exists $u \in PC([t_0,t_1],\mathbb{R}^m)$ such that

$$x(t_1,x_0,t_0,u) = 0. \tag{14}$$

It is well known [9] that these concepts of controllability can be described in terms of the Kalman controllability matrix defined as follows

$$W_1(t_0,t_1) = \int_{t_0}^{t_1} \Phi_A(t_0,t)B(t)B^T(t)\Phi_A^T(t_0,t)\,\mathrm{d}t \tag{15}$$

$$W_2(t_0,t_1) = \int_{t_0}^{t_1} \Phi_A(t_1,t)B(t)B^T(t)\Phi_A^T(t_1,t)\,\mathrm{d}t \tag{16}$$

for $t_1 \geq t_0 \geq 0$. From the definitions it directly follows that $W_1(t_0,t_1)$ and $W_2(t_0,t_1)$ are $n \times n$-dimensional symmetric nonnegative definite matrices. The most frequently used conditions for controllability verification are those formulated and proved in [9].

Theorem 2. System (6):

1) is controllable at t_0 if and only if there exists $t_1 > t_0$ such that
$$W_1(t_0,t_1) > 0,$$
2) is completely controllable if and only if for each $t_0 \in \mathbb{R}_+$ there exists $t_1 > t_0$ such that
$$W_1(t_0,t_1) > 0,$$
3) is uniformly completely controllable if and only if there exist $\alpha, T \in (0,\infty)$ such that
$$W_1(t_0,t_0+T) \geq \alpha I_n$$
for each $t_0 \in \mathbb{R}_+$.

It can be easily shown [9] that the assumption $A \in L^\infty(\mathbb{R}_+,\mathbb{R}^{n\times n})$ implies that the matrix W_1 may be replaced by W_2 in the above theorem.

We now introduce the concepts of stabilizability under investigation.

Definition 9. System (6) is called (uniformly exponentially) exponentially stabilizable if there exists a feedback control $u(t) = U(t)x(t)$, $U \in L^\infty_{loc}(\mathbb{R}_+,\mathbb{R}^{m\times n})$ such that the closed-loop system (9) is (uniformly exponentially) exponentially stable.

Definition 10. [30] System (6) is completely stabilizable if and only if for any $t_0 \in \mathbb{R}_+$ and any continuous and monotonically nondecreasing function $\delta(\cdot,t_0)\colon [t_0,\infty] \to \mathbb{R}_+$ such that $\delta(t_0,t_0) = 0$ there exist a feedback control $u(t) = U(t)x(t)$, $U \in L^\infty_{loc}(\mathbb{R}_+,\mathbb{R}^{m\times n})$ and a constatnt $\alpha(t_0) > 0$ such that

$$\|\Phi_{A+BU}(t,t_0)\| \leq \alpha(t_0)exp(-\delta(t,t_0))$$

for all $t \geq t_0$.

In case that in any of the above two definitions we may choose $U \in L^\infty(\mathbb{R}_+,\mathbb{R}^{m\times n})$, we say that system is uniformly exponentially, exponentially, completely stabilizable by a bounded feedback. The next theorem, taken from [30], describes the relations between complete satbilizability and complete controllability for system with possibly unbounded coefficients.

Theorem 3. If $A \in L^\infty_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times n})$ and $B \in L^\infty_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times m})$ then system (6) is completely stabilizable if and only if it is completely controllable. Moreover, uniform complete controllability of system (6) implies uniform exponential stabilizablity.

If we consider system (6) with bounded coefficients then we have the following relation.

Theorem 4. [30] System (6) is uniformly completely stabilizable by a bounded feedback if and only if it is uniformly completely controllable.

Together with system (6) let us consider an infinite time cost functional of the following form:

$$J(x_0,t_0,u) = \int_{t_0}^{\infty} \left(\|x(s,x_0,t_0,u)\|^2 + \|u(s)\|^2\right) \mathrm{d}s.$$

Definition 11. System (6) is called optimizable if for all $t_0 \geq 0$ there exists $C(t_0) \geq 0$ such that for all $x_0 \in \mathbb{R}^n$ there exists a control $u \in L^2([t_0,\ \infty),\ \mathbb{R}^m)$ such that $J(x_0,t_0,u) \leq C(t_0)\|x_0\|^2$ and system (6) is called uniformly optimizable if the constant $C(t_0)$ may be chosen independently on t_0, i.e. there exists $C > 0$ such that for all $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ there exists $u \in L^2([t_0,\ \infty),\ \mathbb{R}^m)$ such that $J(x_0,t_0,u) \leq C\|x_0\|^2$.

The next theorem contains two conditions which are equivalent to optimizability of system (6) (see [31]).

Theorem 5. The following conditions are each equivalent to optimizability:

1) for each $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ there exists a control $u \in L^2([t_0,\ \infty),\ \mathbb{R}^m)$ such that $J(x_0,t_0,u)$ is finite,
2) for each $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ there exists a control $u \in L^2([t_0,\ \infty),\ \mathbb{R}^m)$ such that
$$x(\cdot,x_0,t_0,u) \in L^2([t_0,\ \infty),\ \mathbb{R}^n)$$
and
$$\lim_{t\to\infty} x(t,x_0,t_0,u) = 0.$$

In the next subsection we will present relations between optimizability and possibility of shifting the Lyapunov and the Bohl exponents by a linear feedback.

2.3. Assignability. We will start this subsection with a discussion about relation between stabilizability, controllability, optimizability and possibility of placement of Lyapunov and Bohl exponents of system (9) by an appropriate feedback.

The relations can be roughly stated as follows:

- complete controllability is equivalent to the possibility to make by a feedback (not necessarily bounded) the Lyapunov exponent of the closed-loop system arbitrary small,
- uniform complete controllability is equivalent to the possibility to make by a bounded feedback the Bohl exponent of the closed-loop system arbitrary small,
- exponential stabilizability is equivalent to the existence of a feedback such that the corresponding closed-loop system has the greatest Lyapunov exponent negative,

- uniform exponential stabilizability is equivalent to the existence of a feedback such that the corresponding closed-loop system has the Bohl exponent negative and all of these conditions are equivalent to uniform optimizability.

We have the following theorem.

Theorem 6. [31] System (6) is completely controllable if and only if for all $\lambda > 0$ there exists a feedback control $U \in L^{\infty}_{loc}(\mathbb{R}_+, \mathbb{R}^{m\times n})$ such that

$$\lambda_n(A+BU) \leq -\lambda.$$

Notice that the feedback control U from the above theorem does not have to be bounded. From this theorem it is clear that complete controllability implies optimizability. It has been shown in [31], by an example, that opposite implication is not true. It also follows from point 2 of Theorem 5 that optimizability implies possibility of finding locally integrable feedback control such that the closed-loop system (9) is asymptotically stable (but not necessarily exponentially stable).

The next theorem proved in [30] (see also [31]) provides relations between uniform compltete controllability and Bohl exponents.

Theorem 7. System (6) is uniformly completely controllable if and only if for each $P > 0$ there exists a feedback control $U \in L^{\infty}(\mathbb{R}_+, \mathbb{R}^{m\times n})$ such that

$$\beta(A+BU) \leq -P.$$

The relation between uniform optimizability and uniform exponential stability is given by the following theorem.

Theorem 8. [31] System (6) is uniformly optimizable if and only if there exists a feedback control $U \in L^{\infty}(\mathbb{R}_+, \mathbb{R}^{m\times n})$ such that

$$\beta(A+BU) < 0.$$

In the next part of this subsection we will present necessary and/or sufficient conditions for the Lyapunov spectrum assignability, defined below in the Definitions 12–18. The problem of construction of a feedback providing the placement of certain characteristics of the closed-loop system in a priori given points is refered to as the problem of the assignability of the characteristics. It is generalization of the well known pole-placement problem (see [32]) to the case of time-varying system. The assignability problem for Lyapunov exponents of continuous time-varying linear system was stated in the first time in [33], where it is also shown that this problem can be naturally considered under the assumption of uniform complete controllability. In the framework of that approach a number of authors obtained various conditions for assignability of different numerical characteristics of continuous time-varying systems. These results are summarized in recent monograph of Makarov and Popova [29] where presented below concepts of assignability of the Lyapunov spectrum have been introduced and investigated.

The next definition expresses one of the possible way of formulation of the Lyapunov spectrum assignability problem.

Definition 12. The Lyapunov spectrum of system (9) is called globally assignable if for each $\mu \in \mathbb{R}^n_{\leq}$ there exists a feedback control $U \in PC(\mathbb{R}_+, \mathbb{R}^{m\times n})$ such that

$$\lambda(A+BU) = \mu. \tag{17}$$

In this definition there is in general no bound on the norm of the feedback control. In some practical applications it is desirable to have a bound on the control which tends to zero in case the placed Lyapunov spectrum tends to the Lyapunov spectrum of the free system. This requirement is the base for the following definition.

Definition 13. The Lyapunov spectrum of system (9) is called proportionally globally assignable if there exists $\ell > 0$ such that for any sequence $\mu \in \mathbb{R}^n_{\leq}$ there exists a feedback control $U \in PC(\mathbb{R}_+, \mathbb{R}^{m\times n})$, satisfying the estimate

$$\|U\|_{\infty} \leq \ell \max_{j=1,2,\ldots,n} |\lambda_j(A) - \mu_j| \tag{18}$$

and such that equality (17) is satisfied.

One may also consider the local and local proportional version of assignability of the Lyapunov spectrum.

Definition 14. The Lyapunov spectrum of system (9) is called locally assignable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\mu \in O_{\delta}(\lambda(A))$ there exists a feedback control $U \in PC(\mathbb{R}_+, \mathbb{R}^{m\times n})$ such that $\|U\|_{\infty} < \varepsilon$ and equality (17) is satisfied.

Definition 15. The Lyapunov spectrum of system (9) is called proportionally locally assignable if there exist $\ell > 0$ and $\delta > 0$ such that for all $\mu \in O_{\delta}(\lambda(A))$ there exists a feedback control $U \in PC(\mathbb{R}_+, \mathbb{R}^{m\times n})$, such that estimate (18) and equality (17) are satisfied.

Finally, we present a definition of nonmultiply proportional local assignability

Definition 16. The Lyapunov spectrum of system (9) is called nonmultiply proportionally locally assignable if for some $\beta > 0$ and $\delta > 0$ and for each $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_{\leq}$ such that $|\mu_i - \lambda_i(A)| \leq \delta$, $i = 1, 2, \ldots, n$ there exists a feedback control $U \in PC(\mathbb{R}_+, \mathbb{R}^{m\times n})$ satisfying the estimate $\|U\|_{\infty} \leq \beta \max_i |\mu_i - \lambda_i(A)|$ such that equality (17) is satisfied.

The notion of regularity of linear differential equations was introduced in famous paper of Lyapunov [34]. Some facts of regularity of discrete equations may be found in [35, 36]. The regularity is defined by certain numerical characteristics which are called regularity coefficients. In the literature we can find at least three regularity coefficients namely, Lyapunov $\sigma_L(A)$, Perron $\sigma_P(A)$ and Grobman $\sigma_G(A)$ coefficients of system (7). We have already defined Lyapunov regularity coefficients (see Definition 1). For the definitions of the rest of regularity coefficients see [25]. The next definition formulates a problem of assignability of regularity coefficients.

Definition 17. The Lyapunov (Perron, Grobman) regularity coefficient of system (9) is called globally assignable if for each $\sigma \geq 0$ there exists a feedback control $U \in PC(\mathbb{R}_+, \mathbb{R}^{m\times n})$ such that the Lyapunov (Perron, Grobman) regularity coefficient of system (9) is equal to σ.

In the literature also the following very general concept of global assignability of the Lyapunov invariants of system (9) is considered (see [29]).

Definition 18. We say that the Lyapunov invariants of system (9) are globally assignable if for each system

$$\dot{z}(t) = C(t)z(t), \tag{19}$$

with $C \in PC(\mathbb{R}_+, \mathbb{R}^{n\times n})$ there exists a feedback control $U \in PC(\mathbb{R}_+, \mathbb{R}^{m\times n})$ such that system (9) with this control is dynamically equivalent to system (19).

It immediately follows from the above definitions that proportional global assignability implies global assignability. It is also clear that proportional global assignability implies proportional local assignability. It should be pointed out, that the questions about truth of the inverse implications and other relations between them are still open. It is also clear that the global controllability of the set of the Lyapunov invariants implies global assignability of the Lyapunov spectrum.

In order to describe influence of the parametrical inaqurences on values of the greatest and the smallest Lyapunov exponent of system (7) we use upper central exponents $\Omega(A)$ and lower central exponents $\omega(A)$ (see [25, 37, 38]) defined in the following way:

$$\Omega(A) = \lim_{T\to\infty} \limsup_{k\to\infty} \frac{1}{kT} \sum_{i=1}^{k} \ln \|\Phi_A(iT, (i-1)T\| \tag{20}$$

and

$$\omega(A) = \lim_{T\to\infty} \limsup_{k\to\infty} \frac{1}{kT} \sum_{i=1}^{k} \ln \|\Phi_A^{-1}(iT, (i-1)T\|^{-1}. \tag{21}$$

Let us notice, that many results about assignability can be proved employing the concept of scalarizability [29].

Definition 19. System (9) is called scalarizable if for each bounded function $p\colon \mathbb{R}_+ \to \mathbb{R}$ there exists a feedback control $U \in PC(\mathbb{R}_+, \mathbb{R}^{m\times n})$ such that system (9) is dynamically equivalent to system (10) with $G(t) = p(t)I_n$.

A key result that connects the concept of the scalarizability with the problem of assignability and controllability is given by the following result from [39].

Theorem 9. If system (6) is uniformly completely controllable and $B \in PUC(\mathbb{R}_+, \mathbb{R}^{n\times m})$ then system (9) is scalarizable.

Another important property of uniformly completely controllable systems is given by the next theorem proved in [40].

Theorem 10. If system (6) is uniformly completely controllable and $B \in PUC(\mathbb{R}_+, \mathbb{R}^{n\times m})$ then for arbitrary $p_i \in PC(\mathbb{R}_+, \mathbb{R})$, $i = 1, 2, \ldots, n$ there exists a feedback control $U \in PC(\mathbb{R}_+, \mathbb{R}^{m\times n})$ such that closed system (9) is dynamically equivalent to a system with an upper triangular piecewise continuous and bounded matrix function whose diagonal coincides with $(p_1, p_2, \ldots, p_n)$.

Using this theorem the following sufficient condition for global assignability of the Lyapunov spectrum was obtained in [40].

Theorem 11. If system (6) is uniformly completely controllable and $B \in PUC(\mathbb{R}_+, \mathbb{R}^{n\times m})$ then the Lyapunov spectrum of system (9) is globally assignable. Moreover, the feedback control $U \in PC(\mathbb{R}_+, \mathbb{R}^{m\times n})$ that assigns the Lyapunov exponents may be chosen such that system (9) is regular.

It is natural to ask whether the condition about uniform complete controllability of (6) is necessary for the global assignability of the Lyapunov spectrum of system (9). In [40] an example (Example 1) was constructed, which shows that the answer to this question is negative. In this example an one-dimensional system (6) is constructed with the properties that the corresponding system (9) has globally assignable Lyapunov spectrum but system (6) is not uniformly completely controllable.

Even the uniform complete controllability is not a necessary condition for global assignability then there is an interesting relation between uniform complete controllability of certain sets of systems connected to system (6) and complete controllability of system (9). In order to present it let us introduce certain notation.

Consider system

$$\dot{x}(t) = A_0(t)x(t) + B_0(t)u(t) \tag{22}$$

with uniformly continuous and bounded coefficients $A_0 : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $B_0 : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$. We identify the system with the function $\sigma_0 : \mathbb{R}_+ \to \mathbb{R}^{n\times(n+m)}$ given by

$$\sigma_0(t) = (A_0(t), B_0(t)).$$

For given $\tau \in \mathbb{R}_+$ by σ_τ we denote the shift of σ_0 by τ, i.e. $\sigma_\tau(t) = \sigma_0(t+\tau)$.

Consider the set $\Re(\sigma_0)$ that is the closure of the set $\{\sigma_\tau : \tau \in \mathbb{R}_+\}$ in the topology of uniform convergence on closed intervals. The set $\Re(\sigma_0)$ is called Bebutov hull of σ_0 (see [41] for details).

Theorem 12. [40] Suppose that A, B are uniformly continuous and bounded. System σ_0 is uniformly completely controllable if and only if for each system $\sigma = (A, B) \in \Re(\sigma_0)$, the corresponding system (9) has the property of global assignability of the Lyapunov spectrum.

Now we will present some results about global assignability for a special forms of system (9). The next theorem deals with two-dimensional systems.

Theorem 13. [29, p. 325] Suppose that $n = 2$, system (6) is uniformly completely controllable and $B \in PUC(\mathbb{R}_+, \mathbb{R}^{2\times m})$, where $m = 1$ or $m = 2$, then the set of the Lyapunov invariants of system (9) is globally assignable.

It is possible to extend partially the above results to a larger class of two-dimensional systems, namely the class of systems with locally Lebesgue integrable and integrally bounded coefficients matrices A and B. The precise statement is as follows.

Theorem 14. [42] Consider system (6) for $n=2$ with $A:\mathbb{R}_+\to\mathbb{R}^{2\times 2}$, $B\in\mathbb{R}_+\to\mathbb{R}^{2\times m}$, where $m=1$ or $m=2$ satisfying the following inequalities:

$$\sup_{t\geq 0}\int_t^{t+1}\|A(\tau)\|\,\mathrm{d}\tau<+\infty,\quad \sup_{t\geq 0}\int_t^{t+1}\|B(\tau)\|\,\mathrm{d}\tau<+\infty$$

and assume that system (6) is uniformly complete controllable. Then for each $\mu\in\mathbb{R}^2_{\leq}$ there exists a feedback control $U\in L^\infty(\mathbb{R}_+,\mathbb{R}^{m\times 2})$ such that

$$\lambda(A+BU)=\mu.$$

It is worth to emphases that the assignability of the Lyapunov spectrum in the above theorem is understood in the different way than in Definition 13. The difference is in the class of feedback control. In Definition 13 it is required that the feedback control is piecewise continuous and bounded whereas in the Theorem 14 the feedback control is from a larger class of bounded functions.

Consider now system (6) with ω-periodic functions A and B. It was proved in [20], under the assumption on smoothness of the coefficients, that the ω-periodic system (6) is completely controllable if and only if for any arbitrary given real $n\times n$ matrix Λ with positive determinant, there exists an ω-periodic control U such that the multipliers of system (9) with this control are equal to eigenvalues of Λ. In [43] the assumption about smoothness of coefficients has been weaken and the following result has been proved.

Theorem 15. Suppose that system (6) has piecewise continuous ω-periodic coefficients and it is completely controllable, then the set of the Lyapunov invariants is globally assignable.

The next theorem will present sufficient conditions for proportional local assignability of the Lyapunov spectrum of the system (9). These conditions require uniform complete controllability of (6) and one of the following properties of the homogeneous system (7): regularity, diagonalizability or stability of the Lyapunov spectrum.

Theorem 16. [29] If the system (6) is uniformly completely controllable and at least one of the conditions hold:
1) system (7) is regular,
2) system (7) is diagonalizable,
3) the Lyapunov spectrum of system (7) is stable,
then the Lyapunov spectrum of system (9) is proportionally locally assignable.

This theorem has been partially extended in [44] in the following way.

Theorem 17. If system (6) is uniformly completely controllable and system (7) is regular, then there exist $\delta>0$ and $\ell>0$ such that, for all $\mu\in O_\delta(\lambda(A))$ and for each number $\sigma\in[0,\delta]$ there exists a feedback control $U\in PC(\mathbb{R}_+,\mathbb{R}^{m\times n})$ such that

$$\|U\|_\infty\leq \ell\max\{\sigma,\ |\mu_i-\lambda_i(A)|:\ i=1,2,\ldots,n\},$$

$$\lambda(A+BU)=\mu$$

and

$$\sigma_L(A+BU)=\sigma.$$

Using Theorem 9 in [29] the following results about assignability of the improperness coefficients and central exponents were obtained.

Theorem 18. Suppose that system (6) is uniformly completely controllable and $B\in PUC(\mathbb{R}_+,\mathbb{R}^{n\times m})$ then the Lyapunov, Perron and Grobman improperness coefficients are assignable. Moreover, for each numbers $\omega\leq\Omega$ there exists a feedback control $U\in PC(\mathbb{R}_+,\mathbb{R}^{m\times n})$ such that

$$\omega(A+BU)=\omega,\qquad \Omega(A+BU)=\Omega. \tag{23}$$

In order to present conditions for nonmultiply proportional local assignability of the Lyapunov spectrum, let us introduce some properties of the homogeneous system (7) (see [29]). Suppose that $x_1,x_2,\ldots,x_n$ is a fundamental system of solutions (FSS) of (6). For any $i=1,2,\ldots,n$ and $t\in\mathbb{R}_+$ denote by $V_i(t)\subset\mathbb{R}^n$ linear subspace spanned on vectors $x_j(t)$, $j=1,2,\ldots,n$, $j\neq i$ and by $\varphi_i(t)=\sphericalangle(x_i(t),V_i(t))$ the angle between the vector $x_i(t)$ and the linear subspace $V_i(t)$.

Let us take any $\vartheta>0$. For any $\gamma\in\left(0,\frac{\pi}{2}\right)$, $k\in\mathbb{N}$, and $i=1,2,\ldots,n$ we set

$$\Gamma_i^\gamma(\vartheta)=\{j\in\mathbb{N}\colon\ \varphi_i(j\vartheta)\geq\gamma\},$$

$$\Gamma_i^\gamma(k;\vartheta)=\Gamma_i^\gamma(\vartheta)\cap\{1,2,\ldots,k\}.$$

Let $N_i^\gamma(k,\vartheta)$ be the number of elements of the set $\Gamma_i^\gamma(k;\vartheta)$. Let us also introduce the following notation

$$g_i^\gamma(k;\vartheta)=\frac{N_i^\gamma(k,\vartheta)}{k},$$

$$f_i(k;\vartheta)=\frac{\ln\|x_i(k,\vartheta)\|}{k\vartheta}.$$

If the numbers γ and ϑ are given in advance, then the corresponding symbols in the above-introduced notation are omitted.

A sequence $(t_k)_{k\in\mathbb{N}}$ of real numbers strictly increasing to ∞ is referred to as a realizing sequence of a solution $x(\cdot,x_0,t_0)$ of the linear homogeneous system (7) if

$$\lambda(x_0)=\lim_{k\to\infty}\frac{\ln\|x(t_k)\|}{t_k}.$$

Definition 20. We say that a solution x_i occurring in the FSS $x_1,x_2,\ldots,x_n$ is ϑ-bounded away (from the remaining solutions of the FSS) if for a given $\vartheta>0$, there exists $\gamma\in\left(0,\frac{\pi}{2}\right]$ and a realizing sequence $(k_j\vartheta)_{j\in\mathbb{N}}$ for the solution x_i, where $k_j\in\mathbb{N}$, such that

$$\lim_{j\to\infty}g_i^\gamma(k_j;\vartheta)>0.$$

A FSS $x_1,x_2,\ldots,x_n$ is said to be ϑ-separated if each of the solutions in the FSS is ϑ-bounded away.

Let us consider basic properties of the above-introduced notions taken from [45].

Theorem 19. If a solution x_j occurring in the FSS $x_1, x_2, \ldots, x_n$ is ϑ_0-bounded away for some $\vartheta_0 > 0$ then it is ϑ-bounded away for any $\vartheta > 0$.

Having in mind Theorem 19 we say that a solution x_i occurring in a FSS $x_1, x_2, \ldots, x_n$ is bounded away if it is ϑ-bounded away for some $\vartheta > 0$. Accordingly, a FSS is called separated if it is ϑ-separated for some $\vartheta > 0$.

Definition 21. System (7) that has a bounded away normal FSS is called to be bounded away.

Using the concept of bounded away system we may formulate the following results about nonmultiply proportional local assignability and proportional local assignability (see [45] for the proofs).

Theorem 20. If system (6) is uniformly completely controllable and system (7) is bounded away then the Lyapunov spectrum of (9) is nonmultiply proportionally locally assignable. If in addition all the Lyapunov exponents of (7) are distinct then the Lyapunov spectrum of (9) is proportionally locally assignable and then the nonmultiply proportional local assignability is equivalent to proportional local assignability.

It should be mentioned, that taking into account the condition of boundedness away we may obtain certain results for unstability of the Lyapunov spectrum given in the next theorem [29].

Theorem 21. If system (7) has a bounded away FSS which is not normal, then its Lyapunov spectrum is unstable.

3. Discrete-time systems

3.1. Basic notation. For $t_0 \in \mathbb{N}$ denote by $\mathbb{N}_{t_0} = \{t_0, t_0+1, t_0+2, \ldots\}$ and consider the discrete linear time-varying system

$$x(t+1) = A(t)x(t) + B(t)u(t), \tag{24}$$

where $A = (A(t))_{t\in\mathbb{N}_{t_0}}$, $B = (B(t))_{t\in\mathbb{N}_{t_0}}$ are sequences of $n\times n$ and $n\times m$ real matrices, respectively. Moreover, the control sequence $u = (u(t))_{t\in\mathbb{N}_{t_0}}$ is m-dimensional.

The solution of (24), corresponding to the control u and the initial condition $x(t_0) = x_0 \in \mathbb{R}^n$ with $t_0 \in \mathbb{N}$, is denoted by

$$x = (x(t, x_0, t_0, u))_{t\in\mathbb{N}_{t_0}}$$

and is given by the following formula

$$\begin{aligned} x(t, t_0, x_0, u) = {} & \Phi_A(t, t_0)x_0 + \\ & + \sum_{j=t_0}^{t-1} \Phi_A(t, j+1)B(j)u(j), \; t \in \mathbb{N}_{t_0} \end{aligned} \tag{25}$$

where $\Phi_A(t, t_0)$ is the transition matrix of homogeneous system

$$x(t+1) = A(t)x(t) \tag{26}$$

given by

$$\Phi_A(t,t) = I_n,$$

$$\Phi_A(t, j) = A(t-1)\ldots A(j) \quad \text{for } t > j, \; t, j \in \mathbb{N}_{t_0}.$$

Additionally, when $A = (A(t))_{t\in\mathbb{N}_{t_0}}$ consists of invertible matrices we define

$$\Phi_A(j,t) = \Phi_A^{-1}(t, j) \quad \text{for } t > j, \; t, j \in \mathbb{N}_{t_0}.$$

Definition 22. A bounded sequence $(D(t))_{t\in\mathbb{N}_{t_0}}$ of invertible $n\times n$ matrices such that $(D^{-1}(t))_{t\in\mathbb{N}_{t_0}}$ is bounded is called the Lyapunov sequence.

In our further considerations we will assume that A is a Lyapunov sequence and B is bounded. However, occasionally we will consider system (24) with unbounded A, B coefficients or with A consisting of noninvertible matrices but it will be clearly stated.

For a given initial condition $x(t_0) = x_0 \in \mathbb{R}^n$ with $t_0 \in \mathbb{N}$ the solution of (26) is denoted by $x((t, t_0, x_0))_{t\in\mathbb{N}_{t_0}}$ and is given by

$$x(t, t_0, x_0) = \Phi_A(t, t_0)x_0, \quad t \in \mathbb{N}. \tag{27}$$

In the special case when $t_0 = 0$ then we simple write $(x(t, x_0))_{t\in\mathbb{N}_0}$ instead of $(x(t, 0, x_0))_{t\in\mathbb{N}_0}$. For $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$ the Lyapunov exponent $\lambda(x_0)$ of (27) is defined as

$$\lambda(x_0) = \limsup_{t\to\infty} \frac{1}{t} \ln \|x(t, t_0, x_0)\|.$$

It is easy to show that if A is a Lyapunov sequence then the value $\lambda(x_0)$ does not depend on t_0.

Similarly as for the continuous-time system it is well known [22] that the set of Lyapunov exponents of all nontrivial solutions of system (26) contains at most n elements. Moreover, if $A = (A(t))_{t\in\mathbb{N}}$ is the Lyapunov sequence, then all the Lyapunov exponents $\lambda_i'(A)$, $i = 1, 2, \ldots, r$ are finite. They are numbered such that the following inequalities are satisfied

$$-\infty < \lambda_1'(A) < \lambda_2'(A) < \ldots < \lambda_r'(A) < \infty.$$

For each $\lambda_i'(A)$ we consider the linear subspaces of $\mathbb{R}^n$

$$E_i = \{v \in \mathbb{R}^n : \lambda(v) \le \lambda_i'(A)\}$$

and additionally we set $E_0 = \{0\}$. The multiplicity n_i $i = 1, 2, \ldots, r$ of the Lyapunov exponent $\lambda_i'(A)$ is defined as

$$n_i = \dim E_i - \dim E_{i-1}.$$

Similarly as in continuous-time case the sequence

$$(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)),$$

where each Lyapunov exponent $\lambda_i'(A)$ appears n_i times will be called the Lyapunov spectrum of system (26).

It is also well known ([22]) that

$$\lambda_n(A) = \limsup_{t\to\infty} \frac{1}{t} \ln \|\Phi_A(t,t_0)\|. \tag{28}$$

For $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$ the Bohl exponent $\beta(x_0)$ of solution $(x(t,t_0,x_0))_{t\in\mathbb{N}_{t_0}}$ of (26) is defined as follows

$$\beta(x_0) = \limsup_{t-\tau\to\infty} \frac{1}{t-\tau} \ln \frac{\|x(t,x_0,t_0)\|}{\|x(\tau,x_0,t_0)\|}$$

and the Bohl exponent of system (26) is defined by

$$\beta(A) = \limsup_{t-\tau\to\infty} \frac{1}{t-\tau} \ln \|\Phi_A(t,\tau)\|.$$

It is easy to show that the value of $\beta(x_0)$ does not depend on t_0. Similarly to the continuous-time case the structure of the set $\{\beta(x_0)\colon x_0 \in \mathbb{R}^n\setminus\{0\}\}$ and the relation between $\beta(x_0)$ and $\beta(A)$ are much more complicated than in Lyapunov exponents frameworks. The full description of the set of all Bohl exponents of a given system is an open problem. It has been shown in [46] that there exists a system (26) such that $\sup\{\beta(x_0)\colon x_0\in\mathbb{R}^n\setminus\{0\}\} < \beta(A)$.

Definition 23. A bounded sequence $U = (U(t))_{t\in\mathbb{N}}$ of $m\times n$ matrices is said to be an admissible feedback control if $(A(t)+B(t)U(t))_{t\in\mathbb{N}}$ is a Lyapunov sequence.

For an admissible feedback control the Lyapunov spectrum of the closed-loop system

$$x(t+1) = (A(t)+B(t)U(t))\,x(t) \tag{29}$$

will be denoted by

$$(\lambda_1(A+BU), \lambda_2(A+BU), \ldots, \lambda_n(A+BU)).$$

Following continuous-time case we will use certain concepts taken from the theory of linear time-varying difference equations (see [23,47,48]).

Definition 24. The number

$$\sigma_L = \sum_{i=1}^{n} \lambda_i(A) - \liminf_{t\to\infty} \frac{1}{t} \ln|\det(\Phi_A(t,t_0))|$$

is called the Lyapunov regularity coefficient of system (26). System (26) is called regular (in the Lyapunov sense) if $\sigma_L = 0$.

Definition 25. Suppose that $L = (L(t))_{t\in\mathbb{N}_{t_0}}$ is a sequence of invertible $n\times n$ matrices such that

$$\sup_{t\in\mathbb{N}_{t_0}} \left(\|L(t)\| + \|L^{-1}(t)\|\right) < \infty,$$

then the linear transformation

$$y = L(t)x$$

is called the discrete Lyapunov transformation.

Definition 26. System (26) is called dynamically equivalent to system

$$y(t+1) = G(t)y(t) \tag{30}$$

if there exists a discrete Lyapunov transformation L such that

$$G(t) = L(t+1)A(t)L^{-1}(t)$$

for all $t\in\mathbb{N}_{t_0}$. If there exists a discrete Lyapunov transformation L such that all the matrices $G(t)$, $t\in\mathbb{N}_{t_0}$ are diagonal then system (26) is called diagonalizable.

Definition 27. System (26) is called a system with the integral separation if it has a basis of solutions $\left(x\left(t,x_0^{(i)}\right)\right)_{t\in\mathbb{N}}$ such that for some $a>1, b>0$ and all natural numbers $t>\tau$, $i\le n-1$ the inequalities

$$\frac{\left\|x\left(t,x_0^{(i+1)}\right)\right\|}{\left\|x\left(\tau,x_0^{(i+1)}\right)\right\|} \geq ba^{t-\tau} \frac{\left\|x\left(t,x_0^{(i)}\right)\right\|}{\left\|x\left(\tau,x_0^{(i)}\right)\right\|}$$

are satisfied.

Let us consider together with system (26) the following disturbed system

$$x(t+1) = (A(t)+Q(t))\,x(t), \tag{31}$$

where $Q = (Q(t))_{t\in\mathbb{N}_{t_0}}$ is such that $A+Q$ is a Lyapunov sequence.

Definition 28. The Lyapunov spectrum of (26) is called stable if for any $\varepsilon>0$ there exists $\delta>0$ such that for any Q such that $(A+Q)$ is a Lyapunov sequence, the inequality

$$\|Q\|_\infty < \delta$$

implies

$$\lambda(A+Q) \in O_\varepsilon(\lambda(A)),$$

where $\lambda(A+Q)$ is the Lyapunov spectrum of (31).

Let us briefly describe the relationships between the introduced concepts. It was established in [48] that the integral separation of the system (26) is equivalent to the stability and nonmultiplicity of its Lyapunov spectrum, and also the fact that the integral separation implies the diagonalizability of the system (26). There is no connection between the integral separation and the regularity.

3.2. Stability, stabilizability and controllability.

In this subsection we will define the concepts of stability, stabilizability, controllability of discrete time-varying linear systems.

Definition 29. [26] System (26) is called (uniformly exponentially) exponentially stable if there exist positive constants M, $\omega\in\mathbb{R}_+$ such that

$$\left(\|\Phi_A(t_1,t_2)\| \le Me^{-\omega(t_1-t_2)}\right)$$

$$\|\Phi_A(t,0)\| \leq Me^{-\omega t}$$

for all $(t_1 > t_2, t_1, t_2 \in \mathbb{N}_{t_0})$ $t \in \mathbb{N}_{t_0}$.

The defined above concepts of stability are characterized by Lyapunov and Bohl exponents in the following ways.

Theorem 22. [46,47] System (26) is (uniformly exponentially) exponentially stable if and only if

$$(\beta(A) < 0)$$

$$\lambda_n(A) < 0.$$

Among many different kinds of controllability we will consider the following ones, which are based on Kalman proposition for continuous-time systems [9] (see [49, 50] for discrete-time version).

Definition 30. System (24) is called controllable at $t_0 \in \mathbb{N}$ if and only if for each x_0, $x_1 \in \mathbb{R}^n$ there exists $t_1 \geq t_0$, $t_1 \in \mathbb{N}$ and a control $u = (u(t))_{t\in\mathbb{N}_{t_0}}$ such that

$$x(t_1, x_0, t_0, u) = x_1\,. \tag{32}$$

Definition 31. System (24) is called completely controllable if and only if for each $(t_0, x_0, x_1) \in \mathbb{N}\times\mathbb{R}^n\times\mathbb{R}^n$ there exists $t_1 > t_0$ and a control $u = (u(t))_{t\in\mathbb{N}_{t_0}}$

$$x(t_1, x_0, t_0, u) = x_1\,. \tag{33}$$

Definition 32. System (24) is called uniformly completely controllable if there exist $\ell \in (0,\infty)$ and $T \in \mathbb{N}$ such that for all $(t_0, x_0, x_1) \in \mathbb{N}\times\mathbb{R}^n\times\mathbb{R}^n$ there exists a control $u = (u(t))_{t\in\mathbb{N}_{t_0}}$ such that

$$x(t_1, x_0, t_0, u) = x_1\,. \tag{34}$$

and

$$\|u(n)\| \leq \ell \max\{\|x_0\|, \|x_1\|\}$$

for all $n = t_0, t_0+1, \ldots, t_0+T-1$.

It should be noted that in contrast to continuous-time case we can not, in general, replace x_1 by zero on the right-hand sides of (32)–(34) without losing generality but if we know that the sequence A is a Lyapunov sequence, then such a modification leads to equivalent concepts of controllability (see [51]). However, simirarly as in the continous-time case we have the following remark [49].

Remark 2. System (24) is controllable at $t_0 \in \mathbb{N}$ if and only if there exists $t_1 \geq t_0$, $t_1 \in \mathbb{N}$ such that for each x_0, $x_1 \in \mathbb{R}^n$ there exists a control $u = (u(t))_{t\in\mathbb{N}_{t_0}}$ such that

$$x(t_1, x_0, t_0, u) = x_1\,. \tag{35}$$

In controllability investigation of discrete-time systems also a crucial role is played by the Kalman controllability matrices defined for system (24) as follows

$$W_1(t,\tau) = \sum_{j=\tau}^{t-1} \Phi_A(t, j+1)B(j)B^T(j)\Phi_A^T(t, j+1),$$

$$W_2(t,\tau) = \sum_{j=\tau}^{t-1} \Phi_A(\tau, j+1)B(j)B^T(j)\Phi_A^T(\tau, j+1),$$

where $t > \tau$, t, $\tau \in \mathbb{N}$.

The next theorem (see [49, 51], [52, Proposition 3, p. 34]) gives, in the terms of the Kalman controllability matrix, the necessary and sufficient conditions for the above defined concepts of controllability.

Theorem 23. System (24):

1) is controllable at $t_0 \in \mathbb{N}$ if and only if there exists $t_1 > t_0$, $t_1 \in \mathbb{N}$ such that
$$W_1(t_0, t_1) > 0,$$
2) is completely controllable if and only if for each $t_0 \in \mathbb{N}$ there exists $t_1 > t_0$, $t_1 \in \mathbb{N}$ such that
$$W_1(t_0, t_1) > 0,$$
3) is uniformly completely controllable if and only if there exist $\ell \in (0,\infty)$ and $T \in \mathbb{N}$ such that
$$W_1(t_0, t_0+T) \geq \alpha I_n$$
for each $t_0 \in \mathbb{N}$.

It can be easily shown that the assumption that A is a Lyapunov sequence implies that the theorem remains true if we replace W_1 by W_2.

We now introduce the concepts of stabilizability of discrete-time systems.

Definition 33. System (24) is called (uniformly exponentially) exponentially stabilizable if there exists a feedback control $u(t) = U(t)x(t)$, $U \in L^\infty_{loc}(\mathbb{R}_+, \mathbb{R}^{m\times n})$ such that the closed-loop system (26) is (uniformly exponentially) exponentially stable.

Definition 34. [53] System (24) is called completely stabilizable if for each $t_0 \in \mathbb{N}$ and each nondecreasing sequence $(\delta(t,t_0))_{t\in\mathbb{N}}$ satisfying $\delta(t_0,t_0) = 0$ there exist a feedback matrix $(U(t))_{t\in\mathbb{N}}$ and a constant $\alpha(t_0) > 0$ such that

$$\|\Phi_{A+BU}(t,t_0)\| \leq \alpha(t_0)\exp(-\delta(t,t_0))\,.$$

In case that in any of the above two definitions we may choose the sequence $(U(t))_{t\in\mathbb{N}}$ being bounded, we say that system is uniformly exponentially, exponentially, completely stabilizable by a bounded feedback.

The next two theorems proved in [53] are a discrete counterpart of Theorem 3. Notice that we neither require A is a Lyapunov sequence nor that B is bounded.

Theorem 24. If system (24) with arbitrary A, B is completely controllable then it is completely stabilzable.

Theorem 25. If system (24) with A consisting of invertible matrices and arbitrary B is completely stabilzable then it is completely controllable.

3.3. Assignability. For continuous-time systems in the Subsections 2.2 and 2.3 (Definition 11, Theorems 6, 7 and 8), we presented relations between possibility of shifting the greatest Lyapunov exponent and Bohl exponent, complete controllability, uniform complete controllability and optimalizability. According to our best knowledge, analogical results for discrete-time systems are unknown. However, it is worth to mention that in [54] the concept of optimalizability for discrete jump linear systems was introduced and connected to properties of solution of appropriate Riccati equation.

The next four definitions of assignability are the discrete-time versions of Definitions 12–15 for continuous-time case. They are originally formulated in series of papers [55–58] and [59].

Definition 35. The Lyapunov spectrum of system (29) is called globally assignable if for each $\mu \in \mathbb{R}^n_{\leq}$ there exists an admissible feedback control U such that

$$\lambda(A+BU) = \mu. \tag{36}$$

Definition 36. The Lyapunov spectrum of system (29) is called proportionally globally assignable if for all $\Delta > 0$ there exists $\ell = \ell(\Delta) > 0$ such that for any sequence $\mu = (\mu_1, \ldots, \mu_n) \in O_\Delta\big(\lambda(A)\big)$ there exists an admissible feedback control U, satisfying the estimate

$$\|U\|_\infty \leq \ell \max_{j=1,\ldots,n} |\lambda_j(A) - \mu_j| \tag{37}$$

and such that equality (36) is satisfied.

Definition 37. The Lyapunov spectrum of system (29) is called locally assignable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\mu \in O_\delta\big(\lambda(A)\big)$ there exists an admissible feedback control U such that

$$\lambda(A+BU) = \mu \quad \text{and} \quad \|U\|_\infty < \varepsilon.$$

Definition 38. The Lyapunov spectrum of system (29) is called proportionally locally assignable if there exist $\ell > 0$ and $\delta > 0$ such that for all $\mu \in O_\delta\big(\lambda(A)\big)$ there exists an admissible feedback control U, such that estimate (37) and equality (36) are satisfied.

Definitions 35, 37 and 38 are direct translations of their continuous counterparts. However, the direct transformation of definition of proportional global assignability is as follows: the Lyapunov spectrum of system (29) is called proportionally globally assignable if there exists $\ell > 0$ such that for all $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n_{\leq}$ there exists a feedback control U, satisfying (36) and (37). The Example 2 below justifies our modification.

The next theorem presents a sufficient condition for global assignability of the Lyapunov spectrum.

Theorem 26. [55] If system (24) is uniformly completely controllable, then the Lyapunov spectrum of system (29) is globally assignable.

The following example shows that uniform complete controllability is not a necessary condition for global assignability of the Lyapunov spectrum.

Example 1. Let us define a sequence $(n_k)_{k\in\mathbb{N}}$ by the recurrent formulae

$$n_1 = 1, \quad n_{2m} = mn_{2m-1}, \quad n_{2m+1} = m + n_{2m}$$

for all $m \in \mathbb{N}$. The sequence $(n_k)_{k\in\mathbb{N}}$ is strictly increasing for $k \geq 2$, tends to $+\infty$, and satisfies the relations

$$\lim_{m\to\infty} \frac{n_{2m-1}}{n_{2m}} = \lim_{m\to\infty} \frac{1}{m} = 0,$$

$$\lim_{m\to\infty} \frac{m}{n_{2m}} = \lim_{m\to\infty} \frac{1}{n_{2m-1}} = 0$$

and

$$\lim_{m\to\infty} \frac{n_{2m}}{n_{2m+1}} = \lim_{m\to\infty} \frac{1}{1+m/n_{2m}} = 1.$$

Put

$$b(n) = \begin{cases} 1 & \text{for} \quad n = 1, \\ 1 & \text{for} \quad n \in [n_{2m-1}, n_{2m} - 1], \\ 0 & \text{for} \quad n \in [n_{2m}, n_{2m+1} - 1], \end{cases}$$

for $m = 2, 3, \ldots$, and consider a scalar linear control equation

$$x(n+1) = x(n) + b(n)u(n). \tag{38}$$

System (38) is not uniformly completely controllable. Indeed, for each $K \in \mathbb{N}$, there exists a number $m = K$ such that the Kalman controllability matrix of (38) is equal to zero on the interval $[n_{2m}, n_{2m}+K]$, that is,

$$W(n_{2m}, n_{2m}+K) = \sum_{j=k}^{n-1} \Phi_A(k, j+1) b^2(j) \Phi_A^T(k, j+1) = \\ = \sum_{j=n_{2m}}^{n_{2m}+K-1} b^2(j) = 0.$$

The closed-loop equation corresponding to (38) has the form

$$x(n+1) = \big(1 + b(n)U(n)\big)x(n). \tag{39}$$

Now, let us show that the above equation has the assignability property of the Lyapunov spectrum.

Fix any $\alpha \in \mathbb{R}$, denote $\beta = e^\alpha - 1$ and define $U(n) \equiv \beta$, $n \in \mathbb{N}$. The Lyapunov exponent of each nontrivial solution to

(39) with the defined $U(\cdot)$ coincides with the upper mean value of the function $\ln(1+\beta b(\cdot))$, that is, with the value

$$\mu = \limsup_{n\to\infty} n^{-1}\sum_{j=1}^{n-1}\ln\big(1+\beta b(j)\big).$$

Our aim is to prove that $\mu = \alpha$. Put $\varphi(1) = 0$ and

$$\varphi(n) = \frac{1}{n}\sum_{j=1}^{n-1}\ln\big(1+\beta b(j)\big)$$

for natural number $n > 1$. It is clear that

$$\ln\big(1+\beta b(n)\big) = \begin{cases} \alpha & \text{for} \quad n = 1, \\ \alpha & \text{for} \quad n \in [n_{2m-1}, n_{2m}-1], \\ 0 & \text{for} \quad n \in [n_{2m}, n_{2m+1}-1], \end{cases}$$

for $m = 2,3,\ldots$. Let $\alpha \geq 0$. Then,

$$0 \leq \ln(1+\beta b(n)) \leq \alpha$$

and therefore, $\varphi(n) \leq \alpha$ for all $n \in \mathbb{N}$. Hence, $\mu \leq \alpha$. However,

$$\begin{aligned}\mu &\geq \limsup_{m\to\infty}\varphi(n_{2m}) \\ &= \limsup_{m\to\infty}\frac{1}{n_{2m}}\sum_{j=1}^{n_{2m}-1}\ln\big(1+\beta b(j)\big) \\ &\geq \limsup_{m\to\infty}\frac{1}{n_{2m}}\sum_{j=n_{2m-1}}^{n_{2m}-1}\alpha \\ &= \alpha\lim_{m\to\infty}\frac{n_{2m}-n_{2m-1}}{n_{2m}} = \alpha.\end{aligned}$$

Thus, $\mu = \alpha$. Now let $\alpha \leq 0$. Then,

$$0 \geq \ln(1+\beta b(n)) \geq \alpha$$

and therefore $0 \geq \varphi(n) \geq \alpha$ for all $n \in \mathbb{N}$. Hence, $\mu \geq \alpha$. For each $k \in [n_{2m-1}, n_{2m}]$ with any natural number $m > 1$, we have

$$\begin{aligned}\varphi(k) &= \frac{1}{k}\left(\sum_{j=1}^{n_{2m-1}-1}\ln(1+\beta b(j)) + (k-n_{2m-1})\alpha\right) \\ &= k^{-1}\big(\varphi(n_{2m-1})n_{2m-1} + (k-n_{2m-1})\alpha\big) \\ &= \frac{n_{2m-1}}{k}\varphi(n_{2m-1}) + \alpha\frac{k-n_{2m-1}}{k} \leq \varphi(n_{2m-1}).\end{aligned}$$

In addition, for $k = n_{2m}$, we obtain

$$\varphi(n_{2m}) \leq \alpha\frac{n_{2m}-n_{2m-1}}{n_{2m}} = \alpha\left(1-\frac{1}{m}\right). \tag{40}$$

For each $k \in [n_{2m}, n_{2m+1}]$ with any $m \in \mathbb{N}$, we also have

$$\varphi(k) = k^{-1}\varphi(n_{2m})n_{2m} \leq n_{2m+1}^{-1}\varphi(n_{2m})n_{2m} = \varphi(n_{2m+1}). \tag{41}$$

Thus, $\varphi(k) \leq \varphi(n_{2m+1})$ for all $k \in [n_{2m}, n_{2m+2}]$. Moreover, from (40) and (41) we get

$$\varphi(n_{2m+1}) = \frac{n_{2m}}{n_{2m+1}}\varphi(n_{2m}) \leq \alpha\frac{n_{2m}}{n_{2m+1}}\left(1-\frac{1}{m}\right),$$

so

$$\varphi(k) \leq \alpha\frac{n_{2m}}{n_{2m+1}}\left(1-\frac{1}{m}\right)$$

for all $k \in [n_{2m}, n_{2m+2}]$. Note that

$$\lim_{m\to\infty}\alpha\frac{n_{2m}}{n_{2m+1}}\left(1-\frac{1}{m}\right) = \alpha.$$

Put

$$r(k) = \begin{cases} 0, & k = 1, \\ \dfrac{\alpha n_{2m}}{n_{2m+1}}\left(1-\dfrac{1}{m}\right), & k \in [n_{2m}, n_{2m+2}-1]. \end{cases}$$

It is clear that $r(k) \to \alpha$ as $k \to \infty$. Since $\varphi(k) \leq r(k)$ for all $k \in \mathbb{N}$, we have

$$\mu = \limsup_{k\to\infty}\varphi(k) \leq \limsup_{k\to\infty} r(k) = \alpha.$$

Therefore, $\mu = \alpha$. Thus, system (39) with the defined control $U(\cdot)$ has the Lyapunov spectrum consisting of α, and the Lyapunov spectrum of the equation (39) is assignable.

The example below explains why we changed the definition of proportional global assignability for discrete-time systems with comparison to definition for continuous-time systems.

Example 2. Let us consider a linear discrete-time scalar control system

$$x(t+1) = x(t) + u(t). \tag{42}$$

Here $A(t)$, $B(t)$ are scalars and $A(t) = B(t) = 1$ for all t. Therefore, for the transition matrix of the homogeneous system

$$x(t+1) = x(t)$$

we have $\Phi_A(t,t_0) = 1$ for all $t, t_0 \in \mathbb{N}_0$. Thus, system (42) is uniformly completely controllable. Since every solution $x(t,t_0,x_0)$ of system (42) is constant, it follows that the Lyapunov spectrum coincides with 0. Let us close system (42) by a feedback $u(t) = U(t)x(t)$. Then we get a system

$$x(t+1) = (1+U(t))x(t). \tag{43}$$

By the Theorem 26 the Lyapunov spectrum of system (43) is globally assignable, so for every $\alpha \in \mathbb{R}$ we can construct a control U, such that the Lyapunov spectrum of system (43) coincides with the number α. Let us find out whether it is possible to find a number $\ell > 0$, such that for all $\alpha > 0$ there exists a control U for which we have

$$\lambda(A+BU) = \alpha, \quad \|U\|_\infty \leq \ell\alpha. \tag{44}$$

Here we restrict ourselves to the consideration of positive numbers α, since below we will prove that even for this case it is impossible. Suppose that for each $\alpha > 0$ there exists a control U for which both conditions (44) are satisfied. Then for an arbitrary nontrivial solution $x_U(t,t_0,x_0)$ of system (43) we have estimates

$$\begin{aligned}
\alpha = \lambda(A+BU) &= \limsup_{t\to\infty} \frac{1}{t} \ln |x_U(t,t_0,x_0)| = \\
&= \limsup_{t\to\infty} \frac{1}{t} \ln \left| \prod_{j=1}^{t-1} (1+U(j)) x_0 \right| \leq \\
&\leq \limsup_{t\to\infty} \frac{1}{t} \ln \prod_{j=1}^{t-1} (1+|U(j)|) |x_0| \leq \\
&\leq \limsup_{t\to\infty} \frac{1}{t} \ln \prod_{j=1}^{t-1} (1+\ell\alpha) |x_0| = \\
&= \limsup_{t\to\infty} \frac{1}{t} \ln (1+\ell\alpha)^{t-1} |x_0| = \ln(1+\ell\alpha).
\end{aligned}$$

Thus, there exists $\ell > 0$ such that for each $\alpha > 0$ the inequality $\alpha \leq \ln(1+\ell\alpha)$ holds, that is, $e^\alpha \leq 1+\ell\alpha$. But this is impossible, since the exponential function grows faster than any linear function. However, if we choose an arbitrary $\Delta > 0$, then there exists an $\ell = \ell(\Delta) > 0$ such that for each $\alpha \in \mathbb{R}$, $|\alpha| < \Delta$ there exists a control U for which the conditions (44) are satisfied. Here we can take $U(t) = e^\alpha - 1$. Then

$$\|U\|_\infty = |e^\alpha - 1| \leq e^{|\alpha|} - 1 \leq \ell|\alpha|,$$

where $\ell = \dfrac{e^\Delta - 1}{\Delta}$.

In order to present a deeper relation between global assignability and uniform complete controllability let us introduce the concept of Bebutov hull of a sequence. For any bounded sequence $F_0 = (F_0(t))_{t\in\mathbb{N}} \subset \mathbb{R}^{r\times q}$ and any $h \in \mathbb{N}$, let us consider a sequence $F_h = (F_h(t))_{t\in\mathbb{N}}$, where $F_h(t) = F_0(t+h)$ is a shift of $F_0(t)$ by h. Let us denote by $\mathfrak{R}(F_0)$ the closure in the topology of pointwise convergence on $\mathbb{N}$ of the set $\{F_h(\cdot)\colon h \in \mathbb{N}\}$. It is well known that $\mathfrak{R}(F_0)$ is metrizable by means of the metric

$$\rho\left(F, \widehat{F}\right) = \sup_{t\in\mathbb{N}} \min\left\{ \|F(t) - \widehat{F}(t)\|, t^{-1} \right\}.$$

The space $(\mathfrak{R}(F_0), \rho)$ is compact [60, p. 34] and it is called the Bebutov hull of the sequence F_0 (see [61, p. 32], [62]).

Let us identify system (24) with the sequence $(A,B) = \left(A(t),B(t)\right)_{t\in\mathbb{N}} \subset \mathbb{R}^{n\times(n+m)}$. The space $\mathfrak{R}(A,B)$ will be called the Bebutov hull of system (24).

Theorem 27. [56] System (24) is uniformly completely controllable if and only if for each system from $\mathfrak{R}(A,B)$ the corresponding closed-loop system has globally assignable Lyapunov spectrum.

For a given system (24), which is not uniformly completely controllable, the problem of finding a system from $\mathfrak{R}(A,B)$ such that corresponding closed-loop system does not have assignable Lyapunov spectrum is in general a difficult task. The proof of Theorem 27 does not give a recipe to find a "bad" system from the hull, but only establishes the fact of its existence.

Now we will present a result about local proportional assignability of the spectrum of system (29).

Theorem 28. Let system (24) be uniformly completely controllable and assume that at least one of the following conditions holds:
1) system (25) is regular;
2) system (25) is diagonalizable;
3) the Lyapunov spectrum of system (25) is stable;
4) system (25) is a system with integral separation.

Then the Lyapunov spectrum of system (29) is proportionally locally assignable.

Points 1–3 of this theorem were proved in [57] and point 4 in [59].

Let us consider one more concept of assignability, which is connected to the Lyapunov regularity coefficients.

Definition 39. [58] The Lyapunov spectrum and the Lyapunov irregularity coefficient of system (29) are called simultaneously proportionally locally assignable if there exist $\ell > 0$ and $\delta > 0$ such that for all

$$\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in O(\lambda(A))$$

and $\sigma \in [0,\delta)$ there exists an admissible feedback control U for system (24), such that estimate

$$\|U\|_\infty \leq \ell \max\{\sigma, |\lambda_1(A) - \mu_1|, \ldots, |\lambda_n(A) - \mu_n|\} \tag{45}$$

and equalities

$$\lambda(A+BU) = \mu, \quad \sigma_L(A+BU) = \sigma$$

are satisfied.

Theorem below contains a result concerning the defined above concept of assignability.

Theorem 29. [58] System (29) has the property of simultaneous proportional local controllability of the Lyapunov spectrum and the Lyapunov irregularity coefficient if system (24) is uniformly completely controllable and the free system (26) is regular.

4. Summary

In this paper we presented results concerning problems of location by feedback of Lyapunov invariants of linear discrete and continuous-time controlled systems. We concentrated on one of the most important numerical characteristics – Lyapunov exponents, which describe exponential stability (Theorem 1 and 22). These results were presented in the context of complete controllability and uniform complete controllability (Theorems 4, 6, 11, 12, 16, 26, 27 and 28). Moreover, for continuous-time system we are able to present results about shifting of the Bohl exponent with connection to the concept of uniform complete controllability (Theorem 7) and optimalizability (Theorem 8). Also for continuous-time systems, it can be shown that uniform complete controllability together with the property of boundness away implies nonmultiply proportional local assignability (Theorem 20). Theorems 11, 17, 18 and 29 give sufficient conditions for assignability of another numerical characteristics such as regularity coefficients and central exponents.

When we compare the results for continuous and discrete-time systems then we see that for the latter we don't have the relations between complete stabilizability, optimalizability and Riccati equation as well as we don't have the concept of bounded away systems and its relation with proportional local assignability. Another further direction of research in this area is to solve the problem of assignability of numerical characteristics of time-varying systems which correspond to imaginary parts of eigenvalues. The proposition of such characteristics is formulated in [63]. Finally, an effective methods of designing of the feedback that ensures desired location of the different numerical characteristics should be a subject of further research.

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