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The pointwise completeness and the pointwise degeneracy of fractional descriptor discrete-time linear systems

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Abstract. The Drazin inverse of matrices is applied to analysis of the pointwise completeness and of the pointwise degeneracy of the fractional descriptor linear discrete-time systems. Necessary and sufficient conditions for the pointwise completeness and the pointwise degeneracy of the fractional descriptor linear discrete-time systems are established. It is shown that every fractional descriptor linear discrete-time systems is not pointwise complete and it is pointwise degenerated in one step (for $i = 1$).

Key words: fractional, descriptor, pointwise completeness, pointwise degeneracy.

1. Introduction

A dynamical system described by homogenous equation is called pointwise complete if every final state of the system can be reached by suitable choice of its initial state. A system, which is not pointwise complete is called pointwise degenerated. The pointwise completeness and pointwise degeneracy of linear continuous-time systems with delays have been investigated in [2, 3, 9, 15, 17], the pointwise completeness of linear discrete-time cone systems with delays in [18] and of fractional linear systems are presented in [1, 9, 10]. The pointwise completeness and pointwise degeneracy of standard and positive hydrid systems described by the general model have been analyzed in [7] and of positive linear systems with state-feedbacks in [8]. Some new results in fractional systems have been given in [4, 13, 14 16].

The Drazin inverse of matrices has been applied to analysis of the pointwise completeness and of the pointwise degeneracy of the descriptor linear continuous-time and discrete-time systems in [5] and for fractional standard and descriptor linear continuous-time systems in [12].

In this paper the Drazin inverse of matrices will be applied to analysis of the pointwise completeness and of the pointwise degeneracy of the fractional descriptor discrete-time linear systems.

The paper is organized as follows. In Section 2 the basic definitions and theorems concerning the fractional descriptor linear discrete-time systems and the Drazin inverse of matrices are recalled. The pointwise completeness of the fractional descriptor linear discrete-time systems is investigated in Section 3 and the pointwise degeneracy in Section 4. Concluding remarks are given in section 5. The considerations are illustrated by numerical example of fractional linear discrete-time system.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, I_n – the $n\times n$ identity matrix. Im P is the image of the operator (matrix) P.

2. Fractional autonomous descriptor discrete-time linear systems and their solutions

Consider the fractional autonomous descriptor discrete-time linear system

$$E\Delta^\alpha x_{i+1} = Ax_i,\ i \in Z_+ = \{0, 1, \ldots\}, \tag{1}$$

where, $x_i \in \Re^n$ is the state vector $E, A \in \Re^{n\times n}$ and

$$\Delta^\alpha x_i = \sum_{j=0}^{i} (-1)^j \binom{\alpha}{j} x_{i-j} \tag{2a}$$

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0 \\ \dfrac{\alpha(\alpha-1)\ldots\alpha(\alpha-j+1)}{j!} & \text{for } j = 1, 2, \ldots \end{cases} \tag{2b}$$

is the fractional $\alpha \in \Re$ order difference of x_i.

Substituting (2) into (1) we obtain

$$Ex_{i+1} = A_\alpha x_i + \sum_{j=2}^{i+1} c_j E x_{i-j+1}, \tag{3a}$$

where

$$A_\alpha = A + E\alpha,\ \ c_j = (-1)^j \binom{\alpha}{j}. \tag{3b}$$

It is assumed that $\det E = 0$ and

$$\det[Ez - A_\alpha] \neq 0 \ \text{ for some } \ z \in C, \tag{4}$$

where C is the field of complex numbers.

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Assuming that for some chosen $c \in C$, $\det[Ec - A_\alpha]^{-1} \neq 0$ and premultiplying (3a) by $[Ec - A_\alpha]^{-1}$, we obtain

$$\overline{E}x_{i+1} = \overline{A}_\alpha x_i + \sum_{j=2}^{i+1} c_j \overline{E} x_{i-j+1}, \tag{5a}$$

where

$$\overline{E} = [Ec - A_\alpha]^{-1} E\,, \; \overline{A}_\alpha = [Ec - A_\alpha]^{-1} A_\alpha. \tag{5b}$$

Note that the equations (3a) and (5a) have the same solution x_i, $i \in Z_+$.

Definition 1. [6, 11] The smallest nonnegative integer q is called the index of the matrix $\overline{E} \in \Re^{n\times n}$ if

$$\operatorname{rank} \overline{E}^{\,q} = \operatorname{rank} \overline{E}^{\,q+1}. \tag{6}$$

Definition 2. [6, 11] A matrix $\overline{E}^D$ is called the Drazin inverse of $\overline{E} \in \Re^{n\times n}$ if it satisfies the conditions

$$\overline{E}\overline{E}^D = \overline{E}^D\overline{E}, \tag{7a}$$

$$\overline{E}^D\overline{E}\overline{E}^D = \overline{E}^D, \tag{7b}$$

$$\overline{E}^D\overline{E}^{q+1} = \overline{E}^q, \tag{7c}$$

where q is the index of $\overline{E}$ defined by (6).

The Drazin inverse $\overline{E}^D$ of a square matrix $\overline{E}$ always exists and is unique [5, 6]. If $\det\overline{E} \neq 0$ then $\overline{E}^D = \overline{E}^{-1}$. Some methods for computation of the Drazin inverse are given in [5, 9].

Theorem 1. The matrices $\overline{E}$ and $\overline{A}_\alpha$ defined by (5b) satisfy the following equalities

1. $\overline{A}_\alpha\overline{E} = \overline{E}\overline{A}_\alpha$ and $\overline{A}_\alpha^D\overline{E} = \overline{E}\overline{A}_\alpha^D$, $\overline{E}^D\overline{A}_\alpha = \overline{A}_\alpha\overline{E}^D$, $\overline{A}_\alpha^D\overline{E}^D = \overline{E}^D\overline{A}_\alpha^D$, (8a)

2. $\ker \overline{A}_\alpha \cap \ker \overline{E} = \{0\}$, (8b)

3. $\overline{E} = T\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix}T^{-1}$, $\overline{E}^D = T\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}T^{-1}$, (8c)

$\det T \neq 0$, $J \in \Re^{n_1\times n_1}$ is nonsingular, $N \in \Re^{n_2\times n_2}$ is nilpotent, $n_1 + n_2 = n$,
Proof is given in [6].

Theorem 2. Let

$$P = \overline{E}\overline{E}^D, \tag{9a}$$

$$Q = \overline{E}^D\overline{A}_\alpha. \tag{9b}$$

Then:

1) $P^k = P$ for $k = 2, 3, \ldots,$ (10)

2) $PQ = QP = Q,$ (11)

3) $P\overline{E}^D = \overline{E}^D.$ (12)

Proof is given in [6].

Theorem 3. The solution to the equation (5a) is given by

$$\begin{aligned} x_i = \big(&Q^i + c_2Q^{i-2} + c_3Q^{i-3} + \ldots + \\ &+ 2c_{i-1}Q + I_nc_i\big)Pw = T_ix_0, \;\; i \in Z_+, \end{aligned} \tag{13a}$$

$$\begin{aligned} T_i = &Q^i + c_2Q^{i-2} + c_3Q^{i-3} + \ldots + \\ &+ 2c_{i-1}Q + I_nc_i, \;\; x_0 \in \operatorname{Im} P, \end{aligned} \tag{13b}$$

where Q and P are defined by (9), coefficient c_j can be computed using (3b) and $w \in \Re^n$ is arbitrary.
Proof is given in [6].

Theorem 4. Let

$$\Phi_0(i) = Q^i + \sum_{k=2}^{i} c_k\overline{A}_\alpha P, \tag{14}$$

where Q and P are defined by (9).
Then

$$P\Phi_0(i) = \Phi_0(i). \tag{15}$$

Proof is given in [5].

3. Pointwise completeness of fractional descriptor discrete-time linear systems

In this section conditions for the pointwise completeness of fractional descriptor linear discrete-time linear systems will be established.

Definition 3. The fractional descriptor discrete-time linear system (1) is called pointwise complete for $i = q$ if for final state $x_f \in \Re^n$, there exists an initial condition $x_0 \in \operatorname{Im} P$ such that

$$x_f = x_q \in \operatorname{Im} P \tag{16}$$

where P is defined by (9a).

Theorem 5. The fractional descriptor discrete-time system (1) is pointwise complete for any $i = q$ and every $x_f \in \Re^n$ if and only if

$$\operatorname{rank} T_q = n, \tag{17}$$

where T_q is defined by (13b) for $i = q$.

Proof. Note that there exists the inverse matrix T_q^{-1} if and only if the condition (17) is satisfied. In this case from (13a) for $i = q$ we obtain

$$x_0 = T_q^{-1} x_f. \tag{18}$$

Therefore, for every x_f there exists $x_0 \in \operatorname{Im} P$ such that $x_q = x_f$ if and only if the condition (17) is satisfied. □

Theorem 6. Every fractional descriptor linear discrete-time system (1) is not pointwise complete for $q = 1$.

Proof. From the assumption $\det E = 0$ it follows that

$$\det \bar{E} = \det\left\{[Ec - A]^{-1} E\right\} = \det[Ec - A]^{-1} \det E = 0$$

and this implies $\det E^D = 0$. Using (9b) we obtain

$$\det Q = \det\left[\bar{E}^D \bar{A}_\alpha\right] = \det \bar{E}^D \det \bar{A}_\alpha = 0. \tag{19}$$

From (13b) for $i = 1$ we have $T_1 = Q$ and $\det T_1 = \det Q = 0$. Therefore, by Theorem 5 every fractional descriptor linear discrete-time system is not pointwise complete for $q = 1$. □

Example 1. Consider the fractional descriptor linear system (1) with $\alpha = 0.6$ and the matrices

$$E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1.6 & 0 & -1.6 \\ 0 & -1.6 & -1.6 \\ 1 & 1 & -1 \end{bmatrix}. \tag{20}$$

Note that the matrix A_α for A and E given by (20) has the form

$$A_\alpha = A + E\alpha = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \tag{21}$$

and it is nonsingular. Therefore, we choose in (4) $z = c = 0$ and we obtain

$$\det[Ec - A_\alpha] = \det[-A_\alpha] = 3 \tag{22}$$

and

$$\begin{aligned} \bar{E} &= [-A_\alpha]^{-1} E = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \\ \bar{A}_\alpha &= [-A_\alpha]^{-1} A_\alpha = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \end{aligned} \tag{23}$$

Note that $\det \bar{E} = 0$ and $\det \bar{A}_\alpha \neq 0$.

The Drazin inverse of the matrix $\bar{E}$ given by (23) has the form

$$\bar{E}^D = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \bar{E} \tag{24}$$

$$P = \bar{E}\bar{E}^D = \bar{E}^2 = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \tag{25}$$

$$Q = \bar{E}^D \bar{A}_\alpha = \frac{1}{3}\begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}. \tag{26}$$

Note that $\det P = \det \bar{E}^2 = 0$ and $\det Q = \det \bar{E}^D \det A_\alpha = 0$ since $\det \bar{E}^D = 0$.

Therefore, by Theorem 6 the fractional descriptor system with $\alpha = 0.6$ and the matrices (20) is not pointwise complete for $q = 1$, since $\det T_1 = \det Q = 0$.

Using (13) and (26) we obtain

$$\begin{aligned} T_2 = Q^2 + c_2 I_3 &= \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - 0.12 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} 0.5467 & -0.3333 & 0.3333 \\ -0.3333 & 0.5467 & 0.3333 \\ 0.3333 & 0.3333 & 0.5467 \end{bmatrix} \end{aligned} \tag{27}$$

and

$$\det T_2 = 0.0929. \tag{28}$$

Therefore, by Theorem 5 the descriptor system for $q = 2$ is pointwise complete.

$$\begin{aligned} T_3 = Q^3 + 2c_2 Q + c_3 I_3 &= \left(\frac{1}{3}\begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}\right)^3 - \\ &- \frac{0.24}{3}\begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} - 0.056 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} -0.5627 & 0.2533 & -0.2533 \\ 0.2533 & -0.5627 & -0.2533 \\ -0.2533 & -0.2533 & -0.5627 \end{bmatrix} \end{aligned} \tag{29}$$

and

$$\det T_3 = -0.0373. \tag{30}$$

Therefore, by Theorem 5 the descriptor system for $q = 3$ is also pointwise complete.

4. Pointwise degeneracy of fractional descriptor discrete-time linear systems

In this section conditions for the pointwise degeneracy of fractional descriptor linear discrete-time linear systems will be established.

Definition 4. The fractional descriptor discrete-time linear system (1) is called pointwise degenerated in the direction $v \in \Re^n$ for $i = q$ if there exists a nonzero vector v such that for all initial conditions $x_0 \in \operatorname{Im} P$ the solution x_i of the system (1) for $i = q$ satisfies the condition

$$v^T x_q = 0. \tag{31}$$

Theorem 7. The fractional descriptor discrete-time linear system (1) is pointwise degenerated in the direction $v \in \Re^n$ for $i = q$ if and only if

$$\det T_q = 0 \tag{32}$$

where T_q is defined by (13b) for $i = q$.

Proof. From (31) and (13a) for $i = q$ we have

$$v^T T_q x_0 = 0. \tag{33}$$

Note that there exists nonzero vector $v \in \Re^n$ such that the condition (33) is satisfied for all $x_0 \in \operatorname{Im} P$ if and only if the matrix T_q is singular. Therefore, the fractional descriptor system (1) is pointwise degenerated in the direction $v \in \Re^n$ for $i = q$ if and only if the condition (32) is satisfied. □

Theorem 8. Every fractional descriptor linear discrete-time system (1) is pointwise degenerated for $q = 1$.

Proof. From the assumption $\det E = 0$ if follows that $\det \bar{E} = 0$ and $\det \bar{E}^D = 0$ this implies that $\det Q = 0$ (see (19)). Taking into account that $T_1 = Q$ we obtain $\det T_1 = 0$. Therefore, by Theorem 7 every fractional descriptor discrete-time system (1) is pointwise degenerated for $q = 1$. □

Example 2. (Continuation of Example 1). Consider the fractional descriptor system (1) with $\alpha = 0.6$ and the matrices E, A given by (20).

In this case the matrix Q is given by (26) and

$$\det T_1 = \det Q = \begin{vmatrix} -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \end{vmatrix} = 0. \tag{34}$$

Therefore, by Theorem 8 the fractional descriptor system for $q = 1$ is pointwise degenerated.

The vector $v \in \Re^3$ in which the system is pointwise degenerated can be computed from the equation

$$Q^T v = \frac{1}{3}\begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{35}$$

and its solution is $v^T = [v_1, v_2, v_3]^T = [-a, -a, a]$ for any number a.

5. Concluding remarks

The Drazin inverse of matrices has been applied to analysis of the pointwise completeness and of the pointwise degeneracy of the fractional descriptor linear discrete-time systems. Necessary and sufficient conditions for the pointwise completeness and the pointwise degeneracy of the fractional descriptor linear discrete-time systems have been established (Theorems 5 and 7). It is shown that every fractional descriptor linear discrete-time systems is not pointwise complete and it is pointwise degenerated for $i = 1$ (Theorems 6 and 8).

The considerations have been illustrated by numerical example of the fractional descriptor linear discrete-time system. The considerations can be extended to the fractional different orders linear continuous-time and discrete-time linear systems.

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