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ARCHIVE OF MECHANICAL ENGINEERING

Volume 67 2020 Number 4

DOI: 10.24425/ame.2020.131700

**Key words:** free vibration, variable elastic foundation, axially functionally graded beam, Euler-Bernoulli beam, Timoshenko beam

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# Dynamic behaviour of axially functionally graded beam resting on variable elastic foundation

In this paper, a comprehensive study is carried out on the dynamic behaviour of Euler–Bernoulli and Timoshenko beams resting on Winkler type variable elastic foundation. The material properties of the beam and the stiffness of the foundation are considered to be varying along the length direction. The free vibration problem is formulated using Rayleigh-Ritz method and Hamilton's principle is applied to generate the governing equations. The results are presented as non-dimensional natural frequencies for different material gradation models and different foundation stiffness variation models. Two distinct boundary conditions viz., clamped-clamped and simply supported-simply supported are considered in the analysis. The results are validated with existing literature and excellent agreement is observed between the results.

## 1. Introduction

In this era, advanced materials are continuously being researched and developed to improve the strength to weight ratio of structures. One such material is the functionally graded material (FGM) in which the material properties vary continuously along spatial directions. The advantage of FGMs over traditional composites is that they retain most of the properties of their constituent materials because of the continuous transition of materials. With these characteristics, FGMs naturally attracted the attention of various structural engineers and researchers. As beams are the most basic components of any engineering structure, many research works are focussed on analysis of FGM beams. Neuringer and Elishakoff [1] derived the first closed form solutions for natural frequency of FGM beams and obtained

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some unexpected results which were extensively investigated in their later work[2]. A new, fast and accurate approach was presented in [3] for determining the natural frequencies of FGM beams in free vibration. In this approach, the governing equation was transformed to Fredholm integral equations and its nontrivial solution gave the natural frequencies. Dynamic analysis on FGM Euler–Bernoulli beams subjected to moving harmonic load was carried out by Simsek et al. [4] where they used Lagrange's equations to derive the equation of motion. Modified couple stress theory along with Rayleigh–Ritz method were used by Akgoz and Civalek [5] to determine an approximate solution for natural frequencies of FGM micro beams. Sarkar and Ganguli [6] obtained closed form solutions for Timoshenko FGM beams in which axial variation of material properties with different polynomial functions was considered. Rezaiee-Pajand and Hozhabrossadati [7] provided analytical solutions for FGM double beam systems connected by mass-spring system. Javid and Hemmatnezhad [8] used finite element method to perform nonlinear free vibration analysis on FGM beams and direct numerical integration technique was used to determine the natural frequencies. Chen et al. [9] investigated the thermal buckling behaviour of such beams using transformed-section method.

The foundations of complex structures also play a vital role in engineering. Over the years, various models have been presented by researchers for modelling the elastic foundations. One such model is given by Winkler, which is used in this study. It is the most basic model where it is assumed that numerous independent and linear elastic springs make up the foundation, such that the vertical displacement becomes proportional to the contact pressure at an arbitrary point [10]. The Winkler models and its later variations have been extensively studied by various researchers. Ying et al. [11] presented exact solutions for bending and free vibration of FGM beams using 2D theory of elasticity and state space method. The research work by Yan et al. [12] was focussed on dynamic response of cracked FGM beams subjected to a constantly moving load. Fallah and Aghdam [13, 14] studied the nonlinear free vibration and thermo-mechanical buckling of FGM beams subjected to axial load and obtained the closed form solutions using He's variational method. Yaghoobi and Torabi [15] investigated the imperfect FGM beams for post-buckling and nonlinear vibration behaviour using Galerkin method. Kanani et al. [16] carried out nonlinear free and forced vibration analysis on FGM beams using Galerkin method and derived the approximate solutions in closed form with the help of a variational iteration method. Wattanasakulpong and Mao [17] used Chebyshev collocation method to carry out free vibration analysis on FGM beams subjected to different boundary conditions. Calim [18] investigated the dynamic response of FGM beams supported on viscoelastic foundation using complimentary functions method. Deng et al. [19] performed vibration and buckling analysis on FGM double-beam system where Wittrick-William algorithm was utilised to compute the natural frequency and buckling load. Lohar et al. [20] conducted nonlinear forced vibration analysis on axially functionally graded beams using a semi-analytical approach based on Rayleigh-Ritz method. Karami and Janghorban [21] investigated the size dependent behaviour of FGM nanobeams where the governing equations were obtained by Hamilton's principle and solutions were obtained using Navier series technique. FGM Timoshenko beam subjected to moving mass was investigated by Esen [22] and effects of material constituents, foundation parameters and inertia of the moving mass on the response of beam were studied. Chaabane et al. [23] conducted static and dynamic analysis using hyperbolic shear deformation theory on FGM beams with different material variation models.

Apart from these, a few research works on beams resting on variable elastic foundation have also been conducted by various researchers, but most of these are on homogeneous beams. Eisenberger and Clastornik [24] presented the first work on free vibration of beams resting on variable elastic foundation where they proposed two methods for obtaining the natural frequency of the beam. Kacar et al. [25] conducted free vibration analysis on beams resting on elastic foundation with varying stiffness using differential transform method. Mirzabeigy and Madoliat [26] investigated nonlinear free vibration behaviour of beams subjected to axial loads using energy method along with Hamilton's principle and the solutions were obtained using homotopy perturbation method. Zhang et al. [27] carried out buckling and free vibration analysis on non-uniform beams using Hencky bar-chain model. A few works on FGM beams resting on variable elastic foundation are also available in the literature. Yas et al. [28] performed free vibration analysis on Euler-Bernoulli FGM beams using generalized differential quadrature method. The work by Jena et al. [29] is concerned with dynamic behaviour of FGM nanobeams. Euler-Bernoulli beam theory was used to model the nanobeam and differential quadrature method was used for the analysis.

The literature review presented above suggests that, despite the presence of numerous works on beams resting on elastic foundation, research works on FGM beams are extremely rare. Therefore, it is essential to thoroughly investigate the vibration characteristics of the FGM beams with axial material gradation resting on variable elastic foundations, which is the objective of this paper.

## 2. Mathematical formulation

The present paper discusses the dynamic behaviour of FGM beams resting on variable elastic foundation. The free vibration problem is formulated based on Euler-Bernoulli beam theory and Timoshenko beam theory separately. The methodology used for the formulation for both the beam theories is based on Rayleigh-Ritz approach.

An axially functionally graded beam of length $L$ and cross-sectional area $A$ ($= b \times h$), moment of inertia $I$ ($= (b \times h^3)/12$) is shown in Fig. 1.

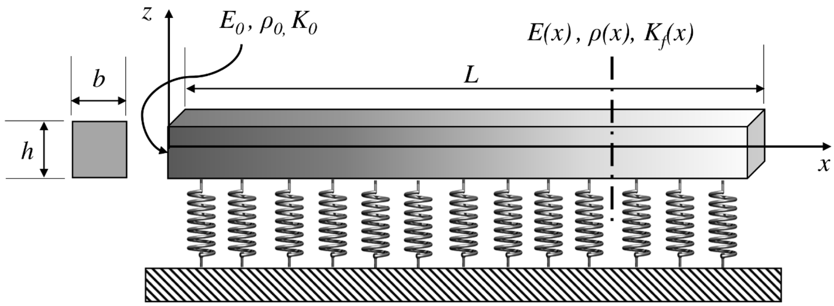


Fig. 1. FGM beam resting on Winkler foundation

The material of the beam is considered to be varying along the length direction following three different material models:

$$\begin{aligned}
&\textit{Material 1}: \quad E(x) = E_0, \quad \rho(x) = \rho_0, \\
&\textit{Material 2}: \quad E(x) = E_0\left(1+\frac{x}{L}\right), \quad \rho(x) = \rho_0\left(1+\frac{x}{L}+\left(\frac{x}{L}\right)^2\right), \\
&\textit{Material 3}: \quad E(x) = E_0 e^{(x/L)}, \quad \rho(x) = \rho_0 e^{(x/L)}.
\end{aligned} \tag{1}$$

Here, $E_0$ and $\rho_0$ are the Young's modulus and material density, respectively, at one end of the beam. Poisson's ratio $\mu$ remains constant throughout the beam and the modulus of rigidity is given by $G(x) = E(x)/2(1+\mu)$.

The stiffness of elastic foundation is also considered to be varying axially following three different functions:

$$\begin{aligned}
&\textit{Constant}: \quad K_f(x) = K_0, \\
&\textit{Linear}: \quad K_f(x) = K_0\left(1+\lambda\frac{x}{L}\right), \\
&\textit{Parabolic}: \quad K_f(x) = K_0\left(1+\lambda\left(\frac{x}{L}\right)^2\right).
\end{aligned} \tag{2}$$

Here, $K_0$ is the foundation stiffness at one end of the beam and $\lambda$ is stiffness variation coefficient.

### 2.1. Euler-Bernoulli beam theory (EBBT)

The assumptions of the Euler-Bernoulli beam theory that the beam is sufficiently thin, the effects of shear deformation and rotary inertia are negligible, and the plane sections remain plane after deformation are followed. The displacement at a general point in the beam can be expressed in terms of mid-plane deformations as,

$$\begin{aligned}
\bar{u}(x,z) &= u(x) - z\frac{\mathrm{d}v(x)}{\mathrm{d}x}, \\
\bar{v}(x,z) &= v(x),
\end{aligned} \tag{3}$$

where, $\bar{u}$ and $\bar{v}$ are displacement fields at a general point and $u$ and $v$ are displacement components at the neutral axis ($z = 0$) of the beam along the axial and transverse directions, respectively. The strain-displacement relation can now be expressed as,

$$\varepsilon_{xx} = \frac{\mathrm{d}u}{\mathrm{d}x} - z\left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right). \tag{4}$$

The total strain energy can be obtained by summation of the strain energy stored in the beam and the strain energy of the foundation,

$$U = \frac{1}{2}\int\limits_{vol} \sigma_{xx}\varepsilon_{xx}\,\mathrm{d}V + \frac{1}{2}\int\limits_0^L K_f(x)v^2\,\mathrm{d}x. \tag{5}$$

Substituting the expression of strain and $\sigma_{xx} = E(x)\varepsilon_{xx}$ in the above equation yields,

$$U = \frac{1}{2}A\int\limits_0^L \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 E(x)\,\mathrm{d}x + \frac{1}{2}I\int\limits_0^L \left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right)^2 E(x)\,\mathrm{d}x + \frac{1}{2}\int\limits_0^L K_f(x)v^2\,\mathrm{d}x. \tag{6}$$

The kinetic energy of the dynamic system is expressed as,

$$T = \frac{1}{2}A\int\limits_0^L \left(\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2\right)\rho(x)\,\mathrm{d}x. \tag{7}$$

### 2.2. Timoshenko beam theory (TBT)

The effects of shear deformation and rotary inertia are considered in Timoshenko beam theory. Therefore, the displacements at a general point in the beam are given as,

$$\begin{aligned} \bar{u}(x,z) &= u(x) - z\theta, \\ \bar{v}(x,z) &= v(x), \end{aligned} \tag{8}$$

where $\theta$ is the rotary displacement. The strain displacement relation can now be expressed as

$$\varepsilon_{xx} = \frac{\mathrm{d}u}{\mathrm{d}x} - z\frac{\mathrm{d}\theta}{\mathrm{d}x}, \qquad \gamma_{xz} = \frac{\mathrm{d}v}{\mathrm{d}x} - \theta. \tag{9}$$

Here, $\varepsilon_{xx}$ is the axial strain and $\gamma_{xz}$ is the shear strain.

The total strain energy will be the summation of strain energy stored in the beam due to axial deformation and shear, and the strain energy of the foundation

$$U = \frac{1}{2}\int\limits_{vol} \sigma_{xx}\varepsilon_{xx}\,\mathrm{d}V + \frac{1}{2}\int\limits_{vol} \tau_{xz}\gamma_{xz}\,\mathrm{d}V + \frac{1}{2}\int\limits_0^L K_f(x)v^2\,\mathrm{d}x. \tag{10}$$

Substituting the expression of strain from Eq. (9) along with the expression of stress $\sigma_{xx} = E(x)\varepsilon_{xx}$, $\tau_{xz} = G(x)\gamma_{xz}$ in the above equation yields

$$\begin{aligned} U &= \frac{1}{2}A\int_0^L \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 E(x)\,\mathrm{d}x + \frac{1}{2}I\int_0^L \left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 E(x)\,\mathrm{d}x \\ &+ \frac{k_{sh}}{2}A\int_0^L \left[\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 + \theta^2 - 2\theta\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\right] G(x)\,\mathrm{d}x + \frac{1}{2}\int_0^L K_f(x)v^2\,\mathrm{d}x, \end{aligned} \tag{11}$$

where $k_{sh}$ is the shear correction factor. The expression of the kinetic energy of the beam is written as

$$T = \frac{1}{2}A\int_0^L \left(\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 + z^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right)\rho(x)\,\mathrm{d}x. \tag{12}$$

### 2.3. Dynamic analysis

Normalised coordinate system ($\xi = x/L$) is followed and a number of reference points known as the Gauss points are generated throughout the domain to carry out the computations. As per the Rayleigh–Ritz method, the displacement fields $u$, $v$ and $\theta$ for the beam are represented as linear combinations of unknown parameters ($d_i$) and orthogonal admissible functions ($\alpha_i$), ($\beta_i$) and ($\phi_i$) as follows

$$\begin{aligned} u(\xi) &= \sum_{i=1}^{nu} d_i\alpha_i(\xi)\, e^{j\omega t}, \\ v(\xi) &= \sum_{i=nu+1}^{nu+nv} d_i\beta_i(\xi)\, e^{j\omega t}, \\ \theta(\xi) &= \sum_{i=nu+nv+1}^{nu+nv+ns} d_i\phi_i(\xi)\, e^{j\omega t}, \end{aligned} \tag{13}$$

It is to be noted that, in the case of Euler-Bernoulli beam, only the first two expressions of the above equation are utilised. In the above expression, $\omega$ denotes the natural frequency of the beam and, $nu$, $nv$ and $ns$ are number of orthogonal functions for $u$, $v$ and $\theta$, respectively. These functions not only satisfy the boundary conditions of the beam but are also continuous and differentiable within the domain. The first set of the series of orthogonal function is known as start functions and it is generated by satisfying the flexural, membrane and rotational boundary conditions of the beam. The start functions for the two boundary conditions considered in this study are furnished in Table 1.

Table 1.

Start functions for different boundary conditions

| Displacement field | Boundary | End conditions | Function |
|---|---|---|---|
| $u$ | CC | $u = 0$ at $x = 0,\ L$ | $\alpha_1(\xi) = \xi(1-\xi)$ |
| | SS | | |
| $v$ | CC | $v = 0$ at $x = 0,\ L$ | $\beta_1(\xi) = \{\xi(1-\xi)\}^2$ |
| | SS | | $\beta_1(\xi) = \sin(\pi\xi)$ |
| $\theta$ | CC | $\theta = 0$ at $x = 0,\ L$ | $\phi_1(\xi) = \sin(\pi\xi)$ |
| | SS | $\theta \neq 0$ at $x = 0,\ L$ | $\phi_1(\xi) = \cos(\pi\xi)$ |

These start functions are used to generate appropriate sets of higher order functions using Gram-Schmidt orthogonalisation procedure, which is discussed in detail in [30].

The set of governing differential equations of the dynamic problem are derived using Hamilton's principle, which has the mathematical expression

$$\delta\left(\int\limits_{t_1}^{t_2}(T-U)\,\mathrm{d}t\right) = 0. \tag{14}$$

Putting the expressions of U and T along with displacement fields from Eq. (13) in above equation reduces the governing set of equations in the following form

$$\Big[[K] - \omega^2[M]\Big]\{d\} = 0, \tag{15}$$

where $[K]$ is the stiffness matrix and $[M]$ is the mass matrix of the beam. The elements of these matrices for both beam models are given in the Appendix. The above equation is a standard eigenvalue problem, the solution of which gives the natural frequency and mode shape of the beam.

## 3. Results and discussion

Numerical results for free vibration of FGM beam resting on variable elastic foundation are generated based on the methodology discussed above. Two different boundary conditions are considered, namely, clamped-clamped (CC) and simply supported-simply supported (SS). The number of Gauss points is taken as 24 and the number of higher order functions for each displacement parameter is taken as 8. The results are presented in the form of non-dimensional parameters which are given as follows

$$\bar{\omega} = \omega L^2\sqrt{\frac{\rho_0 A}{E_0 I}}, \qquad \bar{K}_f = \frac{K_0 L^4}{E_0 I}, \qquad \text{and} \qquad a_r = \frac{h}{L}.$$

First of all, it is essential to validate the feasibility of the methodology presented in this paper. In order to do that, a comparative study is carried out where the results obtained through current method are compared with the results already available in literature. The non-dimensional frequency parameter of homogeneous beam supported on elastic foundation for clamped-clamped and simply supported-simply supported boundary conditions are validated with the results provided by Chen et al. [10] and Kacar et al. [25] and furnished in Table 2. It can be seen that the present results show a decent agreement with the published results, which validates the accuracy of the methodology presented in this paper.

Table 2.
Comparison of fundamental frequency parameters of homogeneous beam (*Material 1*)

| Boundary condition | $\bar{K}_f$ | $\sqrt{\bar{\omega}}$ | | |
|---|---|---|---|---|
| | | EBBT | TBT | |
| | | Constant elastic foundation | | Chen et al. [10] |
| CC | 0 | 4.7212 | 4.7217 | 4.7314 |
| | $10^2$ | 4.9427 | 4.9433 | 4.9515 |
| | $10^4$ | 10.1220 | 10.1220 | 10.1227 |
| SS | 0 | 3.1416 | 3.1415 | 3.1414 |
| | $10^2$ | 3.7484 | 3.7483 | 3.7482 |
| | $10^4$ | 10.0243 | 10.0242 | 10.0240 |
| | $10^6$ | 31.6235 | 31.6234 | 31.6217 |
| | | Linear elastic foundation ($\lambda = -0.2$) | | Kacar et al. [25] |
| CC | 10 | 4.7424 | 4.7431 | 4.7511 |
| | $10^2$ | 4.9219 | 4.9225 | 4.9296 |
| | $10^3$ | 6.1132 | 6.1135 | 6.1172 |
| SS | 10 | 3.2118 | 3.2117 | 3.2117 |
| | $10^2$ | 3.6999 | 3.6999 | 3.6999 |
| | $10^3$ | 5.6185 | 5.6195 | 5.6185 |
| | | Parabolic elastic foundation ($\lambda = -0.2$) | | Kacar et al. [25] |
| CC | 10 | 4.7435 | 4.7442 | 4.7522 |
| | $10^2$ | 4.9314 | 4.9320 | 4.9391 |
| | $10^3$ | 6.1625 | 6.1628 | 6.1664 |
| SS | 10 | 3.2150 | 3.2150 | 3.2150 |
| | $10^2$ | 3.7212 | 3.7212 | 3.7211 |
| | $10^3$ | 5.6788 | 5.6797 | 5.6787 |

The non-dimensional frequency parameters for the first four modes of vibration are presented in Table 3 and Table 4 for linearly FGM beam (*Material 2*) and exponentially FGM beam (*Material 3*), respectively. The results are presented for Euler–Bernoulli beam and Timoshenko beam for different variable foundations (constant, linear and parabolic), foundation stiffness ($\bar{K}_f$) and boundary conditions. The value of foundation stiffness variation coefficient ($\lambda$) is kept constant at 0.5 for the generation of results. The results indicate that the natural frequency increases with the increase in foundation stiffness for all cases. It is also observed that the increase in natural frequency is lowest for constant stiffness and highest for linear stiffness. To clearly observe the behaviour of FGM beam with foundation stiffness, some graphical results are also presented in Fig. 2. These graphs show the relation-

Table 3.
First four natural frequency parameters of linearly functionally graded beams (*Material 2*)

| Boundary condition | $\bar{K}_f$ | $\sqrt{\bar{\omega}_1}$ | | $\sqrt{\bar{\omega}_2}$ | | $\sqrt{\bar{\omega}_3}$ | | $\sqrt{\bar{\omega}_4}$ | |
|---|---|---|---|---|---|---|---|---|---|
| | | EBBT | TBT | EBBT | TBT | EBBT | TBT | EBBT | TBT |
| | | Constant elastic foundation | | | | | | | |
| CC | 0 | 4.5161 | 4.5166 | 7.5043 | 7.5061 | 10.5077 | 10.5095 | 13.4877 | 13.5174 |
| | $10^2$ | 4.6638 | 4.6643 | 7.5387 | 7.5405 | 10.5205 | 10.5223 | 13.4938 | 13.5234 |
| | $10^4$ | 8.7425 | 8.7426 | 9.7876 | 9.7882 | 11.6125 | 11.6139 | 14.0598 | 14.0868 |
| SS | 0 | 3.0047 | 3.0075 | 6.0304 | 6.0348 | 9.0369 | 9.0398 | 12.0359 | 12.0382 |
| | $10^2$ | 3.4270 | 3.4274 | 6.0973 | 6.1013 | 9.0571 | 9.0621 | 12.0445 | 12.0483 |
| | $10^4$ | 8.4444 | 8.6020 | 9.3439 | 9.6632 | 10.6449 | 10.9911 | 12.8248 | 13.1383 |
| | | Linear elastic foundation ($\lambda = 0.5$) | | | | | | | |
| CC | 0 | 4.5161 | 4.5166 | 7.5043 | 7.5061 | 10.5077 | 10.5095 | 13.4877 | 13.5174 |
| | $10^2$ | 4.6979 | 4.6983 | 7.5464 | 7.5482 | 10.5232 | 10.5250 | 13.4951 | 13.5247 |
| | $10^4$ | 9.2533 | 9.2534 | 10.1136 | 10.1141 | 11.8092 | 11.8103 | 14.1720 | 14.1977 |
| SS | 0 | 3.0047 | 3.0075 | 6.0304 | 6.0348 | 9.0369 | 9.0398 | 12.0359 | 12.0382 |
| | $10^2$ | 3.5107 | 3.5113 | 6.1114 | 6.1148 | 9.0613 | 9.0656 | 12.0463 | 12.0493 |
| | $10^4$ | 9.0305 | 9.1291 | 9.6991 | 9.8122 | 10.8831 | 10.9273 | 12.9674 | 13.2400 |
| | | Parabolic elastic foundation ($\lambda = 0.5$) | | | | | | | |
| CC | 0 | 4.5161 | 4.5166 | 7.5043 | 7.5061 | 10.5077 | 10.5095 | 13.4877 | 13.5174 |
| | $10^2$ | 4.6821 | 4.6826 | 7.5431 | 7.5448 | 10.5221 | 10.5239 | 13.4945 | 13.5242 |
| | $10^4$ | 9.0583 | 9.0584 | 9.9625 | 9.9630 | 11.7246 | 11.7259 | 14.1253 | 14.1520 |
| SS | 0 | 3.0047 | 3.0075 | 6.0304 | 6.0348 | 9.0369 | 9.0398 | 12.0359 | 12.0382 |
| | $10^2$ | 3.4748 | 3.4755 | 6.1055 | 6.1089 | 9.0596 | 9.0663 | 12.0456 | 12.0498 |
| | $10^4$ | 8.8584 | 8.9376 | 9.5218 | 9.6846 | 10.7787 | 10.8922 | 12.9089 | 13.1302 |

Table 4.
First four natural frequency parameters of exponentially functionally graded beams (*Material 3*)

| Boundary condition | $\bar{K}_f$ | $\sqrt{\bar{\omega}_1}$ | | $\sqrt{\bar{\omega}_2}$ | | $\sqrt{\bar{\omega}_3}$ | | $\sqrt{\bar{\omega}_4}$ | |
|---|---|---|---|---|---|---|---|---|---|
| | | EBBT | TBT | EBBT | TBT | EBBT | TBT | EBBT | TBT |
| | | Constant elastic foundation | | | | | | | |
| CC | 0 | 4.7358 | 4.7363 | 7.8489 | 7.8507 | 10.9788 | 10.9806 | 14.0860 | 14.1251 |
| | $10^2$ | 4.8739 | 4.8744 | 7.8809 | 7.8827 | 10.9906 | 10.9925 | 14.0916 | 14.1307 |
| | $10^4$ | 8.9461 | 8.9463 | 10.0353 | 10.0359 | 12.0166 | 12.0186 | 14.6186 | 14.6532 |
| SS | 0 | 3.1262 | 3.1372 | 6.2900 | 6.3806 | 9.4287 | 9.4330 | 12.5621 | 12.6835 |
| | $10^2$ | 3.5334 | 3.5381 | 6.3525 | 6.6344 | 9.4475 | 9.7124 | 12.5701 | 12.7790 |
| | $10^4$ | 8.6457 | 8.7687 | 9.5152 | 9.6293 | 10.9478 | 11.1336 | 13.2966 | 13.4354 |
| | | Linear elastic foundation ($\lambda$ = 0.5) | | | | | | | |
| CC | 0 | 4.7358 | 4.7363 | 7.8489 | 7.8507 | 10.9788 | 10.9806 | 14.0860 | 14.1251 |
| | $10^2$ | 4.9053 | 4.9058 | 7.8880 | 7.8898 | 10.9932 | 10.9950 | 14.0928 | 14.1319 |
| | $10^4$ | 9.4482 | 9.4482 | 10.3531 | 10.3537 | 12.2048 | 12.2063 | 4.7247 | 14.7581 |
| SS | 0 | 3.1262 | 3.1372 | 6.2900 | 6.3806 | 9.4287 | 9.4330 | 12.5621 | 12.6835 |
| | $10^2$ | 3.6142 | 3.6193 | 6.3656 | 6.4002 | 9.4515 | 9.4934 | 12.5718 | 12.7905 |
| | $10^4$ | 9.2305 | 9.2970 | 9.8601 | 9.9928 | 11.1814 | 11.2768 | 13.4334 | 13.4997 |
| | | Parabolic elastic foundation ($\lambda$ = 0.5) | | | | | | | |
| CC | 0 | 4.7358 | 4.7363 | 7.8489 | 7.8507 | 10.9788 | 10.9806 | 14.0860 | 14.1251 |
| | $10^2$ | 4.8905 | 4.8909 | 7.8849 | 7.8867 | 10.9921 | 10.9939 | 14.0923 | 14.1314 |
| | $10^4$ | 9.2452 | 9.2452 | 10.2071 | 10.2077 | 12.1245 | 12.1262 | 14.6809 | 14.7155 |
| SS | 0 | 3.1262 | 3.1372 | 6.2900 | 6.3806 | 9.4287 | 9.4330 | 12.5621 | 12.6835 |
| | $10^2$ | 3.5790 | 3.5845 | 6.3601 | 6.3905 | 9.4499 | 9.4887 | 12.5711 | 12.7461 |
| | $10^4$ | 9.0468 | 9.0969 | 9.6879 | 9.9263 | 11.0813 | 11.2621 | 13.3783 | 13.4550 |

ship between the first natural frequency of vibration and foundation stiffness. Three material models, three variable stiffness models and two boundary conditions are considered for generating the graphs. The value of foundation stiffness variation coefficient ($\lambda$) is kept constant at 0.5. The general trend is clearly evident from these figures, which is the increase in natural frequency with increasing foundation stiffness, but the nature of the relationship is nonlinear. Also, the difference in natural frequency of different foundation models is clearly evident as the foundation stiffness increases. Fig. 3 presents similar graphs, where the vibration characteristics of the three material models are compared for linear foundation model. It is seen from these figures that the natural frequency of *Material 1* (Homogeneous beam) is greater than those of *Material 2* (Linear FGM beam) and *Material 3*

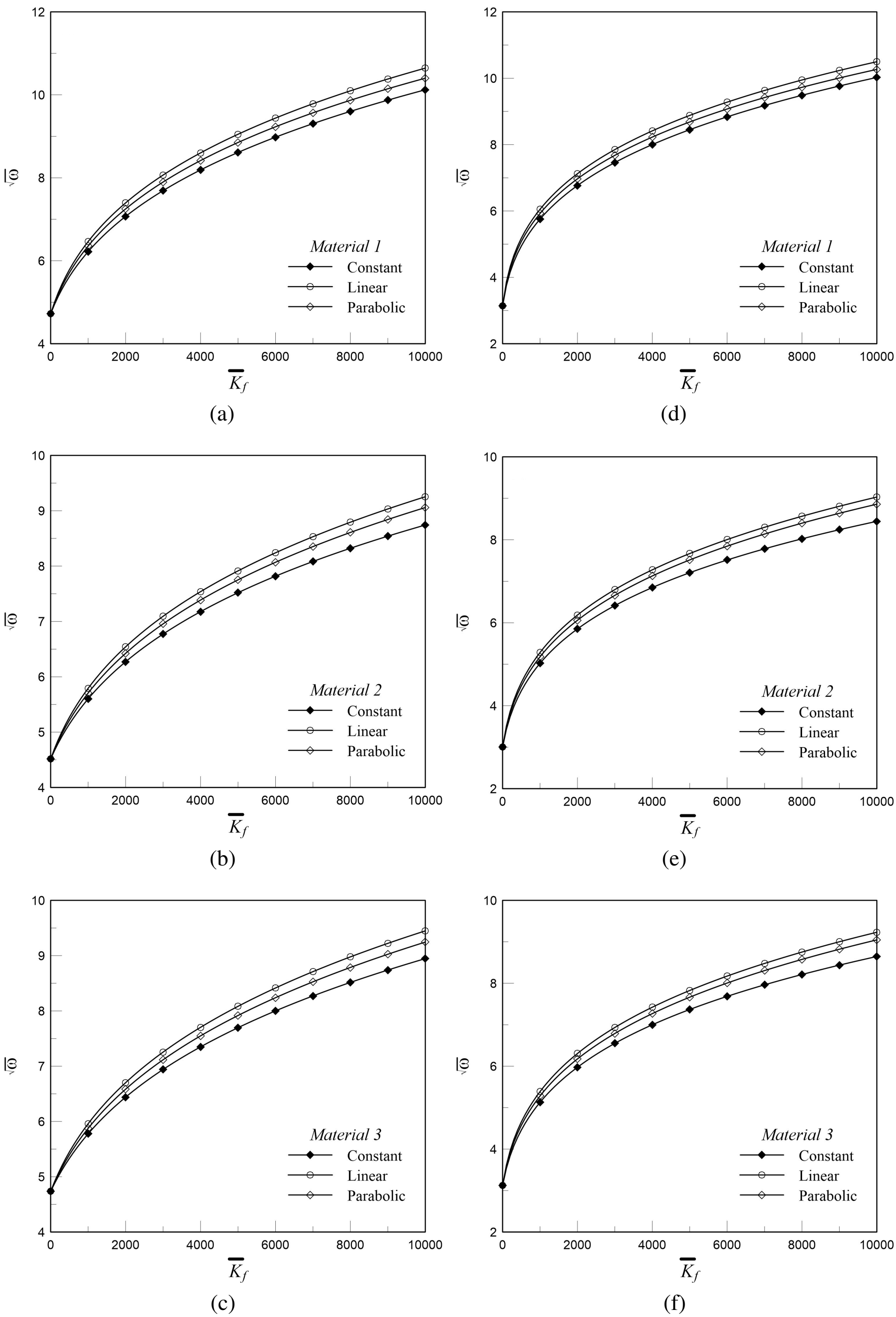


Fig. 2. Effect of foundation stiffness on natural frequency of FGM beam (a, b, c) clamped-clamped, (d, e, f) simply supported-simply supported

(Exponential FGM beam) and the difference gets larger as the foundation stiffness increases.

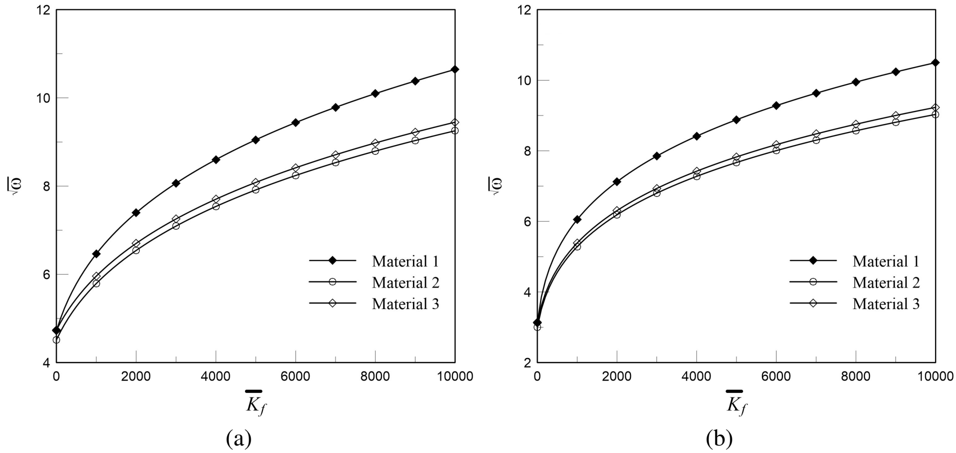


Fig. 3. Effect of different material models on natural frequency of FGM beam (a) clamped-clamped, (b) simply supported-simply supported

Fig. 4 shows a comparative study of Linear FGM beam (*Material 2*) with different end supports for the linear foundation model. It can be observed that the natural frequency of CC beam is greater than that of SS beam for all values of foundation stiffness, but the difference becomes smaller as the value of stiffness increases. This indicates that the effect of boundary condition on the dynamic behaviour of FGM beams reduces as the foundation stiffness increases.

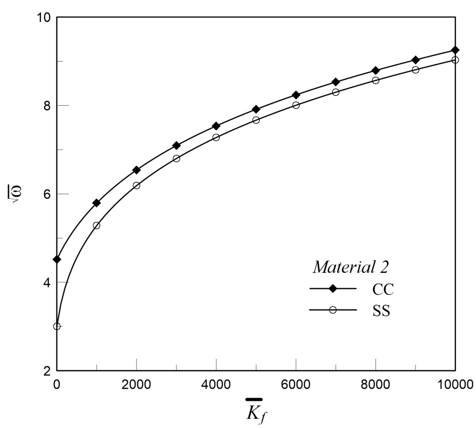


Fig. 4. Effect of boundary condition on natural frequency of FGM beam

The effect of foundation stiffness variation parameter ($\lambda$) on the dynamic behaviour of FGM beam is depicted in Fig. 5. The results are presented for linear and parabolic foundation and all three material models and two boundary conditions

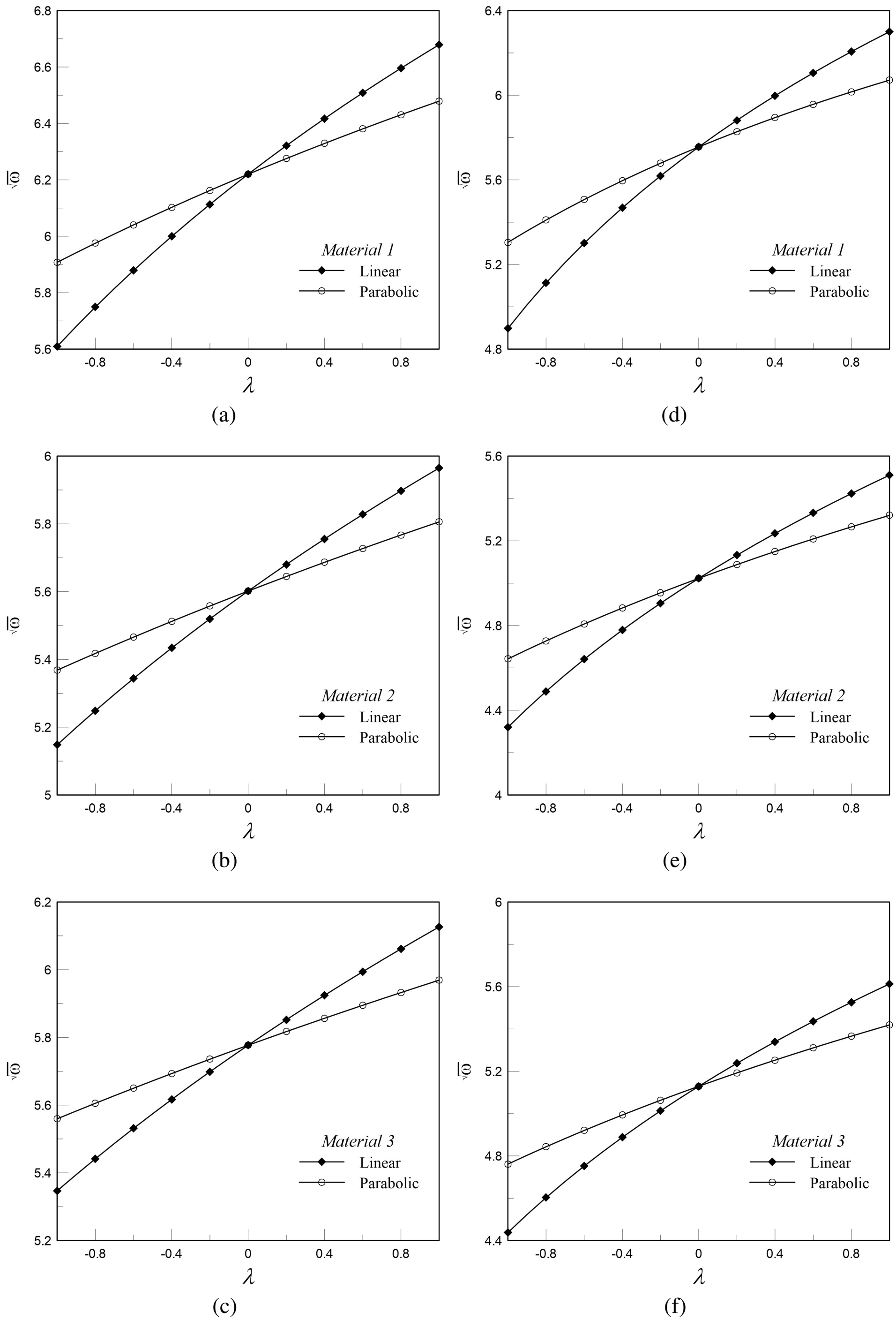


Fig. 5. Effect of foundation stiffness parameter ($\lambda$) on natural frequency of FGM beam (a, b, c) clamped-clamped, (d, e, f) simply supported-simply supported

are considered. The value of foundation stiffness $\bar{K}_f$ is kept constant at 1000 and the values of $\lambda$ are varied from $-1$ to $+1$. It is seen form the figure that, when the value of $\lambda$ is on the negative side, the natural frequency for linear foundation is lower than that of parabolic foundation, but the trend reverses as the value of $\lambda$ reaches the positive side. Also, the difference in the natural frequency of both the foundations becomes more evident as the value of $\lambda$ increases on positive or negative side, and at the maximum value of $\lambda = 1$ and $-1$, the difference is 2% to 3%.

Fig. 6 and Fig. 7 show the effect of variable foundation and foundation variation parameter ($\lambda$) on mode shape of FGM beam, respectively. To generate the mode shapes the eigenvectors are normalised by dividing them by their maximum

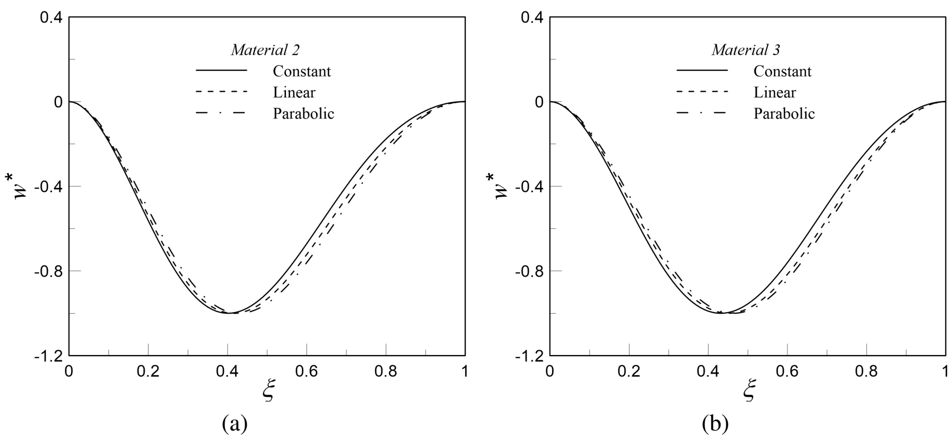


Fig. 6. Effect of variable foundation on the mode shapes of CC FGM beam (a) *Material 2* (b) *Material 3*

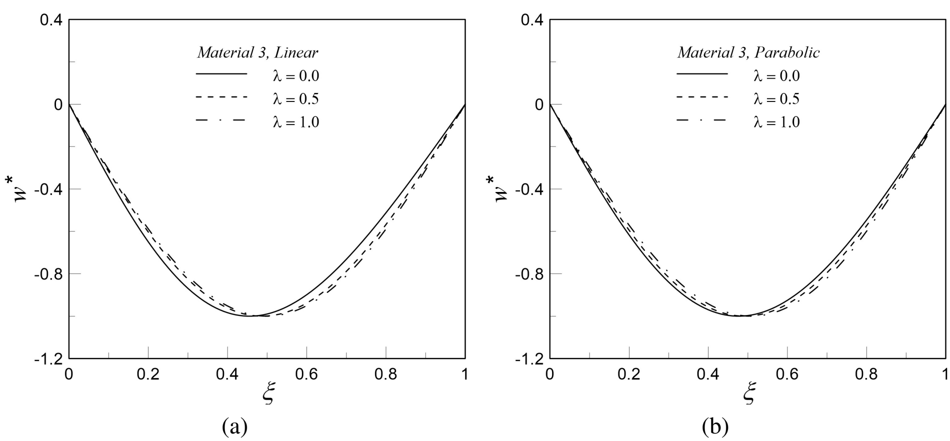


Fig. 7. Effect of foundation stiffness parameter ($\lambda$) on the mode shapes of SS FGM beam (a) *Linear* (b) *Parabolic*

value. The ordinate represents the normalised displacement ($w^*$) and the abscissa represents the beam's length. Fig. 6 is generated for CC beam and the values of foundation stiffness ($\bar{K}_f$) and foundation stiffness parameter ($\lambda$) are fixed at 10000 and 0.5, respectively. Whereas, Fig. 7 is generated for SS beam made of *Material 3* where the value of $\bar{K}_f$ is fixed at 1000 and three values of $\lambda$ (0, 0.5 and 1.0) are selected for comparison. The variations in mode shapes (shifting of the point of maximum deflection) for different foundation models and foundation stiffness parameters are observed in the figures.

The effect of aspect ratio ($a_r = h/L$) on the natural frequency of FGM beam is shown in Table 5. The results are generated using Timoshenko beam theory and provided for four values of aspect ratio (1/100, 1/50, 1/10 and 1/5) for all material models, foundation models and boundary conditions considered in this study. The general trend observed from the table is that the natural frequency gradually decreases as the aspect ratio increases. It can be seen that natural frequency for $a_r = 1/5$ has a visible derivation from $a_r = 1/100$ (3% to 4 %).

Table 5.
The effect of aspect ratio ($a_r = h/L$) on the natural frequency of FGM beam

| Material | Foundation | $\sqrt{\bar{\omega}}$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $a_r = 1/100$ | | $a_r = 1/50$ | | $a_r = 1/10$ | | $a_r = 1/5$ | |
| | | CC | SS | CC | SS | CC | SS | CC | SS |
| *Material 1* | *Constant* | 6.2200 | 5.7556 | 6.2172 | 5.7550 | 6.1501 | 5.7412 | 6.0127 | 5.7043 |
| | *Linear* | 6.4634 | 6.0570 | 6.4608 | 6.0565 | 6.3987 | 6.0418 | 6.2726 | 6.0019 |
| | *Parabolic* | 6.3551 | 5.9308 | 6.3524 | 5.9303 | 6.2883 | 5.9156 | 6.1574 | 5.8759 |
| *Material 2* | *Constant* | 5.6017 | 5.0324 | 5.5986 | 5.0319 | 5.5225 | 5.0153 | 5.3607 | 4.9695 |
| | *Linear* | 5.7920 | 5.2885 | 5.7891 | 5.2880 | 5.7193 | 5.2726 | 5.5737 | 5.2303 |
| | *Parabolic* | 5.7070 | 5.1828 | 5.7040 | 5.1823 | 5.6322 | 5.1673 | 5.4820 | 5.1261 |
| *Material 3* | *Constant* | 5.7769 | 5.1319 | 5.7736 | 5.1314 | 5.6917 | 5.1154 | 5.5181 | 5.0717 |
| | *Linear* | 5.9596 | 5.3885 | 5.9564 | 5.3880 | 5.8808 | 5.3730 | 5.7228 | 5.3322 |
| | *Parabolic* | 5.8758 | 5.2819 | 5.8725 | 5.2814 | 5.7945 | 5.2665 | 5.6310 | 5.2263 |

## 4. Conclusions

Dynamic behaviour of FGM beams resting on variable elastic foundation was investigated in this study. Three different material models were considered, in which the material properties varied along the length direction. The investigations were carried out for different variable foundation models and boundary conditions. The Euler-Bernoulli beam theory and Timoshenko beam theory were used for mathematical description of the beam. The dynamic problem was formulated using Rayleigh-Ritz approach and Hamilton's principle was used to generate the governing equations. From the comprehensive study of the results for different material models and foundation models it is concluded that various parameters like foundation stiffness and its variation, beam aspect ratio and boundary conditions significantly affect the natural frequency of FGM beam. This is especially true for the foundation stiffness, as it is seen that increasing the foundation stiffness increases the natural frequency. Also, the relationship between these two parameters is found to be nonlinear in nature.

## Appendix

### Euler-Bernoulli beam theory

Elements of stiffness matrix

$$[K] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix},$$

$$[K_{11}] = \frac{A}{L} \sum_{j=1}^{nu} \sum_{i=1}^{nu} \int_0^1 \frac{\mathrm{d}\alpha_i}{\mathrm{d}\xi} \frac{\mathrm{d}\alpha_j}{\mathrm{d}\xi} E(\xi)\,\mathrm{d}\xi,$$

$$[K_{12}] = 0, \qquad [K_{21}] = 0,$$

$$[K_{22}] = \frac{I}{L^3} \sum_{j=nu+1}^{nu+nv} \sum_{i=nu+1}^{nu+nv} \int_0^1 \frac{\mathrm{d}^2\beta_i}{\mathrm{d}\xi^2} \frac{\mathrm{d}^2\beta_j}{\mathrm{d}\xi^2} E(\xi)\,\mathrm{d}\xi + L \sum_{j=nu+1}^{nu+nv} \sum_{i=nu+1}^{nu+nv} \int_0^1 \beta_i \beta_j K_f(\xi)\,\mathrm{d}\xi.$$

Elements of mass matrix

$$[M] = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$[M_{11}] = LA \sum_{j=1}^{nu} \sum_{i=1}^{nu} \int_0^1 \alpha_i \alpha_j \rho(\xi)\,\mathrm{d}\xi,$$

$$[M_{12}] = 0, \qquad [M_{21}] = 0,$$

$$[M_{22}] = LA \sum_{j=nu+1}^{nu+nv} \sum_{i=nu+1}^{nu+nv} \int_0^1 \beta_i \beta_j \rho(\xi)\,\mathrm{d}\xi,$$

**Timoshenko beam theory**

Elements of stiffness matrix

$$[K] = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix},$$

$$[K_{11}] = \frac{A}{L} \sum_{j=1}^{nu} \sum_{i=1}^{nu} \int_0^1 \frac{\mathrm{d}\alpha_i}{\mathrm{d}\xi} \frac{\mathrm{d}\alpha_j}{\mathrm{d}\xi} E(\xi)\,\mathrm{d}\xi,$$

$$[K_{12}] = 0, \qquad [K_{13}] = 0, \qquad [K_{21}] = 0,$$

$$[K_{22}] = \frac{k_{sh}A}{L} \sum_{j=nu+1}^{nu+nv} \sum_{i=nu+1}^{nu+nv} \int_0^1 \frac{\mathrm{d}\beta_i}{\mathrm{d}\xi} \frac{\mathrm{d}\beta_j}{\mathrm{d}\xi} G(\xi)\,\mathrm{d}\xi,$$

$$[K_{23}] = -k_{sh}A \sum_{j=nu+nv+1}^{nu+nv+ns} \sum_{i=nu+1}^{nu+nv} \int_0^1 \frac{\mathrm{d}\beta_i}{\mathrm{d}\xi} \phi_j G(\xi)\,\mathrm{d}\xi, \qquad [K_{31}] = 0,$$

$$[K_{32}] = -k_{sh}A \sum_{j=nu+1}^{nu+nv} \sum_{i=nu+nv+1}^{nu+nv+ns} \int_0^1 \phi_i \frac{\mathrm{d}\beta_j}{\mathrm{d}\xi} G(\xi)\,\mathrm{d}\xi,$$

$$[K_{33}] = \frac{I}{L} \sum_{j=nu+nv+1}^{nu+nv+ns} \sum_{i=nu+nv+1}^{nu+nv+ns} \int_0^1 \frac{\mathrm{d}\phi_i}{\mathrm{d}\xi} \frac{\mathrm{d}\phi_j}{\mathrm{d}\xi} E(\xi)\,\mathrm{d}\xi$$

$$+ k_{sh}AL \sum_{j=nu+nv+1}^{nu+nv+ns} \sum_{i=nu+nv+1}^{nu+nv+ns} \int_0^1 \phi_i \phi_j G(\xi)\,\mathrm{d}\xi.$$

Elements of mass matrix

$$[M] = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix},$$

$$[M_{11}] = AL \sum_{j=1}^{nu} \sum_{i=1}^{nu} \int_0^1 \alpha_i \alpha_j \rho(\xi)\,\mathrm{d}\xi,$$

$$[M_{12}] = 0, \qquad [M_{13}] = 0, \qquad [M_{21}] = 0,$$

$$[M_{22}] = AL \sum_{j=nu+1}^{nu+nv} \sum_{i=nu+1}^{nu+nv} \int_0^1 \beta_i \beta_j \rho(\xi)\, \mathrm{d}\xi,$$

$$[M_{23}] = 0, \qquad [M_{31}] = 0, \qquad [M_{32}] = 0,$$

$$[M_{33}] = IL \sum_{j=nu+nv+1}^{nu+nv+ns} \sum_{i=nu+nv+1}^{nu+nv+ns} \int_0^1 \phi_i \phi_j \rho(\xi)\, \mathrm{d}\xi.$$

Manuscript received by Editorial Board, June 22, 2020;
final version, October 19, 2020.

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