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Some Topological Aspects of Sampling Theorem and Reconstruction Formula

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*This paper is dedicated to memory of **Professor Witold Stepowicz**, who passed away on the 28th of October 2019. He will remain in our memories for ever as an excellent teacher as well as a friendly, open-minded, tolerant, and responsible person.*

***Abstract*—We present here a few thoughts regarding topological aspects of transferring a signal of a continuous time into its discrete counterpart and recovering an analog signal from its discrete-time equivalent. In our view, the observations presented here highlight the essence of the above transformations. Moreover, they enable deeper understanding of the reconstruction formula and of the sampling theorem. We also interpret here these two borderline cases that are associated with a time quantization step going to zero, on the one hand, and approaching its greatest value provided by the sampling theorem, on the other**

***Keywords*—topological issues related with reconstruction formula and sampling theorem**

I. Introduction

THE set of continuous time signals, on one side, and the set of discrete-time ones, on another one, are perceived as two different worlds. There are, however, two connecting elements that link them with each other. Those are the reconstruction formula and the sampling theorem [1]-[6]. But, it seems that we do not oft realize the fact that they constitute a basis for consideration of these separate "worlds" mentioned above as only two different perspectives from which one consistent world of signals can be viewed. For some people, this kind of interpretation may be of minor importance. This paper aims in showing just the contrary.

So, let us take a closer look at this issue. And to this end, consider signals shown in Figures 1(a) and 1(b).

Looking at Fig. 1, we see two quite different images. Fig. 1(a) shows an infinite sequence of bars of different heights while Fig. 1(b) depicts a continuous function. So, it is really hard to imagine that they represent the same object. This is, however, true as we know from the reconstruction formula and the sampling theorem [1]-[6]. In more detail, observe that the values of bar heights $x[k]$, $k = ..., -1, 0, 1, ...$, in Fig. 1(a) are equal to the values $x(kT)$ of the continuous function of Fig. 1(a) at the points kT . And, if we assume that the following:

$$1/T = f_s \geq 2 f_m \qquad (1)$$

holds, where T means a sampling period, f_s the corresponding sampling frequency, and f_m stands for the maximal frequency present in the spectrum of the signal $x(t)$, then, by virtue of the sampling theorem, the signals in Figures 1(a) and 1(b) are equivalent to each other in the sense that they can be obtained from each other via the reconstruction formula

$$x(t) = \sum_{k=-\infty}^{\infty} x(kT)\,\mathrm{sinc}(t/T - k) = \sum_{k=-\infty}^{\infty} x[k]\,\mathrm{sinc}(t/T - k) \quad . \qquad (2)$$

In (2), the function $\mathrm{sinc}(t)$ is defined as

$$\mathrm{sinc}(t) = \sin(\pi t)/\pi t \ \text{ for } t \neq 0 \ \text{ and } \ 1 \text{ for } t = 0 \ . \qquad (3)$$

Let us also express the above in some other words. To this end, observe that signals describe in some way objects of a real world. These objects are "visible" for us through measurements of which outcomes are available as just measured signals. So, it is fully justified to speak about signals as representations of real world objects. On the other hand, it is also customary to identify signals with objects which they represent. Here, we follow this convention. And we consider such a scenario in which any real world object can be represented by one signal being a function of a continuous time variable *t* and, equivalently, by an infinite family of appropriate sequences of discrete elements (values). As all these representations represent the same real word object, it is natural to require that they are in some way equivalent to each other. And this is really achieved, as we know, via fulfilment of (2) when inequality (1) is satisfied. We argue here that this can lead to a more consistent viewing of the world of signals.

In viewing of signals proposed here they constitute one coherent world. Moreover, see that the elements of this world can be viewed from infinitely many perspectives. One of them is a representation in form of a function of a continuous time variable *t*. All the others are discrete maps of values of the discrete-time variable into discrete signal amplitudes. In these maps, a set of values of the discrete-time variable can be chosen to be less or more dense (the meaning of this term will be explained in more detail later in this paper). Schematically. a notion of distance between two numbers does not apply Admittedly, ordering of elements of this set of integers can be this point of view is illustrated in Fig. 2.

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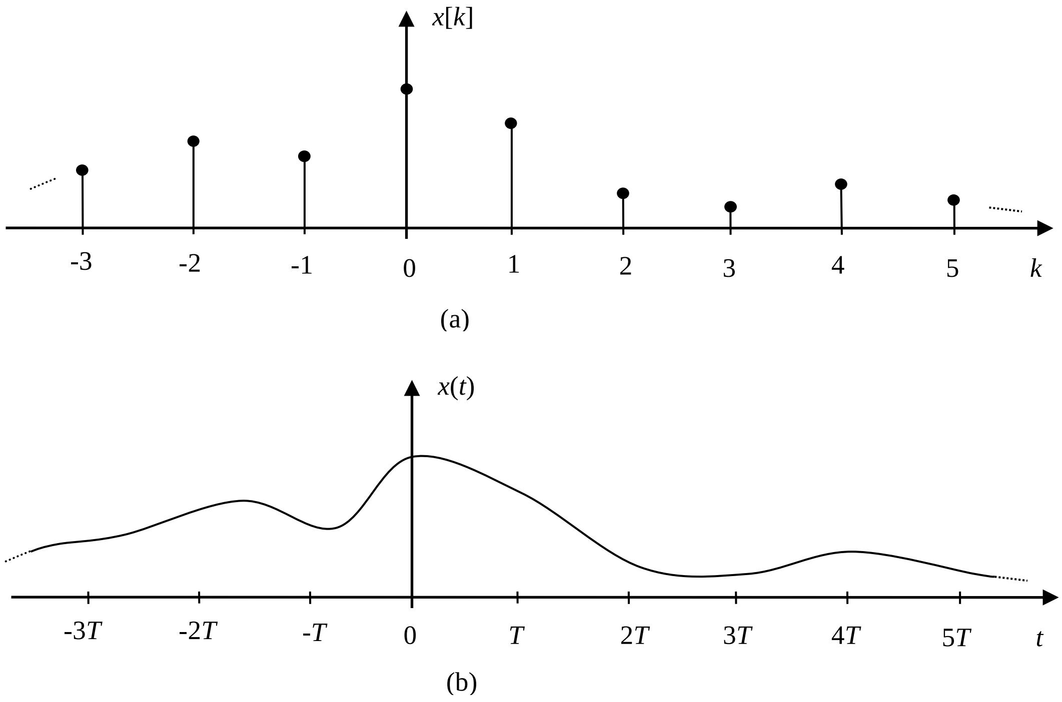

Fig. 1. (a) Example discrete-time signal, where the integers $\ldots,-1,0,1,\ldots$ mean successive values of a discrete time variable *k*. (b) Equivalent signal in the continuous time domain, where *t* stands for a continuous time variable

The most significant difference - between the values, which the continuous time variable *t* in the signal $x(t)$ shown on the top of Fig. 2 assumes, and the discrete-time values *k*, *k′*, *k″* (and so on) in the corresponding discrete-time signals $x[k]$, $x[k']$, $x[k'']$, respectively (and in all the remaining ones that are not explicitly shown in Fig. 2) - is the following: the first ones belong to the set R of real numbers, but all the others to the set Z of integers. The first set is of cardinality $\mathfrak{c}$, but the second of cardinality $\aleph_0$. And obviously, this fact is relevant from the point of view of topology. In this paper, we will study some of its implications.

The rest of this paper is organized as follows. Section 2 contains thorough explanations regarding notion and description of unscaled versus scaled discrete timelines, and also some related material. In Section 3, an informative uniqueness of signals independent of their images in the time domain is discussed. Behavior of the reconstruction formula for sampling periods going to zero is considered in the next section. The paper ends with Section 5 that contains conclusions.

II. UNSCALED VERSUS SCALED DISCRETE TIMELINES AND RELATED MATERIAL

Let us begin this section with the observation that in the set of integers considered in isolation from the set of real numbers carried out, but the determination of distances between them cannot. Obviously, they remain then separate objects, as it should be, but we are not able to say anything more about them.

Note now that such a situation as described above occurs when we write signal samples in the following form: $x[k]=x_k \in \mathrm{R},\ k=\ldots,-1,0,1,\ldots,$ without saying anything about the sampling period T. Then, only the order of the samples occurrences is "visible", but nothing can be said about distances between the times of these occurrences. Obviously, the latter follows from the lack of any accompanying timeline.

Let us consider now the axis of real numbers as a one-dimensional space. And note that using this convention we can view the set $\{x_k\}=\{x[k]=x_k \in \mathrm{R},\ k=\ldots,-1,0,1,\ldots\}$ as a set "immersed" in this space. Further, observe, as indicated above, that this space refers exclusively to the samples of the signal amplitudes. And because of this fact, we will call it a one-dimensional "amplitude-only" space. Moreover, see that distances between elements of the set $\{x_k\}$ will be then naturally determined as $d_{x_{k+i}x_k}=|x_{k+i}-x_k|$ (that is a natural metric of this space).

The amplitude-only space defined above and the set $\{x_k\}$ immersed in it are illustrated in Fig. 3(a).

Note now that in fact Fig. 3(a) shows not only one but an infinite bunch of hidden signals $x[k],\ k=\ldots,-1,0,1,\ldots,$ represented by the set $\{x_k\}$. This clearly follows from (2). To thereby "expand" the one-dimensional object $\{x_k\}$ of Fig. 3(a) into a two-dimensional one in space-time (more precisely,

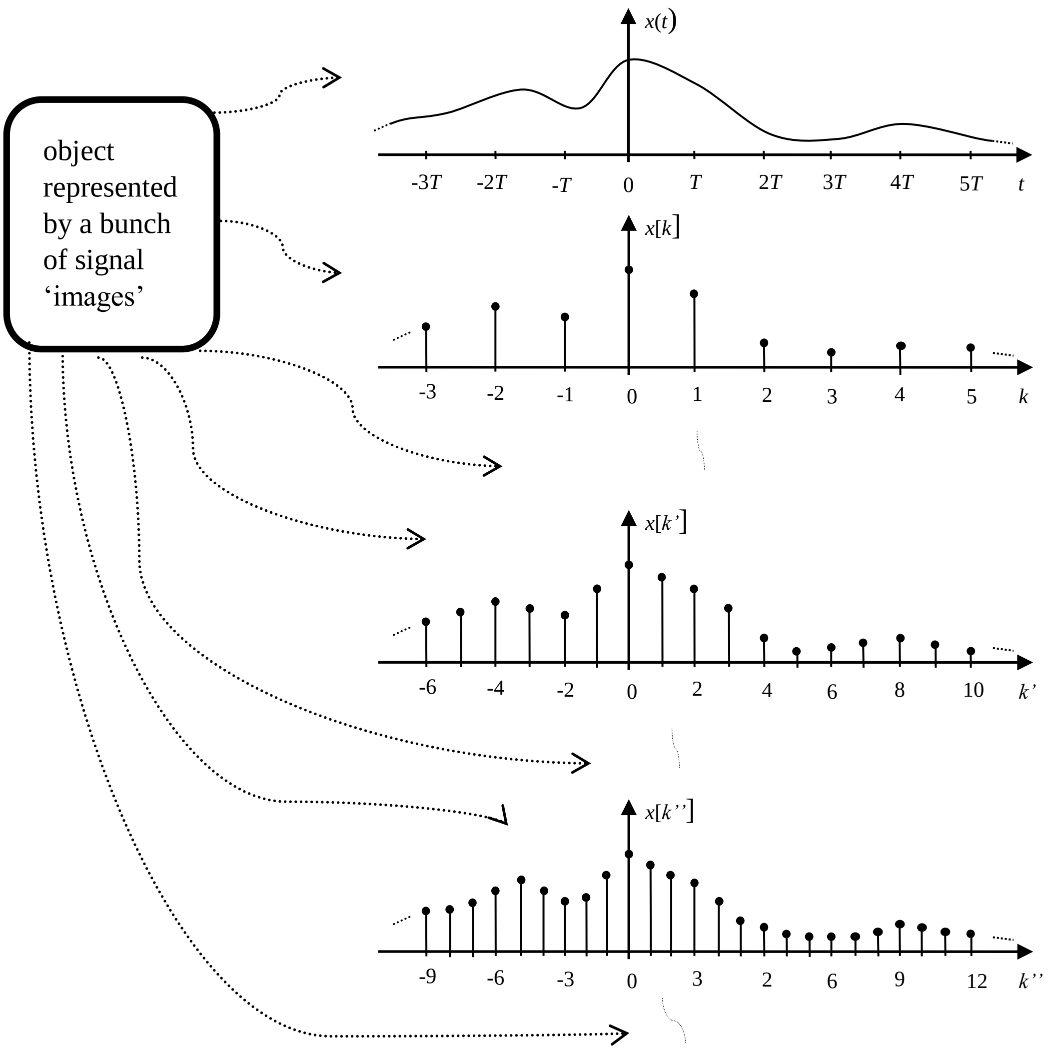

Fig. 2. An object represented by a signal that can viewed from an infinite number of perspectives

this end, see that assuming two different values of T, say T_1 and T_2, in (2), we get two different functions $x_1(t)=\sum_{k=-\infty}^{\infty} x_k \operatorname{sinc}(t/T_1-k)$ and $x_2(t)=\sum_{k=-\infty}^{\infty} x_k \operatorname{sinc}(t/T_2-k)$, respectively. Hence, really, the above observation is valid because the sampling period T is not known for Fig. 3(a); it can assume an infinite number of values.

Let us now return to the set of integers considered in isolation such as those hidden in indices of the elements of the set $\{x_k\}$ in Fig. 3(a). If we "immerse" them in the set R of real numbers treated as a one-dimensional space, we will in an amplitude-time space) as illustrated in Fig. 3(b). The distances between the times of occurrences of elements of the set $\{x_k\}$ will be well defined in this space, as $d_{(k+i)k}=|(k+i)-k|=|i|$. However, in this form, they will not be associated with any sampling period T. So, we will call the timeline associated with these times of occurrences a discrete unscaled timeline. In this context, note that choosing a concrete value of the sampling period T corresponds to picking a one unique function from the infinite bunch of hidden signals mentioned above. In other words, see that this corresponds to scaling the discrete unscaled timeline defined above with a factor T. And this leads to getting a discrete scaled timeline. Further, note that we have to do with such a scaled timeline in Fig. 1(a) because the successive points $....,-2,-1,0,1,2,3,....$ on it stand in fact for $....,-2T,-T,0,T,$ $2T,3T,....$, respectively. (For this, compare the timelines of with signal values. In contrast to this, all the other signal images in Fig. 2 represent functions that possess "free spaces"

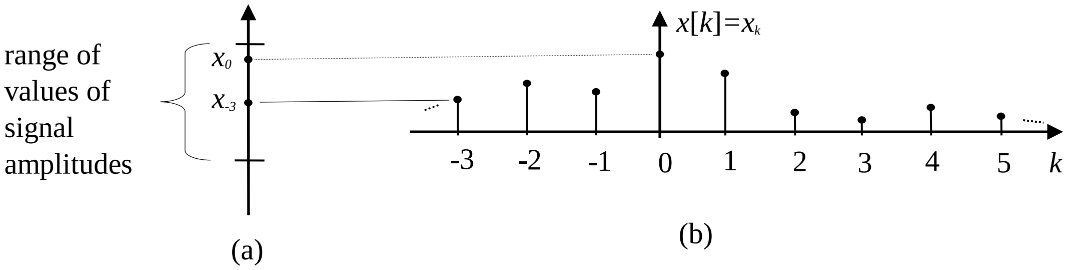

Fig. 3. (a) Illustration of the amplitude-only space and of the set $\{x_k\}$ immersed in it. (b) Expansion of the set $\{x_k\}$ in the space-time after assuming some scaling factor T (sampling period) for the discrete timeline

Figures 1(a) and 1(b).) However, for convenience, the capital letter T was dropped in Fig. 1(a).

In view of the above interpretation, observe also that the distances between the times of occurrences of signal samples in Fig. 1(a) are in fact equal to $d_{((k+i)T)(kT)} = |(k+i)T - kT| = |i|T$.

Further, see that it follows from the material presented hitherto that a perfect recovery of an original signal from its discrete counterpart rolled up as in Fig. 3(a) is possible if, and only if:

1. information about the sampling period T accompanies the set $\{x_k\}$,
2. the sampling period T satisfies inequality (1).

Moreover, the material presented above shows also that it is advisable in some considerations to consider signals viewed as one-dimensional ones (so called, in such a way, in the literature) a little bit differently, as two-dimensional objects in the space-time.

III. Informative Uniqueness of Signals Independent of Their Images in the Time Domain

Let us start considerations of this section with an observation that the notion of signal spectrum, as expressed by the Fourier transform of a signal, can be assumed to play a role of a measure of information contained in the signal. In what sense? In the sense that it provides us with information about the contents of harmonics occurring in it. More precisely, about their amplitudes and relative phases between them.

Having this in mind that the spectrum of all the members of the bunch of signal "images" in Fig. 2 is the same, except periodic repetitions in case of the "sampled images", we can treat them as equivalent to each other with respect to the information measure defined above. Note that this is a very important finding because all these signal images are evidently different pictures of a signal in the time domain. The signal image on the top of Fig. 2 represents a curve that is fully filled between "pillars" (representing signal samples). These free spaces have larger or shorter lengths referred to the time axis. For example, they are equal to T, $T/2$, and $T/3$, respectively, in the case of successive "sampled images" in Fig. 2, where T is defined on the curve representing the "continuous signal image" in this figure. Further, observe that signal amplitudes in the "free spaces" are equal to zero.

Concluding the above finding, we can say shortly that irrespective of "the extent to which a signal image in the time domain is filled with pillars" it is viewed as a unique object from the point of view of the aforementioned information measure. This conclusion is however only true when the sampling period chosen (the length between "pillars") satisfies inequality (1). See that all the choices of values of T larger than $1/(2f_m)$ will interfere in the information contents of the signal considered, leading to the effect that all its "sampling images" for these values will be distorted. At the other extreme, by choosing smaller and smaller values of T, we will shorten the lengths between "pillars" in the signal "sampling images", making thereby points of pillar occurrences denser and denser. This effect will be viewed in the time domain as the signal "sampling images" approaching the "continuous signal image". While in the frequency domain the latter effect can be seen as shifting all the mirrored spectra outside the range of frequencies of interest. This is illustrated in Fig. 4(c); for the sake of completeness, the undistorted and distorted periodically changing spectra of the example signal are also shown - in Figures 4(a) and (b), respectively. $|X_s(f)|$ in Fig. 4 is used to denote the magnitude of the sampled signal.

Observe that there occurs in Figure 4(c) only one "spectrum nonzero pattern" in the range of frequencies in which we are interested (or which is simply "visible" to us). This range is denoted there by f_{ob}.

Let us illustrate implications of the above fact and its possible interpretations. To this end, consider the set of radio frequencies (RF). It is assumed that these are the frequencies whose scope extends from 3 kHz to 3 THz. Therefore, the maximum range of frequencies covered by a "RF spectrum nonzero pattern" will equal approximately $2f_{ob}$ = 6 THz. Further, by identifying f_{ob} with f_m and T_{vs} with T consisting of these "pillars" from its continuous-time counterpart.

In the next step, consider what happens when a value of the period T_{vs} goes to zero and in the limit is equal to zero. Let us

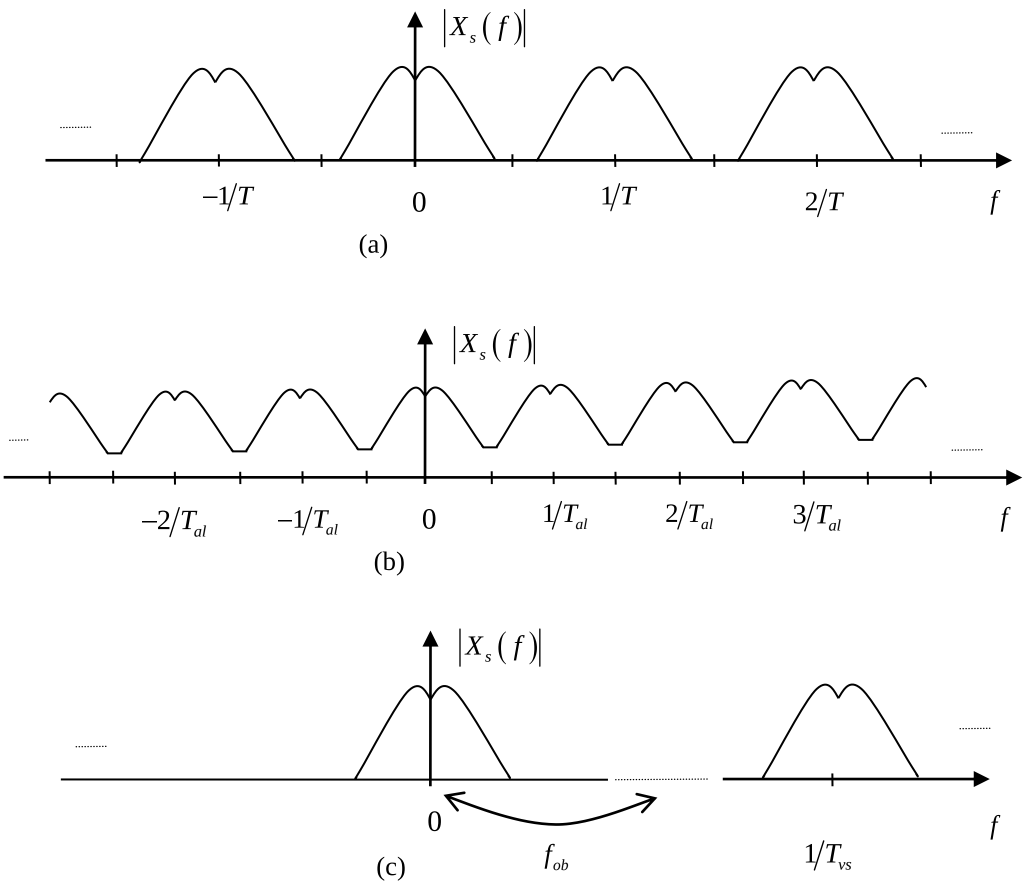

Fig 4. (a) Magnitude of the spectrum of a sampled signal, which does not show aliasing effects. (b) Magnitude of the spectrum of the same signal as in point (a), but sampled here in an appropriately longer sampling period T_{al}. There occur aliasing effects in it. (c) Magnitude of the spectrum of the identical signal as in points (a) and (b), but sampled here with a very short period T_{vs}. In this case, only one "nonzero pattern" occurs in it in the frequency range of interest (observable range of frequencies) f_{ob}.

occurring in (1), we get $T_{vs} \le 1/(2f_{ob})$ after performing substitutions. Finally, substituting the value of f_{ob} given above into the latter inequality, we obtain $T_{vs} \le 0{,}16\ \text{ps} = 160\ \text{fs}$.

Let us now interpret the above result first in the frequency domain. If we sampled any RF signal with the sampling periods smaller or equal to 160 fs, we would not experience any periodicity of the sampled signal spectrum. Simply, this periodicity would not be "visible" in the observed range of frequencies (RFs). Second, in the time domain, if the lengths between the successive signal "pillars" were smaller or equal to 160 fs, they would so densely occur that it would not be practically possible to distinguish between the "signal image" start from the latter. To this end, assume that all the distances between the "pillars" in a signal "sampling image" are exactly equal to zero. This means that all of them are hidden in a one-dimensional space possessing only one dimension "amplitude". So, this case resembles exactly the case already discussed and illustrated in Fig. 3(a).

Now, assume that in the case discussed just above with very short periods T_{vs} the values of these periods go to zero. In other words, we assume then that they are infinitesimal, however remain all the time greater than zero. So, in this case, the image sketched above saying that it is not practically possible to distinguish between the "signal image" consisting of very densely located signal samples from its continuous-time counterpart is valid. Also, we stress here that this picture of a signal is a two-dimensional one, in contrast to the case of $T_{vs} = 0$ considered just before. Furthermore, observe that this picture corresponds to the one shown in Fig. 3(b), but here with the signal samples as close to each other as possible. Moreover, the relation existing between the sets illustrated in Figures 3(a) and (b) extends also to the relation between the signal image for $T_{vs} = 0$ and its images for the infinitesimally small values of T_{vs}. That is the latter ones can be viewed as discrete-time expansions of a one-dimensional space connected with $T_{vs} = 0$.

Let us interpret the two results we arrived at above (one for $T_{vs} = 0$, and second for infinitesimally small values of T_{vs}, but greater than zero). To this end, see that we would expect receiving in the limit (that is when going with T_{vs} to zero) a continuous-time signal rather than a set illustrated in Fig. 3(a). However, note that we could interpret this fact as a lack of a "limit image" of an infinite sequence of "sampling images" discussed above when values of the period T_{vs} go to zero. Such behavior evidently follows from a sudden transition from scaled discrete timelines related with the "sampling images" for $T_{vs} \neq 0$ to a point (on the timeline) related with that one for $T_{vs} = 0$. In other words, this all stems from a "sharp shrinking" of a set of cardinality $\mathcal{C}$ (note that each of the scaled discrete timelines mentioned above is such a set) to a set of zero cardinality (an empty set). As a result, we simply "lose timeline" in performing the operation described above. In a sense, we can regard this as a paradox. Its implications for behavior of the reconstruction formula given by (2) when T in it goes to zero will be discussed in the next section.

However, in the context of the aforementioned paradox, let us comment yet on the following statement: "God made the integers, all else is the work of man" – attributed to a German mathematician Leopold Kronecker [7]. From the discussions presented in this section and in the previous one, it follows clearly that the timeline construction builds on the real numbers, which constitute a set of cardinality $\mathcal{C}$. So, in fact, we can express this in the following way: "if God made time, he have had to create also the reals". Obviously, this contradicts a little bit that what Leopold Kronecker said.

Finally, complementing the above, note that even when the signal sampling moments are expressed by integers, these integers are "immersed" in the set of real numbers (as stressed in Section 2).

IV. SAMPLING BEHAVIOR OF RECONSTRUCTION FORMULA FOR SAMPLING PERIOD GOING TO ZERO

In this section, we will investigate the behavior of the reconstruction formula given by (2) when the sampling period T in it goes to zero. We will check whether it provides, in the limit, that what we expect to get in this case. That is a function $x(t)$ of a continuous time.

To this end, let us denote by $x_m(t)$ the function occurring in the middle of (2). That is

$$x_m(t) = \sum_{k=-\infty}^{\infty} x(kT)\,\mathrm{sinc}(t/T - k)\ . \tag{4}$$

Then, let us carry out some rearrangements in the expression defining $x_m(t)$ that lead to the following form:

$$x_m(t) = \sum_{-k=\infty}^{-\infty} x(kT)\; T\; \frac{\sin\left(\frac{\pi}{T}(t-kT)\right)}{\pi(t-kT)}\ . \tag{5}$$

In the next step, we introduce a new variable $z = kT$ and a differential of this variable, $\Delta z = T$, in (5). And we try to calculate the function $x_m(t)$ for the limiting case of $T = 0$. That is the following:

$$\lim_{\Delta z\to 0}\left(x_m(t)\right) = \sum_{-k=\infty}^{-\infty} x(z)\ \Delta z\ \frac{\sin\left(\frac{\pi}{\Delta z}(t-z)\right)}{\pi(t-z)}. \tag{6}$$

Now, before proceeding further with calculations in (6), let us invoke two results from [8], namely

$$\delta(y) = \lim_{\varepsilon\to 0}\left(\frac{\sin\left(\frac{y}{\varepsilon}\right)}{\pi y}\right) \tag{7a}$$

and

$$\delta(y-y_0) = \lim_{\varepsilon\to 0}\left(\frac{\sin\left(\frac{y-y_0}{\varepsilon}\right)}{\pi(y-y_0)}\right)\ , \tag{7b}$$

where y means some variable, y_0 stands for a value of its shifting, and $\delta(\cdot)$ denotes the so-called delta function (Dirac impulse).

Denoting $\Delta z/\pi$, which occurs in (6), by a symbol ε, and applying then the formula for $\delta(y-y_0)$ given in (7) in (6), we can rewrite the latter equation for sufficiently small values of ε as

$$\lim_{\Delta z\to 0}\left(x_m(t)\right) \cong \lim_{\Delta z\to 0}\left(-\sum_{-k=\infty}^{-\infty} x(z)\ \Delta z\ \delta(z-t)\right) \tag{8}$$

In the next step, recognizing in (8) the definition of an integral and after some manipulations, we arrive finally at

$$\begin{aligned}\lim_{\Delta z\to 0}\left(x_m(t)\right) &\cong -\int_{\infty}^{-\infty} x(z)\delta(z-t)\,dz = \\ &= \int_{-\infty}^{\infty} x(z)\delta(z-t)\,dz = x(t)\end{aligned}\ . \tag{9}$$

Note now that (9) shows that when the sampling period T in the reconstruction formula (given by (2) goes to zero, it provides us with an original continuous-time signal. That is we receive then the signal we expected to arrive at. And this can be obviously viewed as a paradox when compared with the corresponding results obtained in the previous section. But, let us try to explain illustratively the difference existing between these two cases (which is responsible for the different results achieved). To this end, see that we can view the case discussed in Section 3 as such a one in which a set consisting of the signal sampling moments kT, $k = \ldots,-1,0,1,\ldots,$ is "so deeply immersed" in the set of reals standing for the continuous timeline that disappearance of the first of them implies a simultaneous disappearance of the second one. But, in the case of calculations in this section, see that the signal sampling moments kT, $k = \ldots,-1,0,1,\ldots,$ can be seen as "sliding points" on the set of a continuous timeline. And when all these points vanish, it does not mean that the latter set vanishes, too. It remains.

Finally in this section, we remark also on the derivations presented in (8) and (9). Obviously, they can be treated only as a sketch of a proof. A full one is more demanding and needs more advanced mathematics; it will be presented elsewhere. Here, we only draw attention to the fact that it is because of the appearance of a delta function (that is not a simple function but a distribution (generalized function)), the need of the use of an appropriate definition of the integral (we used the Riemann definition, but, it is rather not applicable here), and the occurrence of a limiting operation associated simultaneously with the integrating operation and with a function under the integral symbol that provides us with a distribution.

V. CONCLUSIONS

It seems that nothing can be already said on topics of sampling of continuous-time signals, the sampling theorem, and the reconstruction formula. That is, of course, largely true. However, as shown in this paper, there are still some intriguing points in the above topics that were kept silent in the hitherto literature, but, in our opinion, need to be addressed and explained. Just such a role fulfils this article. Here, we have thoroughly explained the relations between unscaled and scaled discrete timelines, as well as their relation to a continuous-time line. We have done this from the point of view of topology. Furthermore, we have proposed a unique interpretation of the sampled signal images that were sampled with different periods. Even more, we have shown an informative uniqueness of signals independent of their images in the time domain. Attention has been also drawn to occurrence of some paradoxes in cases of the values of the sampling period going to zero. Behavior of the reconstruction formula in such circumstances has been checked, too. We have concluded that it behaved then correctly.

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