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Exact and approximate distributed controllability of processes described by KdV and Boussinesq equations: The Green's function approach

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In this paper, we study the constrained exact and approximate controllability of traveling wave solutions of Korteweg-de Vries (third order) and Boussinesq (fourth order) semi-linear equations using the Green's function approach. Control is carried out by a moving external source. Representing the general solution of those equations in terms of the Frasca's short time expansion, system of constraints on the distributed control is derived for both types of controllability. Due to the possibility of explicit solution provided by the heuristic method, the controllability analysis becomes straightforward. Numerical analysis confirms theoretical derivations.

Key words: KdV equation, Boussinesq equation, Frasca's method, short time expansion, traveling wave, heuristic method, distributed control, constrained controllability

1. Introduction

One of the most challenging topics in contemporary control theory is the development of a universal and efficient method for analysis of controllability of systems with nonlinear constraints. A significant advance in controllability analysis of nonlinear systems has been reported in past several decades (see, for

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instance, [1–6], as well as related references therein). Application of different methods for analysis of particular control systems requires different complexities and/or computational costs. Apparently, computationally efficient numerical methods are mostly desired. This motivates a new research towards improvement of existing methods and development of new ones.

Controllability of a system is its ability to be transmitted from a given state to a desired state within a specified amount of time by means of admissible controls. Depending on the accuracy of the desired state implementation, two types of controllability are mainly distinguished. If the desired state is implemented exactly, the system is referred to as exactly controllable. If the implemented state is, in a certain sense, close to the desired one, then the system is referred to as approximate controllable.

Mathematically, a system whose state is described by $\boldsymbol{w} : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^m$ obeying a set of state constraints (e.g., differential equations and initial/boundary conditions, etc.), is exactly controllable in a given time $0 < T < +\infty$, if for any given initial and terminal states $\boldsymbol{w}_0 : \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{w}_T : \mathbb{R}^n \to \mathbb{R}^m$, there exists an admissible control $\boldsymbol{u} : \mathbb{R}^+ \to \mathbb{R}^m$ such that the residue

$$\mathcal{R}_T(\boldsymbol{u}) = \|\boldsymbol{w}(\boldsymbol{x}, T) - \boldsymbol{w}_T(\boldsymbol{x})\|_{\mathbf{W}_T} = 0, \tag{1}$$

where $\mathbf{W}_T$ is the space of terminal states (appropriate Hilbert space). More generally, if there exists an admissible control $\boldsymbol{u}$ such that

$$\mathcal{R}_T(\boldsymbol{u}) \leqslant \varepsilon \tag{2}$$

for a *given* accuracy ε, then the system is called approximately controllable. Apparently, the exact controllability implies approximate controllability with arbitrarily small accuracy ε, while approximate controllability does not guarantee exact controllability.

In some problems, it is to some extent sufficient to prove the existence of an admissible control providing either exact or approximate controllability. However, more often, it is desirable to determine the corresponding control regimes explicitly. To this aim, the method of moments [2], the norm minimization approach [5] or the heuristic method [7] can be applied. See, e.g., [8–12] for specific applications of the heuristic method in combination with the Green's function approach for the controllability analysis of linear and nonlinear systems.

Admissible controls that provide (1) are called *exactly resolving*. If $\mathcal{U}$ denotes the set of admissible controls, the set of exactly resolving controls is denoted by $\mathcal{U}_{res}^{ex}$. Thus,

$$\mathcal{U}_{res}^{ex} = \{\boldsymbol{u} \in \mathcal{U} : (1)\}.$$

Approximately resolving controls are defined analogously. The set of approximately resolving controls is defined by

$$\mathcal{U}_{res}^{ap} = \{\boldsymbol{u} \in \mathcal{U} : (2)\}.$$

In this terminology, a system is exactly controllable at T if and only if $\mathcal{U}_{res}^{ex} \cap \mathcal{U} \neq \varnothing$. Similarly, a system is approximately controllable if and only if $\mathcal{U}_{res}^{ap} \cap \mathcal{U} \neq \varnothing$.

The difficulty of the controllability analysis and its computational complexity strongly depends on the structure of $\mathcal{U}$. In some pure theoretical studies, there are no any constraints posed on u. However, in practice, controls that any controller can implement is, at least, bounded. Controllability under constraints on control is often referred to as constrained controllability (see the relevant studies [13–16].

In this paper, we aim to establish exact and approximate controllability conditions for externally controlled *nonlinear* Korteweg-de Vries (KdV) and Boussinesq equations. Note that usually boundary (exact and approximate) controllability of both mentioned equations is considered. Exact controllability of the nonlinear KdV equation by means of right-hand side has been studied relatively less. The first has been made in [17], and some further developments are described in the extensive review [18]. Controllability of the Boussinesq equation by means of the right-hand side is studied, e.g., in [19].

We start with a short description of the Green's function approach for studying exact and approximate controllability of systems described by higher order non-homogeneous semi-linear equations. Using the Frasca's short time expansion representation of the solution, the dependence $\mathcal{R}_T$ on u is made explicit allowing to carry out straightforward controllability analysis. Note that nonlinear Green's function of both equations are explicitly found and the error of approximation by corresponding Frasca's solutions is examined numerically in [20]. Assuming that the distributed control represents a moving source, necessary and sufficient conditions for exact controllability are derived. Sufficient conditions for approximate controllability are derived as well. We also address the problem of explicit determination of the sets $\mathcal{U}_{res}^{ex}$ and $\mathcal{U}_{res}^{ap}$ (partly) for both equations.

2. The Green's Function approach

In this section we will briefly outline the Green's function approach to controllability analysis developed in [6]. Let a control process be described by the following semi-linear ordinary differential equation:

$$\frac{\mathrm{d}^n w}{\mathrm{d}t^n} + N\left(\frac{\mathrm{d}^{n-1} w}{\mathrm{d}t^{n-1}}, \ldots, w\right) = f(u,t), \qquad t > 0, \tag{3}$$

where N is a generic non-linearity, f is the external influence on the process, u is a control. Let appropriate Cauchy conditions be attached to (3):

$$\left.\frac{\mathrm{d}^k w}{\mathrm{d}t^k}\right|_{t=0} = w_k, \qquad k = 0, 1, \ldots, n-1. \tag{4}$$

It is numerically established in [20] that the general solution of (3) is represented as follows:

$$w(t) = \int_0^t \hat{G}(t-\tau) f(u,\tau)\, \mathrm{d}\tau, \tag{5}$$

where

$$\hat{G}(t) = G(t) \cdot g(t),$$

with G determined as the general solution of the following Cauchy problem:

$$\frac{\mathrm{d}^n G}{\mathrm{d}t^n} + N\left(\frac{\mathrm{d}^{n-1} G}{\mathrm{d}t^{n-1}}, \ldots, G\right) = s\delta(t), \tag{6}$$

subject to

$$\left.\frac{\mathrm{d}^k G}{\mathrm{d}t^k}\right|_{t=0} = 0, \quad k = 0, 1, \ldots, n-1, \tag{7}$$

function $g \in C^{n-1}[0,T]$ is chosen in numerical purposes to minimize the approximation error, s is a real parameter, δ is the Dirac function. In specific problems, g is expanded into a Taylor series [21] near $t = 0$ and the expansion coefficients are determined in terms of quantities $w^{(k)}(0)$ specified by the attached Cauchy conditions $(k = 1, 2, \ldots, n-1)$ and evaluating (3) and its derivatives at $t = 0$ $(k \geqslant n)$. Moreover, the following result is proved.

Theorem 1 [20] *Let the non-linearity possesses the following generalized homogeneity property:*

$$\theta(t) \cdot N\left(\frac{\mathrm{d}^{n-1} w}{\mathrm{d}t^{n-1}}, \ldots, w\right) = N\left(\theta \frac{\mathrm{d}^{n-1} w}{\mathrm{d}t^{n-1}}, \ldots, \theta w\right), \tag{8}$$

where θ is the Heaviside function and satisfies the existence and uniqueness conditions for (3), (4). Then, the nonlinear Green's function of (3), (4), i.e., the general solution of (6), (7) admits the following representation:

$$G(t) = \theta(t) w_0(t), \tag{9}$$

where w_0 is the general solution of the homogeneous equation

$$\frac{\mathrm{d}^n w_0}{\mathrm{d}t^n} + N\left(\frac{\mathrm{d}^{n-1} w_0}{\mathrm{d}t^{n-1}}, \ldots, w_0\right) = 0, \tag{10}$$

subject to non-homogeneous Cauchy conditions

$$\left.\frac{\mathrm{d}^k w_0}{\mathrm{d}t^k}\right|_{t=0} = 0, \quad k = 0, 1, \ldots, n-2, \quad \left.\frac{\mathrm{d}^{n-1} w_0}{\mathrm{d}t^{n-1}}\right|_{t=0} = s. \tag{11}$$

Remark 1 *The advantage of representation (9) is that because of the Dirac function in the right hand side, the solution of (6), (7), whenever possible, is considerably sophisticated, while the handbooks like [22] mostly contain exact solutions of homogeneous equations like (10). Moreover, since these solutions contain some arbitrary constants, the unique solution satisying (11), in principle, can be found. It is also noteworthy that there are several approaches allowing to reduce quasi-linear PDEs to ODEs of the form (6). See [23] for details.*

Now, once the state function admits the representation (5), then its controllability is established by quantifying the following residue:

$$\mathcal{R}_T(u) = \left| \int_0^T \hat{G}\,(T-\tau)\, f\,(u,\tau)\,\mathrm{d}\tau - w_T \right|, \tag{12}$$

where w_T is the desired value to be achieved at $t = T$. Control function can be determined from (12) using the heuristic method developed in [7].

3. Controllability analysis

In this section we apply the Green's function approach described in the previous section for establishment of controllability conditions for the KdV and Boussinesq equations. For error estimate of the nonlinear Green's function solution of these equations for various source functions, see [20].

3.1. KdV equation

Consider the following KdV equation governed by a distributed control of special form:

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} + 6w\frac{\partial w}{\partial x} = u\,(x - vt)\,, \qquad x \in [0,l],\ t > 0. \tag{13}$$

Here u represents a moving control function with constant velocity $v > 0$. For a given function $w_T \in L^2[0,l]$ $(= \mathbf{W}_T)$ and given time moment $0 < T < +\infty$, let us consider the following residue:

$$\mathcal{R}_T(u) = \|w\,(x,T) - w_T(x)\|_{\mathbf{W}_T}\,. \tag{14}$$

Then, the problem is to find admissible controls

$$u \in \mathcal{U} = \left\{ u \in L^2[0,l] :\ |u| \leqslant \epsilon,\ \mathrm{supp}(u) \subseteq [0,l] \right\},$$

where $\text{supp}(u) = \overline{\{\xi \in \mathbb{R} : u(\xi) \not\equiv 0\}}$ is the support of u, providing

$$\mathcal{R}_T(u) = 0, \tag{15}$$

for exact controllability, or

$$\mathcal{R}_T(u) \leqslant \varepsilon \tag{16}$$

with a *given* precision ε for approximate controllability.

Remark 2 *Thus, the set of admissible controls contains bounded, compactly supported functions from L^2. Therefore, exact and approximate controllability conditions obtained below are for constrained controllability.*

3.1.1. Green's function solution of (13)

The traveling wave solution of (13) is determined from the following nonlinear ODE:

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{\mathrm{d}^2\widetilde{w}}{\mathrm{d}\zeta^2} + 3\widetilde{w}^2 - v\widetilde{w}\right] = u(\zeta), \quad \zeta \in [0, l], \tag{17}$$

where $\widetilde{w}(\zeta) = \widetilde{w}\,(x - vt) = w\,(x, t)$.

Evidently, in this case, (8) is satisfied allowing to write the nonlinear Green's function of (17) as follows:

$$\widetilde{G}(\zeta) = \theta(\zeta)\widetilde{w}_0(\zeta),$$

where $\widetilde{w}_0$ is the general solution of the following *homogeneous* equation:

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{\mathrm{d}^2\widetilde{w}_0}{\mathrm{d}\zeta^2} + 3\widetilde{w}_0^2 - v\widetilde{w}_0\right] = 0, \tag{18}$$

subjected to the Cauchy conditions:

$$\widetilde{w}_0(0) = \left.\frac{\mathrm{d}\widetilde{w}_0}{\mathrm{d}\zeta}\right|_{\zeta=0} = 0, \quad \left.\frac{\mathrm{d}^2\widetilde{w}_0}{\mathrm{d}\zeta^2}\right|_{\zeta=0} = s. \tag{19}$$

The general solution of (18), (19) is given as follows:

$$\widetilde{w}_0(\zeta) = \zeta_3 + (\zeta_2 - \zeta_3)\,\text{sn}^2\left[\frac{1}{2}\sqrt{(4\zeta_3 + 2\zeta_2 - v)\,(\zeta + c_3)^2}, \frac{\zeta_2 - \zeta_3}{\zeta_1 - \zeta_3}\right]. \tag{20}$$

Here sn is the Jacobi snoidal function defined as follows:

$$\text{sn}\,(\sigma, m) = \sin\varphi, \quad \text{where} \quad \sigma = \int_0^{\varphi} \frac{\mathrm{d}\phi}{\sqrt{1 - m\sin^2\phi}},$$

ζ_1, ζ_2, and ζ_3 are the roots of the cubic equation

$$2\zeta^3 - v\zeta^2 - 2c_1\zeta - c_2 = 0,$$

and c_1, c_2, and c_3 are integration constants determined from the Cauchy conditions (19). The expressions of ζ_1, ζ_2 and ζ_3 can be made explicit. However, since they are not important for this study, we will not extend the text by them.

Thus, making use of (5), the general solution of (13) can be represented as follows:

$$w\left(x,t\right) = \int\limits_0^{x-vt} \hat{G}\left(x - vt - \xi\right) u(\xi)\,\mathrm{d}\xi. \tag{21}$$

3.1.2. Constrained exact controllability

Following to [6], the exact controllability of (13) is equivalent to the following equality:

$$\int\limits_0^{x-vT} \hat{G}\left(x - vT - \xi\right) u(\xi)\,\mathrm{d}\xi - w_T(x) = 0 \quad \text{a.e. in} \quad [0,l]. \tag{22}$$

Let $\varphi_n \in L^2[0,l]$ for all $n \in \mathbb{N}$, be a family of functions orthogonal (probably with some weight) in $[0,l]$ forming a basis in the space of terminal states w_T. Then, expanding both sides of the last equality into a series of φ_n, we will eventually derive the following infinite system of equations:

$$\int\limits_0^{l} \left[\int\limits_0^{x-vT} \hat{G}\left(x - vT - \xi\right) u(\xi)\,\mathrm{d}\xi \right] \varphi_n(x)\,\mathrm{d}x = w_{Tn}, \quad n = 1,2,\ldots, \tag{23}$$

where

$$w_{Tn} = \int\limits_0^{l} w_T(x)\varphi_n(x)\,\mathrm{d}x.$$

Due to the linearity of (23), the set

$$\widetilde{\mathcal{U}}^{ex}_{res} = \{u \in \mathcal{U} : \ (23),\ \|u\|_{L^2} \to \min\} \subset \mathcal{U}^{eq}_{res}$$

can be constructed by means of the method of moments [2]. Nonetheless, in order to explore extended possibilities of $\mathcal{U}^{eq}_{res}$, here we employ the heuristic method developed in [7]. Let us assume that the resolving control can be expanded into an orthogonal series of functions $\phi_m \in L^2[0,l]$ for all $m \in \mathbb{N}$, orthogonal (possibly with some weight) in $[0,l]$:

$$u(\xi) = \sum_{m=1}^{\infty} u_m \phi_m(\xi), \tag{24}$$

such that $u \in \mathcal{U}$. Then, from (23) for unknown coefficients u_m we derive the following infinite system of liner algebraic equations:

$$\mathbf{\Phi}(T)\boldsymbol{u} = \boldsymbol{w}_T, \tag{25}$$

with

$$\mathbf{\Phi} = \{\Phi_m^n\}_{m,n=1}^{\infty}, \quad \Phi_m^n(T) = \int_0^l \left[\int_0^{x-vT} \hat{G}\left(x - vT - \xi\right) u(\xi)\,\mathrm{d}\xi\right] \varphi_n(x)\,\mathrm{d}x,$$

$$\boldsymbol{u}^{\mathrm{T}} = \{u_m\}_{m=1}^{\infty}, \quad \boldsymbol{w}_T^{\mathrm{T}} = \{w_{Tn}\}_{n=1}^{\infty}.$$

The solvability of (25) in general case is studied in the well-known fashion of [24]. Recall the following result.

Theorem 2 ([24], p. 27) *If for given l, v, T, w_T, and ϵ, infinite system (25) is regular, i.e.,*

$$\sigma_n(T) = \sum_{m=1}^{\infty} |\Phi_m^n(T)| < 1, \quad n = 1, 2, \ldots, \tag{26}$$

and for a positive constant C

$$|w_{Tn}| \leqslant C\left[1 - \sigma_n(T)\right], \quad n = 1, 2, \ldots, \tag{27}$$

then (25) has a bounded solution

$$|u_m| \leqslant C, \quad m = 1, 2, \ldots. \tag{28}$$

Moreover, if the solution of the majorant system

$$\sum_{m=1}^{\infty} |\Phi_m^n(T)|\, v_m = C\left[1 - \sigma_n(T)\right], \quad m = 1, 2, \ldots, \tag{29}$$

is strictly positive, then (28) is unique.

At this, the solution of (25) is determined by computing the limit of the truncated $N \times N$-dimensional system when $N \to \infty$. Note that if the mentioned conditions hold, the truncated system has a unique bounded solution.

Thus, the following assertion becomes evident.

Theorem 3 (about constrained exact controllability) *If for given l, v, T, w_T, and ϵ, the matrix and free term of infinite system (25) satisfy the conditions of Theorem 2, then KdV equation (13) is constrained exact controllable in T.*

Proof. Indeed, when Theorem 2 holds, then infinite system (25) has a unique bounded solution. The resulting controls (24) satisfy (22) which is necessary and sufficient for exact controllability of (13).

Remark 3 *In other words, in that case, as soon as (24) with (25) are in $\mathcal{U}$, then $\mathcal{U}^{ex}_{res} \neq \varnothing$.*

Remark 4 *Usually, moving controls are modeled as a concentrated load described by Dirac delta function [25–28]. It is one of the advantages of this approach that it is possible to consider more general forms of the moving source provided that for corresponding basis functions ϕ_m, (25) is regular, i.e., (26) holds.*

3.1.3. Constrained approximate controllability

Derivation of constrained approximate conditions for (13) are more straightforward. The following assertion holds true.

Theorem 4 *If for given w_T and desired accuracy ε,*

$$\varepsilon_T = \varepsilon - \|w_T\|_{\mathbf{W}_T} \geqslant 0, \tag{30}$$

then, for given l, v, T, and ϵ,

$$g_T(u) \leqslant \varepsilon_T \tag{31}$$

is sufficient for approximate controllability of (13) in T.

In other words,

$$\widetilde{\mathcal{U}}^{ap}_{res} = \{u \in \mathcal{U} : \; g_T(u) \leqslant \varepsilon_T\} \subseteq \mathcal{U}^{ap}_{res}.$$

Proof. Making use of the triangle inequality, for the residue (14) we derive the following estimate:

$$\mathcal{R}_T(u) = \left\| \int\limits_0^{x-vT} \hat{G}\left(x - vT - \xi\right) u(\xi) \mathrm{d}\xi - w_T(x) \right\|_{\mathbf{W}_T} \leqslant g_T(u) + \|w_T\|_{\mathbf{W}_T},$$

where

$$g_T(u) = \left\| \int\limits_0^{x-vT} \hat{G}\left(x - vT - \xi\right) u(\xi) \mathrm{d}\xi \right\|_{\mathbf{W}_T}. \tag{32}$$

The Green's function approach also has the convenience to consider extensions of $\mathcal{U}$ to accommodate sliding mode controls described by generalized functions. Let, in particular,

$$u(\xi) = u_0 \delta(\xi).$$

Then,

$$g_T(u) = |u_0| \cdot \left\| \hat{G}\,(x - vT) \right\|_{\mathbf{W}_T},$$

and the approximately resolving controls are defined by

$$|u_0| \leqslant \frac{\varepsilon_T}{\left\| \hat{G}\,(x - vT) \right\|_{\mathbf{W}_T}},$$

provided that

$$K_T = \left\| \hat{G}\,(x - vT) \right\|_{\mathbf{W}_T} \neq 0. \tag{33}$$

Note that the existence of a nontrivial solution requires that $K_T < \infty$, equivalent to $\hat{G}\,(x - vT) \in \mathbf{W}_T$.

In view of the apparent inequality

$$g_T(u) \leqslant \epsilon \left\| \int\limits_0^{x-vT} \hat{G}\,(x - vT - \xi)\, \mathrm{d}\xi \right\|_{\mathbf{W}_T},$$

we obtain

Corollary 3 (about constrained approximate controllability) *If for given w_T and desired accuracy ε, (30) holds, and for given l, v, and T, there exists a constant C such that*

$$\left\| \int\limits_0^{x-vT} \hat{G}\,(x - vT - \xi)\, \mathrm{d}\xi \right\|_{\mathbf{W}_T} \leqslant C,$$

then

$$\epsilon \leqslant \frac{\varepsilon_T}{C}$$

is sufficient for constrained approximate controllability of (13) in T.

In other words

$$\overline{\mathcal{U}}^{ap}_{res} = \{u \in \mathcal{U} : C\epsilon \leqslant \varepsilon_T\} \subseteq \mathcal{U}^{ap}_{res}.$$

Evidently, in the case of null-controllabilty, (30) always holds.

3.2. Boussinesq equation

Now let us consider the Boussinesq equation

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial x}\left[w\frac{\partial w}{\partial x}\right] + \frac{\partial^4 w}{\partial x^4} = u\left(x - vt\right), \qquad x \in [0, l],\ t > 0. \tag{34}$$

Its traveling wave solution is determined from the following ODE:

$$\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\left[\frac{\mathrm{d}^2\widetilde{w}}{\mathrm{d}\zeta^2} + \frac{1}{2}\widetilde{w}^2 + v^2\widetilde{w}\right] = u(\zeta). \tag{35}$$

In this case also the nonlinear term possesses homogeneity condition (8), and thus, (9) holds. However, we were able to derive the explicit solution of the homogeneous equation

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{\mathrm{d}^2\widetilde{w}_0}{\mathrm{d}\zeta^2} + \frac{1}{2}\widetilde{w}_0^2 + v^2\widetilde{w}_0\right] = 0$$

as follows:

$$\widetilde{w}_0(\zeta) = \zeta_3 + (\zeta_2 - \zeta_3)\,\mathrm{sn}^2\left[\frac{1}{2}\sqrt{\left(\frac{2}{3}\zeta_3 + \frac{1}{3}\zeta_2 + v^2\right)\left(\zeta + c_3\right)^2}, \frac{\zeta_2 - \zeta_3}{\zeta_1 - \zeta_3}\right],$$

where ζ_1, ζ_2, and ζ_3 are the roots of the following cubic equation:

$$\zeta^3 + 3v^2\zeta - 6c_1\zeta - 3c_2 = 0,$$

constants c_1, c_2, and c_3 are determined from Cauchy conditions (19). The explicit expressions of the roots are too long to include into the text.

Then, the function $\widetilde{G}_1(\zeta) = \theta(\zeta)\widetilde{w}_0(\zeta)$ satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\left[\frac{\mathrm{d}^2\widetilde{G}_1}{\mathrm{d}\zeta^2} + \frac{1}{2}\widetilde{G}_1^2 + v^2\widetilde{G}_1\right] = s\delta'(\zeta),$$

and therefore, $\widetilde{G}_1$ is the first order Green's function of Boussinesq equation (35). This means that the general solution of (35) is given as follows [20]:

$$\widetilde{w}(\zeta) = \int_0^{\zeta} \hat{G}\left(\zeta - \xi\right) U(\xi)\mathrm{d}\xi,$$

where U is the anti-derivative of u:

$$U(\xi) = \int_0^{\xi} u(\eta)\mathrm{d}\eta.$$

The exact and approximate controllability conditions in this case look similar to (23) and (31), but with U substituted instead of u.

Note that in particular case of point moving source, i.e., when $u(\zeta) = u_0\delta(\zeta)$, we get $U(\zeta) = u_0$. Then, the following assertion holds.

Corollary 4 (about constrained approximate controllability) *If for given w_T and desired accuracy ε, (30) holds, and for given l, v, and T (c.f. (33))*

$$B_T = \left\| \int_0^{x-vT} \hat{G}\left(x - vT - \xi\right) \mathrm{d}\xi \right\|_{\mathbf{W}_T} \neq 0,$$

then

$$|u_0| \leqslant \frac{\varepsilon_T}{B_T}, \tag{36}$$

is sufficient for the constrained approximate controllability of Boussinesq equation (34).

In other words,

$$\hat{\mathcal{U}}^{ap}_{res} = \{u \in \mathcal{U} : \ (36)\} \subseteq \mathcal{U}^{ap}_{res} .$$

For the existence of the nontrivial solution, it is required that $\epsilon_T < \infty$.

4. Numerical examples

In this section we carry out a numerical test in order to verify theoretical derivations above. For the sake of simplicity, only the KdV equation is considered. Nonetheless, the analysis is straightforwardly the same also for the Boussinesq equation.

We begin with the controlled KdV equation (17). Assume that at $\zeta = 0$,

$$\widetilde{w}(0) = 1, \qquad \left.\frac{\mathrm{d}\widetilde{w}}{\mathrm{d}\zeta}\right|_{\zeta=0} = \left.\frac{\mathrm{d}^2\widetilde{w}}{\mathrm{d}\zeta^2}\right|_{\zeta=0} = 0.$$

The control problem requires to find admissible controls $u \in \mathcal{U}$ that provide $w_T \equiv 2$ a.e. in $[0, l]$ as $t \geqslant T$. Let us fix $l = 50$, $v = 1$, and $T = 50$. For the sake of simplicity, let us consider the approximate controllability requiring to ensure

$$\mathcal{R}_T(u) \leqslant \varepsilon \tag{37}$$

Since in this case $w_T \equiv 2$, (30) holds only for $\varepsilon > 2$. That is why we need to evaluate the residue $\mathcal{R}_T$ directly. Below we consider the heuristic control [7]

$$u(\zeta) = u_0 \left[\theta\left(\zeta - \zeta_0\right) - \theta\left(\zeta - \zeta_1\right)\right] = u_0 \chi_{[\zeta_0,\zeta_1]}(\zeta). \tag{38}$$

Here, u_0, $0 \leqslant \zeta_0 < \zeta_1 < l$ are arbitrary parameters that need to be determined to provide required accuracy in (37). At this, we consider two cases: when u_0 is a constant and $u_0 \in C[0, l]$ is a function. First, let $u_0 = const$. Then, substituting (38) into (32), Theorem 4 implies that, e.g., when$u_0 = 0.2$, $\zeta_0 = 0$, $\zeta_1 = 40$, then (37) is provided with $\varepsilon = 0.036$. On the other hand, when

$$u_0(\zeta) = \alpha \sin\left(\omega\zeta + \beta\right), \tag{39}$$

where α, ω and β are arbitrary parameters, the accuracy of (37) can be reduced. Indeed, choosing $\alpha = 0.402$, $\omega = 0.1$, $\beta = 0$, $\zeta_0 = 0$ and $\zeta_1 = 30$, it is possible to provide (37) with $\varepsilon = 0.005$ (see Fig. 1). By a proper choice of corresponding parameters, both estimates can be further improved.

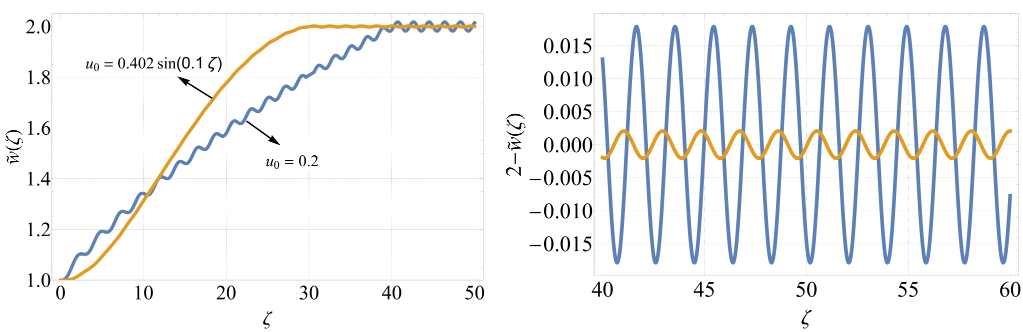

Figure 1: Controlled solution of (17) for $\zeta \in [0, 50]$ (left) and the difference between required and implemented states for $\zeta \in [40, 60]$ (right)

Now, we consider L^2-null-controllability of the KdV equation. In the case of exact null-controllability, one of the trivial solutions when

$$\widetilde{w}(0) = 0, \qquad \left.\frac{d\widetilde{w}}{d\zeta}\right|_{\zeta=0} = \left.\frac{d^2\widetilde{w}}{d\zeta^2}\right|_{\zeta=0} = 0,$$

is

$$u(\zeta) = u_0\delta(\zeta).$$

This control regime corresponds to a point source moving in the direction of the traveling wave with the same velocity.

Consider the approximate null-controllability of the KdV equation attained by the regime (38) with (39). Since in this case $w_T \equiv 0$, then (30) always holds.

Therefore, fixing

$$l = 150,\ \ v = 3,\ \ T = 50,\ \ \widetilde{w}(0) = -0.025,\ \ \frac{\mathrm{d}\widetilde{w}}{\mathrm{d}\zeta}\bigg|_{\zeta=0} = 0.002,\ \ \frac{\mathrm{d}^2\widetilde{w}}{\mathrm{d}\zeta^2}\bigg|_{\zeta=0} = -7{\cdot}10^{-3},$$

and making use of the Theorem 4, we find that $\alpha = 10^{-3}$, $\omega = 0.1$, $\beta = 0$, $\zeta_0 \approx 17.8$, $\zeta_1 \approx 52.2$ ensures (37) with $\varepsilon = 0.0204$.

Moreover, in that case, when

$$\widetilde{w}(0) = -0.01, \qquad \frac{\mathrm{d}\widetilde{w}}{\mathrm{d}\zeta}\bigg|_{\zeta=0} = \frac{\mathrm{d}^2\widetilde{w}}{\mathrm{d}\zeta^2}\bigg|_{\zeta=0} = 0,$$

the conditions of Corollary 3 are satisfied and (38) with $|u_0| \leqslant 2.5 \cdot 10^{-3}$ ensures (37) with $\varepsilon = 0.0185$ (see Fig. 2).

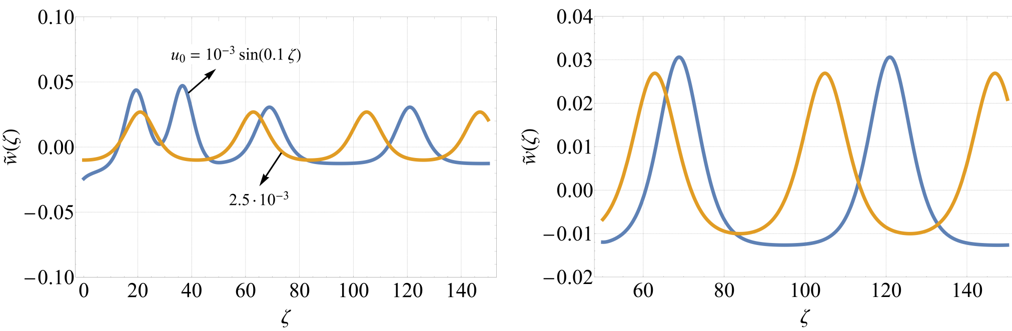

Figure 2: Controlled solution of (17) for $\zeta \in [0, 150]$ (left) and the difference between required and implemented states for $\zeta \in [50, 140]$ (right)

5. Conclusion

Constrained exact and approximate distributed controllability conditions for the KdV (third order) and Boussinesq (fourth order) nonlinear equations are obtained by means of the Green's function approach. Using the traveling wave ansatz, the semi-linear PDEs are reduced to ODEs, the general solution of which is represented by Frasca's short time expansion. Evaluating the residue between the implemented and desired states, an infinite system of equality type constraints on admissible controls providing exact controllability. Sufficient conditions in the form of inequalities are derived for the approximate controllability. In both cases, inequality type constraints on nonlinear Green's function are derived posing appropriate constraints on system parameters and initial and terminal states. The heuristic method allows to construct exactly and approximately resolving controls explicitly. Due to integral relation for the general solution, it becomes possible to consider extensions of the set of admissible controls to contain sliding mode controls expressed by generalized functions. The case point source described by Dirac's delta function is considered. Numerical analysis confirms theoretical derivations and reveals advantages of the Green's function approach.

References

[1] Klamka J.: *Controllability of Dynamical Systems*, Kluwer Academic, Dordrecht, 1991.

[2] Avdonin S.A. and Ivanov S.A.: *Families of Exponentials. The Method of Moments in Controllability Problems for Distributed Parameter Systems*, Cambridge University Press, New York, 1995.

[3] Fursikov A. and Imanuvilov O.Yu.: *Controllability of Evolution Equations*, Lecture Notes Series, vol. 34. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[4] Zuazua E.: *Controllability and Observability of Partial Differential Equations: Some Results and Open Problems*, Handbook of Differential Equations: Evolutionary Differential Equations, vol. 3, Elsevier/North-Holland, Amsterdam, 2006.

[5] Glowinski R., Lions J.-L., and He J., *Exact and Approximate Controllability for Distributed Parameter Systems: A Numerical Approach*, Cambridge University Press, New York, 2008.

[6] Avetisyan A.S. and Khurshudyan As.Zh.: *Controllability of Dynamic Systems: The Green's Function Approach*, Cambridge Scholars Publishing, Cambridge, 2018.

[7] Khurshudyan As.Zh.: Heuristic determination of resolving controls for exact and approximate controllability of nonlinear dynamic systems, *Mathematical Problems in Engineering*, 2018, DOI: 10.1155/2018/9496371.

[8] Khurshudyan As.Zh.: Resolving controls for the exact and approximate controllability of the viscous Burgers' equation: The Green's function approach, *International Journal of Modern Physics C*, **29**(6) (2018), 1850045, 14 pages.

[9] Khurshudyan As.Zh.: Exact and approximate controllability conditions for micro-swimmers deflection governed by electric field on a plane: The Green's function approach, *Archives of Control Sciences*, **28**(3) (2018), 335–347.

[10] Avetisyan A.S. and Khurshudyan As.Zh.: Exact and approximate controllability of nonlinear dynamic systems in infinite time: The Green's function approach, *ZAMM*, 98(11) (2018), 1992–2009.

[11] Khurshudyan As.Zh.: Distributed controllability of heat equation in unbounded domains: The Green's function approach, *Archives of Control Sciences*, **29**(1) (2019), 57–71.

[12] Khurshudyan As.Zh. and Arakelyan Sh.Kh.: Resolving controls for approximate controllability of sandwich beams with uncertainty: The Green's function approach, *Mechanics of Composite Materials*, **55**(1) (2019), 85–94.

[13] Klamka J.: Constrained controllability of nonlinear systems, *Journal of Mathematical Analysis and Applications*, **201**(2) (1996), 365–374.

[14] Klamka J.: Constrained approximate controllability. IEEE Transactions on Automatic Control, 2000, vol. 45, issue 9, pp. 1745–1749.

[15] Klamka J.: Constrained controllability of semilinear systems, *Nonlinear Analysis*, **47**(6) (2001), 2939–2949.

[16] Klamka J.: *Controllability and Minimum Energy Control*, Springer, Cham, 2019.

[17] Russell D.L. and Zhang B.-Y.: Exact controllability and stabilizability of the Korteweg-de Vries equation. Transactions of AMS, 1996, vol. 348, pp. 3643–3672.

[18] Rosier L. and Zhang B.-Y.: Control and stabilization of the Korteweg–de Vries equation: Recent progresses, *Journal of System Science and Complexity*, **22**(4) (2009), 647–682.

[19] Zhang B.-Y.: *Exact controllability of the generalized Boussinesq equation*, In Control and estimation of distributed parameter systems, International Series on Numerical Mathematics, vol. 126, pp. 297–310. Birkhäuser, Basel, 1998.

[20] Frasca M. and Khurshudyan As.Zh.: Green's function for higher order nonlinear equations: Case studies for KdV and Boussinesq equations, International Journal of Modern Physics C, 2018, vol. 29, 1850104, 13 pages.

[21] Khurshudyan As.Zh.: An identity for the Heaviside function and its application in representation of nonlinear Green's function, *Computational & Applied Mathematics*, **39** (2020), DOI: 10.1007/s40314-019-1011-5.

[22] Polyanin A.D. and Zaitsev V.F.: *Handbook of Ordinary Differential Equations: Exact Solutions, Methods, and Problems*, 3rd edition, Chapman & Hall/CRC Press, Boca Raton, 2017.

[23] Polyanin A.D. and Zaitsev V.: *Handbook of Nonlinear Partial Differential Equations*, Chapman & Hall/CRC Press, Boca Raton, 2012.

[24] Kantorovich K.L. and Krylov V.I.: *Approximate Methods of Higher Analysis*, Interscience Publishers, New York, 1958.

[25] Teodorescu P.P., Kecs W.W., and Toma A.: *Distribution Theory: With Applications in Engineering and Physics*, WILEY-VCH Verlag, Weinheim, 2003.

[26] Butkovskii A.G.: Some problems of control of the distributed-parameter systems, *Automation and Remote Control*, **72**(6) (2011), 1237–1241.

[27] Arakelyan Sh.Kh. and Khurshudyan As.Zh.: The Bubnov-Galerkin procedure for solving mobile control problems for systems with distributed parameters, *Mechanics. Proc. Nat. Acad. Sci. Armenia*, **68**(3) (2015), 54–75.

[28] Khapalov A.Y.: *Mobile Point Sensors and Actuators in the Controllability Theory of Partial Differential Equations*, Springer, Cham, Switzerland, 2017.