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Numerical checking method for positive invariance of polyhedral sets for linear dynamical systems

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Abstract. Positively invariant sets play an important role in the theory and applications of dynamical systems. The stability in Lyapunov sense of equilibrium $x = 0$ is equivalent to the existence of the ellipsoidal positively invariant sets. The constraints on the state and control vectors of dynamical systems can be formulated as polyhedral positively invariant sets in practical engineering problems. Numerical checking method of positive invariance of polyhedral sets is addressed in this paper. The validation of the positively invariant sets can be done by solving LPs which can be easily implemented numerically. The effectiveness of the proposed checking method is illustrated by examples. Compared with the now existing algebraic methods, numerical checking method is attractive and, importantly, easy to be implemented.

Key words: positively invariant set, linear system, polyhedral set, polyhedral cone, linear programming.

1. Introduction

Positively invariant sets play an important role in the theory and applications of dynamical systems. They appear in many different problems such as constrained control, robustness analysis, synthesis and optimization in the reason that it is closely related to the stability in the sense of Lyapunov and LaSalle theorem. There are many types of positively invariant sets such as polyhedral sets, ellipsoidal sets, Lorenz cones, etc. [1, 2]. We mainly consider convex polyhedral sets in this paper. Positive invariant sets of a linear dynamical system can be traced back to [3]. For a good survey of the art, refer to the survey paper of F. Blanchini [4]. An extension of positive invariance conditions to nonlinear dynamical systems and stochastic systems can be found in [5, 6].

The necessary and sufficient conditions of positively invariant sets for linear dynamical systems are studied in this paper. Positive invariance condition of polyhedral sets for linear systems was first proposed by Bitsoris in [7, 8], in a study considering a class of polyhedral sets that is symmetrical and nonsymmetric with respect to the origin. Hennet [9] has done a detailed research on the positive invariance properties for constrained linear discrete-time systems with feedback control. Recently, Zoltán Horváth [1] presented a unified approach to invariance conditions for a linear dynamical system. Rather than the algebraic conditions, we study the numerical checking method to check whether a given polyhedral set is a positively invariant set or not. Numerical checking is an attractive method which was once used to check the positivity and stability of the dynamical systems because it is easily implemented [10–13]. The considerations of numerically checking method can be extended to the positively invariant sets of fractional systems [14, 15] and the constrained feedback controller synthesis [16].

Notations: The set of real $m \times n$ matrices will be denoted by $R^{m\times n}$. A matrix $A = [a_{ij}] \in R_+^{m\times n}$ will be called nonnegative and denoted by $A \geq 0$ if $a_{ij} \geq 0$ for $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, A^T is the transpose of the matrix A. Vectors $x, y \in R^n$, $x \geq y$ means $x_i \geq y_i$, $i = 1,2,\ldots,n$. I_n is the identity matrix with size $n \times n$. The set of natural number will be denoted by N, all vectors will be the column vectors in this paper.

2. Main results

We consider the following linear dynamical systems described by the following equation.

$$x(k+1) = A_d x(k), \qquad k \in N, \tag{1}$$

$$\dot{x}(t) = A_c x(t), \qquad t \in R, \tag{2}$$

where $A_d, A_c \in R^{n\times n}$, $x(k), x(t) \in R^n$.

Definition 1. A set $S \in R^n$ is a positively invariant set for the discrete-time system (1) if $x(k) \in S$ implies $x(k+1) \in S$, for all $k \in N$.

Definition 2. A set $S \in R^n$ is a positively invariant set for the continuous-time system (2) if $x(0) \in S$ implies $x(t) \in S$, for all $t \geq 0$.

Definition 3. A matrix H is called a nonnegative matrix, denoted by $H \geq 0$, if $H_{ij} \geq 0$ for all i, j. A matrix M is called a Metzler matrix, if $M_{ij} \geq 0$ for $i \neq j$.

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Proposition 1. [17] The set $S \in R^n$ is a positively invariant set for the discrete-time system (1) if and only if $A_d S \in S$. Similarly, the set S is a positively invariant set for the continuous-time system (2) if and only if for all $t \geq 0$, $e^{At} S \in S$.

In this paper, we mainly investigate invariance conditions and checking methods for the polyhedral sets described by the following inequalities systems.

$$P = \{x \in R^n | \; Gx \leq b\}, \tag{3}$$

$$C_P = \{x | \; Gx \leq 0\}, \tag{4}$$

where $G \in R^{m \times n}$, $b \in R^m$.

2.1. Invariance conditions for discrete-time systems. We have the obtained the following results concerning invariance conditions for discrete-time linear systems.

Theorem 1. [9] A polyhedron P given as in (3) is a positively invariant set for the discrete-time linear system (1) if and only if there exists a matrix $H \in R^{m \times m}$ such that $H \geq 0$, $HG = GA_d$ and $Hb \leq b$.

Corollary 1. A polyhedral cone C_P given as in (4) is a positively invariant set for the discrete-time linear system (1) if and only if there exists a matrix $H \in R^{m \times m}$, such that $H \geq 0$ and $HG = GA_d$.

Denote

$$H = \begin{pmatrix} h_1^T \\ h_2^T \\ \vdots \\ h_m^T \end{pmatrix}, \quad G = \begin{pmatrix} g_1^T \\ g_2^T \\ \vdots \\ g_m^T \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \quad x = \begin{pmatrix} h_i \\ \omega \end{pmatrix},$$

where h_i^T, g_i^T, $i = 1, \ldots, m$ are row vectors of matrices H, G respectively, $b_i, \omega \in R$, $i = 1, 2, \ldots, m$.

Proof. Corollary is obvious when we take $b = 0 \in R^n$ in Theorem 1. □

Proposition 2. The inequalities $h_i^T b \leq b_i$, $i = 1, 2, \ldots, m$ hold if and only if there exists an $\omega \in R^m$ such that $h_i^T b_i + \omega_i = b_i$, $\omega \geq 0$, $i = 1, 2, \ldots, m$ hold.

Proof. The result is obvious. If there exists an $\omega \in R^n$, $\omega \geq 0$ such that $h_i^T b_i + \omega_i = b_i$, $i = 1, 2, \ldots, m$, $h_i^T b_i \leq b_i$, $i = 1, 2, \ldots, m$ hold since $\omega_i \geq 0$. If $h_i^T b_i \leq b_i$, $i = 1, 2, \ldots, m$ hold, there exists ω_i, $i = 1, 2, \ldots, m$ such that $h_i^T b_i \leq b_i$, $i = 1, 2, \ldots, m$, take $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$, we can obtain the result. □

By the above partition of matrices H, G and Proposition 2, results in Theorem 1 and Corollary 1 can be rewritten in the following form respectively.

$$\begin{pmatrix} G^T & 0 \\ b^T & 1 \end{pmatrix} \begin{pmatrix} h_i \\ \omega_i \end{pmatrix} = \begin{pmatrix} A_d^T g_i \\ b_i \end{pmatrix},$$

$$G^T h_i = A_d^T g_i, \qquad i = 1, \ldots, m.$$

With the above form and Theorem 1, we have the following theorem.

Theorem 2. A polyhedron P given as in (3) is a positively invariant set for the discrete-time linear system (1) if and only if there exists a nonnegative matrix $H \in R^{m \times m}$ and $\omega_i \geq 0$ such that

$$\begin{pmatrix} G^T & 0 \\ b^T & 1 \end{pmatrix} \begin{pmatrix} h_i \\ \omega_i \end{pmatrix} = \begin{pmatrix} A_d^T g_i \\ b_i \end{pmatrix}, \qquad i = 1, \ldots, m. \tag{5}$$

Proof. For fixed $i, i = 1, 2, \ldots, m$, from Theorem 1 and the partition of matrices H, G, we have $h_i^T G = g_i^T A_d^T$, take transpose operation on both sides get $G^T h_i = A_d^T g_i$, from Proposition 2, $h_i^T b_i \leq b_i$ can be replaced by $h_i^T b_i + \omega_i = b_i$, $\omega \geq 0$, $i = 1, 2, \ldots, m$, rewrite the equations in the form of matrices product, we have the conclusion. □

Corollary 2. A polyhedral cone C_P given as in (4) is a positively invariant set for the discrete-time linear system (1) if and only if there exists a nonnegative matrix $H \in R^{m \times m}$, such that

$$G^T h_i = A_d^T g_i, \qquad i = 1, \ldots, m \tag{6}$$

Proof. The corollary is obvious when by the partition of matrices H, G and Corollary 1. □

Remark 1. Theorem 2 and Corollary 2 can be obtained directly from Theorem 1 and Corollary 1 by means of Proposition 2 and the partition of matrices, the proofs of the following Theorem 3, 4 for continuous-time cases and the corresponding corollary are the same, we omit the proofs there.

2.2. Invariance conditions for continuous-time systems.

Theorem 3. [18] A polyhedron P given as in (3) is a positively invariant set for the continuous-time system (2) if and only if there exists a Metzler matrix $H \in R^{m \times m}$, such that $HG = GA_c$ and $Hb \leq 0$.

Corollary 3. A polyhedral cone C_P given as in (4) is a positively invariant set for the continuous system (2) if and only if there exists a Metzler matrix $H \in R^{m \times m}$, such that $HG = GA_c$.

Proof. Corollary 3 is obvious when we take $b = 0 \in R^m$ in Theorem 2. □

Theorem 4. A polyhedron P given as in (3) is a positively invariant set for the continuous system (2) if and only if there exists a Metzler matrix $H \in R^{m \times m}$ and $\omega_i \geq 0$ such that the following equalities hold.

$$\begin{pmatrix} G^T & 0 \\ b^T & 1 \end{pmatrix} \begin{pmatrix} h_i \\ \omega_i \end{pmatrix} = \begin{pmatrix} A_c^T g_i \\ b_i \end{pmatrix}, \qquad i = 1, \ldots, m. \tag{7}$$

Corollary 4. A polyhedral cone C_P given as in (4) is a positively invariant set for the continuous system (2) if and only if there exists a Metzler matrix $H \in R^{m \times m}$ such that the following equalities hold.

$$G^T h_i = A_c^T g_i, \qquad i = 1, \ldots, m. \tag{8}$$

2.3. Numerical checking methods. The above theorems and corollaries are the algebraic conditions for the positive invariance of the polyhedral sets for linear dynamical systems. All the results are in the form of linear equation systems with nonnegative solution. This motivates us to consider solving nonnegative solution of the inhomogeneous system of linear equations.

$$Jx = e, \tag{9}$$

where J is the coefficient matrix with size $m \times n$, e is a nonzero vector.

$$J = \begin{pmatrix} j_{11} & j_{12} & \dots & j_{1n} \\ j_{21} & j_{22} & \dots & j_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ j_{m1} & j_{m2} & \dots & j_{mn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad e = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix}.$$

Taking into account the problem of nonnegative solution of linear system (9), we can transform (9) into an optimization problem, the nonnegative solution of system (9) can be obtained by solving the following (10).

In the sequel, we will show that the optimal solution of optimization problem (10) is the nonnegative solution of linear equation systems (9). Rewrite matrix $J = [J_1, J_2, \cdots, J_m]$, where J_i, $i = 1, 2, \cdots, m$ is the column vector of matrix J. Define function $f(x) = \sum_{i=1}^{m} \left| e_i - J_i^T x \right|$, the solution (or approximation solution) of systems (9) can be obtained by finding the solution of the following optimization problem (10).

$$\begin{aligned} &\min \;\; f(x) \\ &s.t. \;\; x \geq 0, \qquad x \neq 0, \end{aligned} \tag{10}$$

which means find the nonnegative but not zero solution of (10). We formulate the conclusion in the following Proposition 3.

Proposition 3. Existence of the solutions for optimization problem (10) is equivalent to the existence of the nonnegative solutions of the following optimization problem (11).

$$\begin{aligned} &\min \;\; g(y) = (u^T, \theta^T) \begin{pmatrix} t \\ x \end{pmatrix} = \sum_{i=1}^{m} t_i, \\ &s.t. \;\; B \begin{pmatrix} t \\ x \end{pmatrix} \geq \begin{pmatrix} e \\ -e \end{pmatrix}, \quad \begin{pmatrix} t \\ x \end{pmatrix} \geq 0. \end{aligned} \tag{11}$$

where

$$B = \begin{pmatrix} I & J \\ I & -J \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

$$y = \begin{pmatrix} t \\ x \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0_n \end{pmatrix}.$$

The optimal solution of (11) is $y^{(0)} = \begin{pmatrix} t^{(0)} \\ x^{(0)} \end{pmatrix}$, $x^{(0)}$ is the nonnegative solution of optimization (10), moreover, $f(x^{(0)}) = g(y^{(0)}) = \sum_{i=1}^{m} t_i^{(0)}$.

Proof. From (11), we have

$$t \geq e - Jx, \qquad t \geq -(e - Jx), \tag{12}$$

so we have $t_i \geq e_i - J_i^T x$, $t_i \geq -(e_i - J_i^T x)$, such that

$$t_i \geq \left| e_i - J_i^T x \right|, \quad i = 1, 2, \cdots, m. \tag{13}$$

Suppose $y^{(0)} = \begin{pmatrix} t^{(0)} \\ x^{(0)} \end{pmatrix}$ is the solution of (11), then, we will prove that $x^{(0)}$ is the nonnegative solution of (10) by contradiction. If it is not, there must exist a nonnegative vector x^* such that $f(x^*) < f(x^{(0)})$, let

$$t_i^* = \left| e_i - J_i^T x^* \right|, \quad i = 1, 2, \cdots, m, \tag{14}$$

then we have $y^\star = \begin{pmatrix} t^* \\ x^* \end{pmatrix} \geq 0$,

where $t^* = (t_1^*, t_2^*, \cdots, t_m^*)^T$, $x^* = (x_1^*, x_2^*, \cdots, x_m^*)^T$,

$$By^* = \begin{pmatrix} t^* & Jx^* \\ t^* & -Jx^* \end{pmatrix} \geq \begin{pmatrix} e \\ -e \end{pmatrix}, \tag{15}$$

which means y^* satisfies (11) and

$$g(y^*) = \sum_{i=1}^{m} t_i^* = \sum_{i=1}^{m} \left| e_i - J_i^T x^* \right| = f(x^*).$$

Again because $t^{(0)}$ satisfies (13) and $f(x^*) < f(x^{(0)})$, we have

$$g(y^*) = f(x^*) < f(x^{(0)}) = \sum_{i=1}^{n} \left| e_i - J_i^T x^{(0)} \right| < \sum_{i=1}^{n} t_i^{(0)} = g(y^{(0)}),$$

which is contradictory to the fact that $y^{(0)}$ is the solution of (11).

In the sequel, we shall prove

$$t_i^{(0)} = \left| e_i - J_i^T x^{(0)} \right|, \quad i = 1, 2, \cdots, m. \tag{16}$$

If (16) does not hold for all $i = 1, 2, \cdots, m$, from (13), we can deduce that $g(y^{(0)}) = \sum_{i=1}^{m} t_i^{(0)} = \sum_{i=1}^{m} \left| e_i - J_i^T x^{(0)} \right| > f(x^{(0)})$, then similar to the former proof, we derive a contradiction that $y^{(0)}$ is the solution of (11), then we have (16) holds, i.e.

$$g(y^{(0)}) = \sum_{i=1}^{m} t_i^{(0)} = \sum_{i=1}^{m} \left| e_i - J_i^T x^{(0)} \right| = f(x^{(0)}),$$

which completes the proof. □

By means of Proposition 3 and the algebraic conditions in Theorem 1 to Theorem 4, we have the following conclusions to check the positive invariance of a given polyhedral set for linear dynamical systems.

Theorem 5. A polyhedron P given as in (3) is a positively invariant set for the discrete-time linear system (1) if and only if the following optimization problems have solutions for all $i = 1,2,\ldots,m$.

$$\begin{aligned} &\min \quad g(y) = \left(u^T, \theta^T\right) \begin{pmatrix} t \\ x \end{pmatrix} = \sum_{i=1}^{m+1} t_j \\ &s.t. \;\; B \begin{pmatrix} t \\ x \end{pmatrix} \geq \begin{pmatrix} \begin{pmatrix} A_d^T g_i \\ b_i \end{pmatrix} \\ -\begin{pmatrix} A_d^T g_i \\ b_i \end{pmatrix} \end{pmatrix}, \quad \begin{pmatrix} t \\ x \end{pmatrix} \geq 0. \end{aligned} \tag{17}$$

where

$$B = \begin{pmatrix} I_{(m+1)} & \begin{pmatrix} G^T & 0 \\ b^T & 1 \end{pmatrix} \\ I_{(m+1)} & -\begin{pmatrix} G^T & 0 \\ b^T & 1 \end{pmatrix} \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_{(m+1)} \end{pmatrix},$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{(m+1)} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad y = \begin{pmatrix} t \\ x \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0_{(m+1)} \end{pmatrix}.$$

Proof. The conclusion can be obtained according to Proposition 3 and Theorem 4 by replacing the matrices J with $\begin{pmatrix} G^T & 0 \\ b^T & 1 \end{pmatrix}$ and e with $\begin{pmatrix} A_d^T g_i \\ b_i \end{pmatrix}$ for $i = 1,2,\ldots,m$. □

Corollary 5. A polyhedral cone C_P given as in (4) is a positively invariant set for the discrete-time linear system (1) if and only if the following optimization have solutions for all $i = 1,2,\ldots,m$.

$$\begin{aligned} &\min \quad g(y) = \left(u^T, \theta^T\right) \begin{pmatrix} t \\ x \end{pmatrix} = \sum_{j=1}^{m} t_j \\ &s.t. \;\; B \begin{pmatrix} t \\ x \end{pmatrix} \geq \begin{pmatrix} A_d^T g_i \\ -A_d^T g_i \end{pmatrix}, \begin{pmatrix} t \\ x \end{pmatrix} \geq 0. \end{aligned} \tag{18}$$

where

$$B = \begin{pmatrix} I & G^T \\ I & -G^T \end{pmatrix}, \qquad t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix},$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad y = \begin{pmatrix} t \\ x \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0_m \end{pmatrix}.$$

Proof. The conclusion can be obtained according to Proposition 3 and Corollary 4 by replacing the matrices J with G^T and e with $A_d^T g_i$ for $i = 1,2,\ldots,m$.

Denote $I^{(j)} = diag\{1,1,\ldots,0_j,1,\ldots,1\}$, $j = 1,2,\ldots,m$, i.e. $I^{(j)}$ is an identity matrix with the jth diagonal entry replaced by 0 while other entries remained unchanged. □

Theorem 6. A polyhedron P given as in (3) is a positively invariant set for the continuous system (2) if and only if the following optimization problems have solutions for all $i = 1,2,\ldots,m$.

$$\begin{aligned} &\min \quad g(y) = \left(u^T, \theta^T\right) \begin{pmatrix} t \\ x \end{pmatrix} = \sum_{j=1}^{m+1} t_j \\ &s.t. \;\; B \begin{pmatrix} t \\ x \end{pmatrix} \geq \begin{pmatrix} A_c^T g_i \\ b_i \\ -A_c^T g_i \\ -b_i \end{pmatrix}, \quad I^{(m+1+i)} \begin{pmatrix} t \\ x \end{pmatrix} \geq 0. \end{aligned} \tag{19}$$

where

$$B = \begin{pmatrix} I & \begin{pmatrix} G^T & 0 \\ b^T & 1 \end{pmatrix} \\ I & -\begin{pmatrix} G^T & 0 \\ b^T & 1 \end{pmatrix} \end{pmatrix}, \qquad t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_{m+1} \end{pmatrix},$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad y = \begin{pmatrix} t \\ x \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0_{(m+1)} \end{pmatrix}.$$

Proof. The conclusion can be obtained according to Proposition 3 and Theorem 4 by replacing the matrices J with $\begin{pmatrix} G^T & 0 \\ b^T & 1 \end{pmatrix}$ and e with $\begin{pmatrix} A_c^T g_i \\ b_i \end{pmatrix}$ for $i = 1,2,\ldots,m$. The constraints of being a Metzler matrix can be obtained by multiplying matrices $I^{(m+1+i)}$, $i = 1,2,\ldots,m+1$, which means omitting the constraints of the ith variables.

Corollary 6. A polyhedral cone C_P given as in (4) is a positively invariant set for the continuous system (2) if and only if the following optimization problem have solutions for all $i = 1,2,\ldots,m$.

$$\begin{aligned} &\min \quad g(y) = \left(u^T, \theta^T\right) \begin{pmatrix} t \\ x \end{pmatrix} = \sum_{j=1}^{m} t_j \\ &s.t. \;\; B \begin{pmatrix} t \\ x \end{pmatrix} \geq \begin{pmatrix} A_c^T g_i \\ -A_c^T g_i \end{pmatrix}, \qquad I^{(m+i)} \begin{pmatrix} t \\ x \end{pmatrix} \geq 0. \end{aligned} \tag{20}$$

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where

$$B = \begin{pmatrix} I & G^T \\ I & -G^T \end{pmatrix}, \qquad t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix},$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad y = \begin{pmatrix} t \\ x \end{pmatrix}, \qquad \theta = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0_m \end{pmatrix}.$$

Proof. The conclusion can be obtained according to Proposition 3 and Corollary 4 by replacing the matrices J with G^T and e with $A_c^T g_i$ for $i = 1,2,\dots,m$. The constraints of being a Metzler matrix can be obtained by multiplying matrices $I^{(m+i)}$, $i = 1,2,\dots,m$, which means omitting the constraints of the ith variables. □

3. Numerical examples

We illustrate our method using the following examples.

Example 1. (*discrete-time systems with polyhedral set*)
Consider the second order discrete-time linear systems

$$x(k+1) = \begin{pmatrix} -0.32 & 0.32 \\ -0.42 & -0.92 \end{pmatrix} x(k). \tag{21}$$

with the polyhedral set

$$P_1 = \left\{ x \middle| \begin{pmatrix} 1 & 4 \\ -2 & 2 \\ -1 & -4 \\ 2 & -2 \end{pmatrix} x \le \begin{pmatrix} 1 \\ 0.5 \\ 1 \\ 0.5 \end{pmatrix} \right\}.$$

To check whether the given polyhedral set P_1 is an invariant set for the given discrete-time systems (21). It was verified by Example 3 in [7] that this set is a positively invariant set. By using Theorem 1 and Theorem 5 in this paper, we only need solving the LP problems (17) in Theorem 5 with

$$A_d = \begin{pmatrix} -0.32 & 0.32 \\ -0.42 & -0.92 \end{pmatrix}, \quad G = \begin{pmatrix} g_1^T \\ g_2^T \\ g_3^T \\ g_4^T \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ -2 & 2 \\ -1 & -4 \\ 2 & -2 \end{pmatrix},$$

$$b = \begin{pmatrix} 1 \\ 0.5 \\ 1 \\ 0.5 \end{pmatrix},$$

thus obtained the result matrix

$$H = \begin{pmatrix} 0.4640 & 0 & 0 & 0.1680 \\ 1.0720 & 0.5360 & 0 & 0 \\ 0 & 0.1680 & 0.4640 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

that satisfies Theorem 1 such that $H \ge 0$, $HG = GA_d$, $Hb \le b$. Therefore, compared with the method in [7], our method is readily implemented.

Example 2. (*discrete-time systems with polyhedral convex cone*)
Consider the same second order discrete-time linear systems as that in (21)

$$x(k+1) = \begin{pmatrix} -0.32 & 0.32 \\ -0.42 & -0.92 \end{pmatrix} x(k)$$

with

$$C_{P_1} = \left\{ x \middle| \begin{pmatrix} -1 & -4 \\ 2 & -2 \end{pmatrix} x \le \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

To check whether the given polyhedral set C_{P_1} is an invariant set for the given discrete-time systems (21). It was verified by Example 3 in [7] that this convex cone is a positively invariant set. By using Corollary 1 and Corollary 5 in this paper, we only need solving LP problems (18) with

$$A_d = \begin{pmatrix} -0.32 & 0.32 \\ -0.42 & -0.92 \end{pmatrix}, \quad G = \begin{pmatrix} g_1^T \\ g_2^T \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ 2 & -2 \end{pmatrix},$$

thus obtained

$$H = \begin{pmatrix} 0.8400 & 0.5360 \\ 0 & 0.1680 \end{pmatrix}$$

which satisfies Corollary 1 such that $H \ge 0$, $HG = GA_d$. Therefore, by using our method, we can draw the same conclusion.

Example 3. (*continuous-time systems with polyhedral set*)
Consider the second order continuous-time linear systems

$$\dot{x} = \begin{pmatrix} -1 & -3.2 \\ -0.1 & -0.6 \end{pmatrix} x. \tag{22}$$

with

$$P_2 = \left\{ x \middle| \begin{pmatrix} -0.5 & 1 \\ 1 & -8 \\ 0 & 1 \end{pmatrix} x \le \begin{pmatrix} 1.5 \\ 4 \\ 0.5 \end{pmatrix} \right\}.$$

It was verified in [8] Example 1 that this set is a positively invariant set for systems (22). By using Theorem 4 and Theorem 6 in this paper, we only need solving LP problems (19) with

$$A_c = \begin{pmatrix} -1 & -3.2 \\ -0.1 & -0.6 \end{pmatrix}, \quad G = \begin{pmatrix} g_1^T \\ g_2^T \\ g_3^T \end{pmatrix} = \begin{pmatrix} -0.5 & 1 \\ 1 & -8 \\ 0 & 1 \end{pmatrix},$$

$$b = \begin{pmatrix} 1.5 \\ 4 \\ 0.5 \end{pmatrix},$$

thus obtaining

$$H = \begin{pmatrix} -0.7326 & 0.2835 & 0.2476 \\ 0.0337 & -0.0583 & 0.0238 \\ 2.0022 & 0.8504 & -0.6571 \end{pmatrix},$$

which is a Metzler matrix that satisfies Theorem 4 such that $HG = GA_c$, $Hb \le b$. This means that P_2 is a positively invariant set for continuous-time systems (22).

Example 4. (*continuous-time systems with polyhedral convex cone*)

Consider the second order continuous-time linear systems (22) in Example 3 with

$$C_{P_2} = \left\{ x \middle| \begin{pmatrix} -0.5 & 1 \\ 1 & -8 \\ 0 & 1 \end{pmatrix} x \le \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

By using Corollary 4 and Corollary 6, we only need solving the LP problems (20) with

$$A_c = \begin{pmatrix} -1 & -3.2 \\ -0.1 & -0.6 \end{pmatrix}, \quad G = \begin{pmatrix} g_1^T \\ g_2^T \\ g_3^T \end{pmatrix} = \begin{pmatrix} -0.5 & 1 \\ 1 & -8 \\ 0 & 1 \end{pmatrix},$$

thus obtaining the result matrix of LP (20)

$$H = \begin{pmatrix} -0.3634 & 0.3237 & 0.2785 \\ 0.2183 & -0.0382 & 0.0392 \\ 3.1097 & 0.9711 & -0.5645 \end{pmatrix},$$

satisfies Corollary 4, which means that the convex cone C_{P_2} is a positively invariant set for continuous-time systems (22).

To illustrate the advantages of our approach, we present other dynamical systems with higher dimension (order) and solve the LP problems step by step to obtain the matrix H that satisfies Corollary 4.

Example 5. (*continuous-time systems with polyhedral convex cone*)

Consider the continuous-time linear system (23) which is a Wilson system

$$\dot{x} = \begin{pmatrix} 0 & 0 & 0 & -150 \\ 1 & 0 & 0 & -245 \\ 0 & 1 & 0 & -113 \\ 0 & 0 & 1 & -19 \end{pmatrix} x \tag{23}$$

with the same convex polyhedral cone C_{P_2} in Example 4. To check whether C_{P_2} is a positively invariant convex cone for systems (23), by Corollary 4, we need to check whether the matrix H that satisfies (8) exists or not. By Corollary 6 we only need solving some LP problems in (20) with

$$B = \begin{pmatrix} I_4 & G^T \\ I_4 & -G^T \end{pmatrix}, \quad G = \begin{pmatrix} g_1^T \\ g_2^T \\ g_3^T \end{pmatrix} = \begin{pmatrix} -0.5 & 1 & 1 & 2 \\ 1 & -8 & 0 & 1 \\ 0 & 1 & 3 & 7 \end{pmatrix}.$$

The fist LP problem denoted (LP1) is described by

$$\begin{pmatrix} A_c^T g_i \\ -A_c^T g_i \end{pmatrix} = \begin{pmatrix} A_c^T g_1 \\ -A_c^T g_1 \end{pmatrix}, \quad A_c = \begin{pmatrix} 0 & 0 & 0 & -150 \\ 1 & 0 & 0 & -245 \\ 0 & 1 & 0 & -113 \\ 0 & 0 & 1 & -19 \end{pmatrix},$$

where g_1 is the transpose of the first row vector of matrix G. $I^{(4+1)} = diag\{1,1,1,1,0,1,1\}$. The solution of LP1 is $[-2,0,0]$, which forms the first row of matrix H.

The second LP problem, denoted (LP2) in (20), involves the same matrices B, G, A_c and

$$\begin{pmatrix} A_c^T g_i \\ -A_c^T g_i \end{pmatrix} = \begin{pmatrix} A_c^T g_2 \\ -A_c^T g_2 \end{pmatrix},$$

where g_2 is the transpose of the second row vector of matrix G. $I^{(4+2)} = diag\{1,1,1,1,1,0,1\}$. The solution of LP2 is $[95.6170, 39.8085, 222.8511]$, which forms the second row of matrix H.

The third LP problem, denoted (LP3) in (20), involves the same matrices B, G, A_c and

$$\begin{pmatrix} A_c^T g_i \\ -A_c^T g_i \end{pmatrix} = \begin{pmatrix} A_c^T g_3 \\ -A_c^T g_3 \end{pmatrix},$$

where g_3 is the transpose of the third row vector of matrix G. $I^{(4+3)} = diag\{1,1,1,1,1,1,0\}$. The solution of LP3 is $[147.6000, 0, -144.6000]$, which forms the second row of matrix H. These three solutions form matrix

$$H = \begin{pmatrix} -2 & 0 & 0 \\ 95.6170 & 39.8085 & 222.8511 \\ 147.6000 & 0 & -144.6000 \end{pmatrix},$$

and H is a Metzler matrix that satisfies Corollary 4, hence C_{P_2} is a positively invariant set for system (23).

4. Concluding remarks

Positively invariant sets are important both for the system theory and for the computational practice of dynamical systems. In this paper, necessary and sufficient conditions for the positive invariance of a polyhedral set for linear dynamical systems were addressed. Algebraic conditions of the positively invariant set have been transformed into a solution of LP problems with constraints. By solving LP problems, we can check numerically whether given polyhedral sets are positively invariant sets or not. The effectiveness of the proposed method is illustrated by numerical examples. Compared to existing checking methods, the new method is favourable since it does not require verification of algebraic conditions. It only requires solving LP problems numerically, which can be done on any mathematical software. In this aspect, numerical checking method has proven attractive.

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