Stability and stabilization of positive linear dynamical systems: new equivalent conditions and computations

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Abstract. New equivalent conditions of the asymptotical stability and stabilization of positive linear dynamical systems are investigated in this paper. The asymptotical stability of the positive linear systems means that there is a solution for linear inequalities systems. New necessary and sufficient conditions for the existence of solutions of the linear inequalities systems as well as the asymptotical stability of the linear dynamical systems are obtained. New conditions for the stabilization of the resultant closed-loop systems to be asymptotically stable and positive are also presented. Both the stability and the stabilization conditions can be easily checked by the so-called $l$-rank of a matrix and by solving linear programming (LP). The proposed LP has compact form and is ready to be implemented, which can be considered as an improvement of existing LP methods. Numerical examples are provided in the end to show the effectiveness of the proposed method.

Key words: Positive linear systems, stability, stabilization, linear inequalities systems, consistency, $l$-rank.

1. Introduction

Positive systems are dynamical systems whose state variables are positive (or nonnegative) in values at all times. Systems with nonnegative states are important because in practice, the non-negativity property can be found frequently in numerous fields such as biology, chemistry, physics, ecology, economy or sociology (specific examples can be found in e.g. [1–5] and references therein). Since positive systems are defined on cones, not on linear spaces, many well-established results of standard linear systems cannot be applied directly to positive systems. Therefore, positive systems have gained increasing interest over the past two decades due to their extensive applications in practice and theoretical complexes in control theory [1–11]. Some developments and applications in positive systems theory are given in [8].

Stability and stabilization are basic issues of standard linear dynamical systems as well as of positive linear systems. Both issues have attracted considerable attention during the last decades. There have been some significant results on the asymptotical stability and stabilization of positive systems. Various approaches have been proposed such as algebra methods [12, 13], methods based on LMI techniques [9], method based on $l_1$-induced norm [14] etc. Among the available methods, systems of linear inequalities method is an attractive method [15, 16]. In [15], the stability and stabilization conditions of the positive systems are formulated as a linear inequalities systems and the feedback control law can be obtained by solving linear inequalities systems with any standard numerical software. But there are limited discussions available on two problems: (1) On what conditions can one guarantee the existence of the related systems of linear inequalities? (2) The related linear inequalities systems are not standard linear programming, so how to obtain the feedback control law by solving the linear inequalities systems? These two problems are fundamental and there are few papers discussing these two problems. We further investigate these two problems in this paper.

The goal of this paper is to propose new equivalent conditions for the stability and stabilization of discrete-time and continuous-time systems and methods for checking the stability as well as obtaining the feedback control law while maintaining their positivity and stability. To this end, we present new conditions on the consistence and inconsistence of the linear inequalities systems based on Farkas theorem and $l$-rank of the coefficient matrices. Then we extend the results to the positive linear systems. About the solution of the linear inequalities system, we propose a standard linear programming model. The consistence of the linear inequalities systems and and the solvability of the LP are discussed. Numerical checking method is an attractive method which was once used to check the positivity and stability of the dynamical systems because it is easy to implement. We once discussed this method for the positivity and stability checking of singular systems in [17–19]. Numerical examples are given in the end to show the effectiveness of our method. The advantage of the proposed method lies in the fact that they are not only sufficient and necessary, but also can be verified easily with any standard numerical software.

The paper is organized as follows. Section 2 analyzes the stability of the positive systems by means of linear inequalities systems. Equivalent conditions of consistence of the linear inequalities systems as well as the asymptotical stability of the
linear dynamical systems are presented. We also give a numerical solving method for the the linear inequalities systems in Section 2. Section 3 presents the conditions of the stabilization of the positive linear systems and the numerical methods for finding the feedback control law. Examples are given in Section 4 to verify our conclusions. We end this paper in Section 5 by concluding remarks.

Notations: \( R^n_+ \) denotes the nonnegative orthant of \( R^n \). \( A^T \) denotes the transpose of the real matrix \( A \). \( A_{ij} \) denotes the th \( i \times j \) matrix. For a real matrix \( M \) with elements \( m_{ij} \), \( 1 \leq i, j \leq n \), \( M \geq 0 \) means that its elements are positive (i.e., \( m_{ij} > 0 \), \( 1 \leq i, j \leq n \)) and \( M \geq 0 \) means that the elements are nonnegative (i.e., \( m_{ij} \geq 0 \), \( 1 \leq i, j \leq n \)). \( M \) is a Metzler matrix if \( m_{ij} \geq 0 \), \( i \neq j \), which means all the off-diagonal elements are nonnegative if \( M \) is a Metzler matrix, \( I \) is the identity matrix with proper dimension.

2. Stability analysis

2.1. Stability analysis. Consider the following autonomous discrete-time and continuous-time linear systems:

\[
\begin{align*}
  x(k+1) &= A_d x(k), & x(0) &= x_0 \in R^n_+, \quad (1) \\
  \dot{x}(t) &= A_c x(t), & x(0) &= x_0 \in R^n_+, \quad (2)
\end{align*}
\]

where \( A_c, A_d \in R^{n \times n}_+ \), \( x \in R^n \), \( k \in \mathbb{Z}_+ \), \( Z_+ = \{ 0, 1, 2, \ldots \} \), \( t \in [0, +\infty) \). System (1) is a positive system if for any initial conditions \( x_0 \in R^n_+ \), \( x(k) \geq 0 \) for all \( k \in \mathbb{Z}_+ \). System (2) is a positive system if for any initial conditions \( x_0 \in R^n_+ \), \( x(t) \geq 0 \) for any \( t \geq 0 \). A necessary and sufficient condition for the system (1) and (2) to be positive is given by the following Theorem 1.

**Theorem 1.** [5, 20]

(1) System (1) is a positive system for any \( x_0 \in R^n_+ \) if and only if \( A_d \notin R^{n \times n}_+ \).

(2) System (2) is a positive system for any \( x_0 \in R^n_+ \) if and only if \( A_c \) is a Metzler matrix.

**Theorem 2.** [15, 16] Assume that system (1) (or (2)) is positive, or equivalently that the matrix \( A_d \) is positive (or matrix \( A_c \) is a Metzler matrix), then the following statements are equivalent:

(i) System (1) is asymptotically stable for every initial condition \( x_0 \in R^n_+ \) if and only if there exists a \( \lambda \in R^n \) such that

\[
(A_d - I) \lambda < 0, \quad \lambda > 0. \tag{3}
\]

(ii) System (2) is asymptotically stable for every initial condition \( x_0 \in R^n_+ \) if and only if there exists a \( \lambda \in R^n \) such that

\[
A_c \lambda < 0, \quad \lambda > 0. \tag{4}
\]

Rewrite (3) into the form of the following system of linear inequalities:

\[
\begin{pmatrix} A_d - I \\ -I \end{pmatrix} \begin{pmatrix} \lambda \\ 0 \end{pmatrix} < \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5}
\]

For simplicity, denote \( S = \begin{pmatrix} A_d - I \\ -I \end{pmatrix} \) and the set \( \Omega_1 = \{ \lambda | S \lambda < 0, \lambda > 0 \} \) for (3) and \( \Omega_2 = \{ \lambda | A_c \lambda < 0, \lambda > 0 \} \) for (4), both of them have the same form.

**Lemma 1.** [21] (Farkas Lemma) Let \( M \in R^{m \times n}_+ \), \( \lambda \in R^n \). Then exactly one of the following two statements is true:

(i) There exists an \( x \in R^n \) such that \( M \lambda = b \) and \( x \geq 0 \).

(ii) There exists a \( y \in R^m \) such that \( M^T y \geq 0 \) and \( b^T y < 0 \).

**Theorem 3.** Let \( M \in R^{m \times n}_+ \), then exactly one of the following two statements is true:

\[
\begin{align*}
  & (i) \quad M x \leq 0, & (6) \\
  & (ii) \quad M^T y \geq 0, \quad y \geq 0, \quad \text{and} \quad y \neq 0. \tag{7}
\end{align*}
\]

**Proof.** If there exist solutions for (6), i.e., there exists at least a \( \lambda \in R^n \) such that \( M \lambda \geq 0 \), then for all \( \gamma \geq 0, \gamma \neq 0 \), we have \( \gamma^T M \lambda \geq 0 \), i.e., \( \gamma^T M^T \gamma \leq 0 \), which shows that (7) has no solutions otherwise contradicts to the hypotheses. If there exists no solutions for (6), then there exists no \( \alpha < 0, x \in R^n \) such that \( M x \leq (\alpha, \ldots, \alpha) \). Denote \( M \lambda = (M, -e) \), \( \beta = (0, \ldots, 0, -1) \), where \( e = (1, \ldots, 1)^T \in R^n \). Then there exist no \( \alpha < 0, x \in R^n \) such that

\[
\begin{pmatrix} \gamma^T \\ \alpha \end{pmatrix} \leq 0, \quad \begin{pmatrix} \gamma^T \\ \alpha \end{pmatrix} > 0.
\]

From Farkas Lemma 1, there exist solution for \( M \lambda = b, \lambda = 0 \), i.e., there exists solution for \( M^T y = 0, e^T \gamma = 1, y \geq 0 \), which is equivalent to that fact that (7) has solutions.

**Definition 1.** If there exists at least a \( x \in R^n \) such that system (6) holds, then system (6) is called consistent.

**Theorem 4.** System (6) is consistent if and only if system \( M^T y = 0, y \geq 0 \) in (7) has only one solution \( y = 0 \).

**Proof.** From Lemma 1, (6) has solutions if and only if (7) has no solutions, which means if \( M^T y = 0, y \geq 0 \) in (7) has solution, it holds only when \( y = 0 \).

The above alternative Theorem 3 and Theorem 4 give two sufficient and necessary conditions for the consistency of system (6), it is also a sufficient and necessary conditions for the asymptotical stability of systems (1) (or (2)). But the algebraic conditions are not easy to verify. Next we present another sufficient and necessary condition for checking the consistency of system (6) by means of the so-called \( I \)-rank in [22] of the coefficient matrix \( M \) of the linear inequality system. In the sequel, we firstly give a review of the corresponding definitions and results about \( I \)-rank of a given matrix.

**Definition 2.** [22] A real matrix \( M \in R^{m \times n}_+ \) is \( I \)-positive (or \( I \)-negative) with respect to the \( r \)th \((1 \leq r \leq n)\) column if all elements of the \( r \)th column of \( M \) are positive (or negative). In both cases \((I \)-positive or \( I \)-negative), the matrix \( M \) will be said to be \( I \)-definite with respect to the \( r \)th column. A matrix will be said to be \( I \)-positive (or \( I \)-negative, or \( I \)-definite) if it possesses at least a column whose entries are all positive (or negative).
Definition 3. [22] For the real matrix \( M \in \mathbb{R}^{m \times n} \), define the I-minors of \( M \) according to different cases.

1) If the matrix \( M \) is not I-definite with respect to the given \( r \)th column \( r = 1, 2, \ldots, n \), then the elements of that column are divided into three classes:
   (i) those which are positive: \( m_{ir}, i = i_1, i_2, \ldots, i_p \);
   (ii) those which are negative: \( m_{jr}, j = j_1, j_2, \ldots, j_q \);
   (iii) those which are zeros: \( m_{kr}, k = k_1, k_2, \ldots, k_{q'} \);

Define the elements of the matrix \( M_1^{(r)} \) from matrix \( M \) as the following, for any pair of elements \( m_{ir}, m_{jr} \) in the \( r \)th column of matrix \( M \) (\( m_{ir} \) is positive and \( m_{jr} \) is negative), we have the second order determinants:

\[
\begin{vmatrix}
  m_{ir} & m_{ir} & m_{ir} & \cdots & m_{ir} \\
  m_{jr} & m_{jr} & m_{jr} & \cdots & m_{jr} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  m_{ir} & m_{ir} & m_{ir} & \cdots & m_{ir} \\
  m_{jr} & m_{jr} & m_{jr} & \cdots & m_{jr}
\end{vmatrix}
\]

The above row corresponds to one row in matrix \( M_1^{(r)} \), thus we have \( P_N + Z \) rows in the derived matrix \( M_1^{(r)} \). The zero elements \( m_{nr} \) in matrix \( M \) corresponds to one row of the matrix \( M_1^{(r)} \), we have \( Z \) rows in the matrix. Then the size of the matrix \( M_1^{(r)} \) will be \((P_N + Z) \times (n - 1)\). The sequential order of the rows will be determined by the convention that: (1) each \( (ij) \) row precedes every \( (k) \) row; and (2) two \( (ij) \) rows that one precedes which has the smaller \( i \) or (in case of equal \( i \)) that one which has the smaller \( j \).

(2) If the matrix \( M \) is I-definite with respect to the given \( r \)th column, define the derived matrix as the matrix of one row and \( n - 1 \) columns, all elements of which are \(+1\) or \(-1\) according to \( M \) is I-positive or I-negative with respect to the \( r \)th column.

In both cases, whether the matrix \( M \) is not I-definite or I-definite respect to the given \( r \)th column \( (r = 1, 2, \ldots, n) \), we will have \( n \) derived matrices \( M_1^{(r)}(r = 1, \ldots, n) \) with size \((P_N + Z) \times (n - 1)\). The \( n \) matrices \( M_1^1, M_1^2, \ldots, M_1^n \) will be called the I-minors of \( n - 1 \) columns of the matrix \( M \).

Theorem 5. [22] The sufficient and necessary condition of the set \( \Omega_1 \) (or \( \Omega_2 \)) is nonempty is that the I-rank of the matrix \( S \) (or \( A \)) is \( k > 0 \).

Proposition 1. The I-rank of a matrix \( M \) is not altered if

(i) Any two rows or any two columns are interchanged.

(ii) All elements of any row or any column are multiplied by the same positive constant.

Example. Given matrix \( M = \begin{pmatrix} 1 & -1 & 2 \\ -5 & 7 & 0 \\ 2 & -6 & -4 \end{pmatrix} \), we compute I-minors of the matrix as the following.

\[
M_1^{(1)} = \begin{pmatrix} 2 & 10 \\ -16 & -20 \end{pmatrix}, \quad M_1^{(2)} = \begin{pmatrix} 2 & 14 \\ -16 & -28 \end{pmatrix}
\]

\[
M_2^{(3)} = \begin{pmatrix} 8 & -16 \\ 0 & 0 \end{pmatrix}, \quad M_2^{(11)} = \begin{pmatrix} 2 \\ -16 \end{pmatrix}, \quad M_2^{(12)} = \begin{pmatrix} 14 \\ -28 \end{pmatrix}, \quad M_2^{(21)} = \begin{pmatrix} 2 \\ -16 \end{pmatrix}, \quad M_2^{(22)} = \begin{pmatrix} -16 \\ 0 \end{pmatrix}.
\]

From the above I-minors and the definition of the I-rank, the I-rank of the above matrix \( M \) is 0.


To transform the linear inequalities systems into linear equalities systems, we need the following Lemma 2.

Lemma 2. Suppose \( \mathbf{c} = (c_1, \ldots, c_n)^T \in \mathbb{R}^n \), the inequalities \( a_i^T \lambda \leq c_i, i = 1, 2, \ldots, n \) hold if and only if there exists an \( \omega \in \mathbb{R}^s \), \( \omega \geq 0 \), such that \( a_i^T \lambda + \omega = c_i, i = 1, 2, \ldots, m \).

Proof. The result is trivial and obvious but important for our conclusion in the following. If there exists an \( \omega \in \mathbb{R}^s \), \( \omega \geq 0 \) such that \( a_i^T \lambda + \omega = c_i, i = 1, 2, \ldots, n \) hold since \( \omega \geq 0 \). If \( a_i^T \lambda \leq c_i, i = 1, 2, \ldots, n \) hold, there exists \( \omega, i = 1, 2, \ldots, n \) such that \( a_i^T \lambda \leq c_i, i = 1, 2, \ldots, n \), take \( \omega = (\omega_1, \omega, \omega_3, \ldots, \omega_m)^T \), thus we get the result.

Denote \( J^{(1)} = (A, J), x = (\lambda^T, a_0^T)^T \), then the asymptotic stability of system (1) (or (2)) is equivalent to the existence of the nonnegative solution for linear equations systems \( J^{(1)} x = \omega \), for \( \varepsilon < 0 \).
Theorem 7. Let $A \in \mathbb{R}^{m \times n}$. The positive system (1) (or system (2)) is asymptotically stable if and only if for any $\varepsilon \in \mathbb{R}^n$, $\varepsilon < 0$, linear equations system

\[
(A, I_n) \begin{pmatrix} \lambda \\ \omega \end{pmatrix} = \varepsilon
\]  

(8)

has nonnegative solution $\begin{pmatrix} \lambda \\ \omega \end{pmatrix}$, i.e. the linear equations system is consistent in nonnegative variables $\begin{pmatrix} \lambda \\ \omega \end{pmatrix}$ (for system (1), $A$ takes $A_d - I$ and for system (2), $A$ takes $A_c$).

Proof. The results can be obtained by Lemma 2 and Theorem 2 directly if we take $(A, I_n)$ as the coefficient matrix and $\begin{pmatrix} \lambda \\ \omega \end{pmatrix}$ as the unknown, then we have

\[
J^{(1)} x = \varepsilon,
\]  

(9)

and then from the matrix equation and linear equalities systems we obtain the result of Theorem 7.

Taking into account the problem of nonnegative solution of linear system (9), we can transform (9) into an optimization problem, the nonnegative solution of system (9) can be obtained by solving the following optimization (10).

The nonnegative solution of systems (10) can be obtained by solving the following optimization problem (11). The optimal solution of optimization (11) is the nonnegative solution of system (10), moreover, $f(x^{(0)}) = g(y^{(0)}) = \sum_{i=1}^{m} t_i^{(0)}$.

Proof. From (9) and (11), we have

\[
t \geq \varepsilon - J^{(1)} x, \quad t \geq -(\varepsilon - J^{(1)} x),
\]  

(12)

thus we have $t_i \geq \varepsilon_i - (J^{(1)})_i^T x$, $t_i \geq -(\varepsilon_i - (J^{(1)})_i^T x)$, such that

\[
t_i \geq |\varepsilon_i - (J^{(1)})_i^T x|, \quad i = 1, 2, \ldots, m.
\]  

(13)

Suppose $y^{(0)} = \begin{pmatrix} \theta^{(0)} \\ x^{(0)} \end{pmatrix}$ is the solution of (11), then, we will prove that $x^{(0)}$ is the nonnegative solution of (10) by contradiction. If it is not, there must exist a nonnegative vector $x^*$ such that $f(x^*) < f(x^{(0)})$, let

\[
t_i^* = |\varepsilon_i - (J^{(1)})_i^T x^*|, \quad i = 1, 2, \ldots, m.
\]  

(14)

then we have

\[
y^* = \begin{pmatrix} t^* \\ x^* \end{pmatrix} \geq \begin{pmatrix} \varepsilon \\ -\varepsilon \end{pmatrix},
\]  

(15)

which means $y^*$ satisfies (11) and

\[
g(y^*) = \sum_{i=1}^{m} t_i^* = \sum_{i=1}^{m} |\varepsilon_i - (J^{(1)})_i^T x^*| = f(x^*).
\]

Again because $y^{(0)}$ satisfies (13) and $f(x^{(0)}) < f(x^{(0)})$, we have

\[
g(y^*) = f(x^*) < f(x^{(0)}) = \sum_{i=1}^{m} |\varepsilon_i - (J^{(1)})_i^T x^{(0)}| < \sum_{i=1}^{m} t_i^{(0)} = g(y^{(0)}),
\]

which is contradictory to the fact that $y^{(0)}$ is the solution of (11).

In the sequel, we shall prove

\[
t_i^{(0)} = |\varepsilon_i - (J^{(1)})_i^T x^{(0)}|, \quad i = 1, 2, \ldots, n.
\]  

(16)

If (16) does not hold for all $i = 1, 2, \ldots, m$, from (13), we can deduce that

\[
g(y^{(0)}) = \sum_{i=1}^{m} t_i^{(0)} = \sum_{i=1}^{m} |\varepsilon_i - (J^{(1)})_i^T x^{(0)}| > f(x^{(0)}),
\]
then similar to the former proof, we derive a contradiction that \( y^{(0)} \) is the solution of (11), then we have (16) holds, i.e.

\[
g(y^{(0)}) = \sum_{i=1}^{n} t_i^{(0)} = \sum_{i=1}^{n} |e_i - (J^{(1)}_i)^T x^{(0)}| = f(x^{(0)})
\]

which completes the proof. □

From Theorem 7 and Proposition 2, we have the following Theorem 8 which is a necessary and sufficient condition of the asymptotically stability for linear systems (1) and (2).

**Theorem 8.** The positive system (1) (or system (2)) is asymptotically stable if and only if for any \( \epsilon \in \mathbb{R}^+, \epsilon < 0 \), the following optimization has a solution.

\[
\min \ g(y) = (u^T, \theta^T) \begin{pmatrix} t \\ x \end{pmatrix} = \sum_{i=1}^{m} t_i
\]

s.t. \( B \begin{pmatrix} t \\ x \end{pmatrix} \geq \begin{pmatrix} \epsilon \\ -\epsilon \end{pmatrix}, \begin{pmatrix} t \\ x \end{pmatrix} \geq 0 \)

where

\[
B = \begin{pmatrix} I & f^{(1)} \\ I & -f^{(1)} \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},
\]

\[
y = \begin{pmatrix} t \\ x \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

**Remark 1.** From Theorem 8, the autonomous positive system (1) (or (2)) is asymptotically stable while guaranteeing the positivity if and only if the corresponding optimization problems (17) has solutions. Theorem 8 shows that the asymptotical stability of the system (1) (or (2)) can be checked by the existence of solution for the LP optimization (17). It is an attractive method for checking the asymptotic stability of the positive linear dynamical systems.

3. Stabilization of the positive linear dynamical systems

3.1. Equivalent conditions of stabilization. This section studies the stabilization problem of the positive linear dynamical systems. In the following forced system:

\[
x(k+1) = A_d x(k) + B_d u(k), \quad x(0) = x_0 \in \mathbb{R}^n_+ \quad \text{for (18)}
\]

\[
\dot{x}(t) = A_c x(t) + B_c u(t), \quad x(0) = x_0 \in \mathbb{R}^n_+ \quad \text{for (19)}
\]

The control law \( u \in \mathbb{R}^p \) is assumed to be a constant state-feedback \( u(k) = K_d x(k) \) for (18) \( u(t) = K_c x(t) \) for (19), where \( K_d \) (or \( K_c \)) is the gain matrix. The control law must be designed in such way that the resulting governed system is positive and asymptotically stable. The stability synthesis of the systems (18) and (19) can be obtained by the following Theorem 11 and Theorem 12.

**Definition 5.** [20]

(1) The discrete-time (18) is called (internally) positive systems if \( x(k) \in \mathbb{R}^n_+ \), \( k \in \mathbb{Z}_+ \) for any initial conditions \( x_0 \in \mathbb{R}^n_+ \) and all inputs \( u(k) \in \mathbb{R}^m_+ \), \( k \in \mathbb{Z}_+ \).

(2) The continuous-time (19) is called (internally) positive systems if \( x(t) \in \mathbb{R}^n_+ \), \( t \geq 0 \) for any initial conditions \( x_0 \in \mathbb{R}^n_+ \) and all inputs \( u(t) \in \mathbb{R}^m_+, t \geq 0 \).

**Theorem 9.** [5, 20] For the systems (18) and (19), the positivity conditions of the systems are as the following:

(1) The systems (18) is a (internally) positive systems for any \( x_0 \in \mathbb{R}^n_+ \) if and only if \( A_d \in \mathbb{R}^{n \times n}_+ \), \( B_d \in \mathbb{R}^{n \times m}_+ \).

(2) The systems (19) is a (internally) positive systems for any \( x_0 \in \mathbb{R}^n_+ \) if and only if \( A_c \) is a Metzler matrix and \( B_c \in \mathbb{R}^{n \times m}_+ \).

**Theorem 10.** [15] For the positive discrete-time system (18), the following statements are equivalent:

(1) There exist \( n + 1 \) vectors \( d = (d_1, d_2, \ldots, d_n)^T \in \mathbb{R}^p \) and \( z_1, z_2, \ldots, z_n \in \mathbb{R}^p \) such that

\[
(A_d - I) d + B_d \sum_{i=1}^{n} z_i < 0; \quad d > 0;
\]

\[
a_{ij} d_j + b_{ij} z_j \geq 0, \quad 1 \leq i, j \leq n.
\]

with \( B_d = \begin{pmatrix} (b_{11})^T, (b_{12})^T, \ldots, (b_{1n})^T \end{pmatrix} \), \( b_{ij} = \begin{pmatrix} b_{ij} \end{pmatrix} \), \( i = 1, 2, \ldots, n \) denotes the \( i \)-th row of matrix \( B_d \) in system (18).

(2) There exists a state-feedback law \( u(k) = K x(k) \) such that the closed-loop system is positive and asymptotically stable, where \( K \) can be calculated as follows:

\[
K = (d_1^{-1} z_1, d_2^{-1} z_2, \ldots, d_n^{-1} z_n)
\]

For the positive continuous-time systems (19), we have the following analogous Theorem 11 as Theorem 10 for discrete-time systems.

**Theorem 11.** [16] For the positive continuous-time systems (19), the following statements are equivalent:

(1) There exist \( n + 1 \) vectors \( d = (d_1, d_2, \ldots, d_n)^T \in \mathbb{R}^p \) and \( z_1, z_2, \ldots, z_n \in \mathbb{R}^p \) such that

\[
A_c d + B_c \sum_{i=1}^{n} z_i < 0; \quad d > 0;
\]

\[
a_{ij} d_j + b_{ij} z_j \geq 0, \quad 1 \leq i, j \leq n, \quad i \neq j.
\]

with \( B_c = \begin{pmatrix} (b_{11})^T, (b_{12})^T, \ldots, (b_{1n})^T \end{pmatrix} \), \( b_{ij} = \begin{pmatrix} b_{ij} \end{pmatrix} \), \( i = 1, 2, \ldots, n \) denotes the \( i \)-th row of matrix \( B_c \) in system (19).

(2) There exists a state-feedback law \( u(t) = K x(t) \) such that the closed-loop system is positive and asymptotically stable,
where $K$ can be calculated as follows:

$$ K = (d_1^{-1}z_1, d_2^{-1}z_2, \ldots, d_n^{-1}z_n). $$

The proof in above Theorem 11 and Theorem 12 can be obtained by replacing $A_d$ in system (1) with $A_d + B_dKd$ and let $B_dKd = B_d \sum_{i=1}^{n} z_i$ (for continuous-time system by replacing $A_c$ in system (2) with $A_c + B_cKd$ and let $B_cKd = B_c \sum_{i=1}^{n} z_i$), detailed proofs can be found in reference [15] and [16]. In the sequel, we derive the new conditions according to these two theorems.

We can rewrite the form of (20) in Theorem 11 as the following:

$$\begin{pmatrix}
A_d - I & B_d & \cdots & B_d \\
-B_d & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{pmatrix}
\begin{pmatrix}
(a_d)_{11} \\
(a_d)_{12} \\
\vdots \\
(a_d)_{1n} \\
(a_d)_{21} \\
\vdots \\
(a_d)_{2n} \\
\vdots \\
(a_d)_{n1} \\
\vdots \\
(a_d)_{nn} \\
\end{pmatrix}
\begin{pmatrix}
d_1 \\
d_2 \\
\vdots \\
d_n \\
z_1 \\
\vdots \\
z_n \\
\end{pmatrix}
< 0; \quad (22)$$

$$\begin{pmatrix}
(a_d)_{11} \\
(a_d)_{12} \\
\vdots \\
(a_d)_{1n} \\
(a_d)_{21} \\
\vdots \\
(a_d)_{2n} \\
\vdots \\
(a_d)_{n1} \\
\vdots \\
(a_d)_{nn} \\
\end{pmatrix}
\begin{pmatrix}
-0.1 \\
0 \\
\vdots \\
0.1 \\
-0.1 \\
\vdots \\
0.1 \\
\vdots \\
-0.1 \\
\vdots \\
0.1 \\
\end{pmatrix}
\begin{pmatrix}
d_1 \\
d_2 \\
\vdots \\
d_n \\
z_1 \\
\vdots \\
z_n \\
\end{pmatrix}
< 0. \quad (23)$$

It should be kept in mind that in the above equation (22), the size of the coefficient matrix in (25) is $(2n) \times (n + np)$, and the variable $\omega = (d_1, d_2, \ldots, d_n, z_1, z_2, \ldots, z_n)^T$ is $(n + np)$ dimensional, in brevity, we write the coefficient matrix as $M_d$ and $R_d$ in (22), (23). The size of the coefficient matrix $R_d$ is $n^2 \times (n + np)$ dimensional and the variable is the same as that in (22), we rewrite the coefficient matrix as $R_d$. Then we can reformulate the equations (22) and (23) as:

$$\Omega_3 = \{ \omega, \omega \in R^{(p+1)n} \mid M_d \omega < 0, \quad R_d \omega \leq 0 \}. \quad (24)$$

We can also rewrite the form of (21) in Theorem 12 as the following:

$$\begin{pmatrix}
A_c & B_c & \cdots & B_c \\
-I & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{pmatrix}
\begin{pmatrix}
(a_c)_{11} \\
(a_c)_{12} \\
\vdots \\
(a_c)_{1n} \\
(a_c)_{21} \\
\vdots \\
(a_c)_{2n} \\
\vdots \\
(a_c)_{n1} \\
\vdots \\
(a_c)_{nn} \\
\end{pmatrix}
\begin{pmatrix}
d_1 \\
d_2 \\
\vdots \\
d_n \\
z_1 \\
\vdots \\
z_n \\
\end{pmatrix}
< 0; \quad (25)$$

The size of the coefficient matrix in (25) is $(2n) \times (n + np)$, and the variable $\omega = (d_1, d_2, \ldots, d_n, z_1, z_2, \ldots, z_n)^T$ is $(n + np)$ dimensional, in brevity, we write the coefficient matrix as $M_c$ and $R_c$ in (25), (26). The size of the coefficient matrix $R_c$ is $(n - 1)^2 \times (n + np)$ dimensional and the variable is the same as that in (25), we rewrite the coefficient matrix as $R_c$. Then we can reformulate the equations (25) and (26) as:

$$\Omega_4 = \{ \omega, \omega \in R^{(p+1)n} \mid M_c \omega < 0, \quad R_c \omega \leq 0 \}. \quad (27)$$

**Lemma 3.** [23] (Motzkin, 1936) Let $A, C$ and $D$ be given matrices, with $A$ being nonvacuous. Then exactly one of the following is true.

(i) $Ax > 0$, $
Cx \geq 0$, $Dx = 0$ has solution.

(ii) $A^T y_1 + C^T y_2 + D^T y_3 = 0$, $y_1 \geq 0$, $y_3 \geq 0$, $y_3 \neq 0$ has a solution $y_1, y_2, y_3$.

**Lemma 4.** [24] Let $A, B$ be given matrices, with $A$ being nonvacuous. Then exactly one of the following is true.

(i) $Ax > 0$, $Bx \geq 0$ has a solution $x$.

(ii) $A^T y_1 + B^T y_2 = 0$, $y_1 \geq 0$, $y_2 \geq 0$, $y_2 \neq 0$ has a solution $y_1, y_2$.

**Proof.** This lemma can be obtained by taking $D = 0$ and replace $A, B$ in the above Motzkin Lemma 3 as $-A, -B$. □

**Theorem 12.** The existence condition of a state-feedback law $u_d(k) = K_d x(k)$ for discrete-time systems (18) $(u_c(t) = K_c x(t)$ for continuous-time systems (19)) while guaranteeing the closed-loop system is positive and asymptotically stable, where $K_d$ (or $K_c$) can be calculated as in Theorem 11(2) (or in Theorem 12(2)) if and only if the I-rank of matrix $J^2 = \begin{pmatrix} M_d \\ R_d \end{pmatrix}$ (or $J^3 = \begin{pmatrix} M_c \\ R_c \end{pmatrix}$) is greater than 0.

**Proof.** From (2) in Theorem 11 (or Theorem 11 for continuous-time systems), we know that the existence condition of a state-feedback law $u_d(k) = K_d x(k)$ for discrete-time systems (18) $(u_c(t) = K_c x(t)$ for continuous-time systems (19)) is equivalent to the condition of (20) for discrete-time systems (or (21) for continuous-time systems (19)), which is equivalent to the nonemptiness of the set $\Omega_3$ (or $\Omega_4$) in (24), and in (27) for continuous-time systems (19). From Theorem 5, the nonempty
of the set $\Omega_3$ (or $\Omega_4$) is equivalent to the $I$-rank of matrix $J^{(2)}$ (or $J^{(3)}$) is greater than 0, which completes the proof. □

3.2. Numerical solving methods. From (23) and (26), we can see that the coefficient matrices don’t have the compact form and are not readily implemented when the state vector is large. Analogous to the above Theorem 8, we have the following theorem to compute the feedback matrix $K$, the methods here have compact form and are ready to be implemented.

Denote $J^{(2)} = \left( \begin{array}{c} M \\ R \end{array} \right)$, $x = \omega$, $e' = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$. By means of Proposition 3, we have

**Theorem 13.** For the positive systems (18), the following statements are equivalent:

1. There exists a nonnegative solution for linear equations systems $J^{(2)}x = e'$, the nonnegative solution can be obtained by solving the following optimization problem.

$$
\min g(y) = (u^T, \theta^T) \begin{pmatrix} t \\ x \end{pmatrix} = \sum_{i=1}^{m} t_i
$$

s.t.

$$
B \begin{pmatrix} t \\ x \end{pmatrix} \geq \begin{pmatrix} -e' \\ -e' \end{pmatrix}, \quad \begin{pmatrix} t \\ x \end{pmatrix} \geq 0,
$$

where $B, u, t, x, \theta$ are as that in (17). The optimal solution of (28) is $y^{(0)} = \begin{pmatrix} t^{(0)} \\ x^{(0)} \end{pmatrix}$, $x^{(0)}$ is the nonnegative solution of systems $J^{(2)}x = e'$, where

$$
B = \begin{pmatrix} I & J^{(2)} \\ I & -J^{(2)} \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix},
$$

$$
u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad y = \begin{pmatrix} t \\ x \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
$$

2. There exists a state-feedback law $u(k) = Kx(k)$ such that the closed-loop system is positive and asymptotically stable, where $K$ can be calculated as follows:

$$
K = \left( d_1^{-1}z_1, d_2^{-1}z_2, \ldots, d_n^{-1}z_n \right),
$$

where $d_i, z_i, i = 1, 2, \ldots, n$ can be obtained from the solution $x^*$ of the optimization (28).

**Theorem 14.** For the positive systems (19), the following statements are equivalent:

1. There exists a nonnegative solution for linear equations systems $J^{(3)}x = e'$, the nonnegative solution can be obtained by solving the following optimization problem.

$$
\min g(y) = (u^T, \theta^T) \begin{pmatrix} t \\ x \end{pmatrix} = \sum_{i=1}^{m} t_i
$$

s.t.

$$
B \begin{pmatrix} t \\ x \end{pmatrix} \geq \begin{pmatrix} -e' \\ -e' \end{pmatrix}, \quad \begin{pmatrix} t \\ x \end{pmatrix} \geq 0,
$$

where $B, u, t, x, \theta$ are as that in (17). The optimal solution of (29) is $y^{(0)} = \begin{pmatrix} t^{(0)} \\ x^{(0)} \end{pmatrix}$, $x^{(0)}$ is the nonnegative solution of systems $J^{(3)}x = e'$, where

$$
B = \begin{pmatrix} I & J^{(3)} \\ I & -J^{(3)} \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},
$$

$$
y = \begin{pmatrix} t \\ x \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
$$

2. There exists a state-feedback law $u(k) = Kx(k)$ such that the closed-loop system is positive and asymptotically stable, where $K$ can be calculated as follows:

$$
K = \left( d_1^{-1}z_1, d_2^{-1}z_2, \ldots, d_n^{-1}z_n \right),
$$

where $d_i, z_i, i = 1, 2, \ldots, n$ can be obtained from the solution $x^*$ of the optimization (11).

**Remark 2.**

1. The results of Theorem 13 and Theorem 14 are from Theorem 8 and Theorem 12 directly, we omit the proof here for brevity.

2. From Theorem 8 and Theorem 12 (or Theorem 13), the closed-loop system (18) (or (19)) is asymptotically stable while guaranteeing the positivity if and only if the corresponding LP optimization problems (28) (or (29)) has solutions.

4. Examples

In this section, two simple examples are given to illustrate the effectiveness of methods proposed in this paper. We present three methods to check the asymptotic stability of the given positive linear dynamical systems.

**Example 1.** Consider the discrete-time linear system

$$x(k+1) = A_dx(k) + B_du(k),$$

...
where

\[
A_d = \begin{pmatrix}
0.5 & 0 & 0.6 \\
0.6 & 0.8 & 1.2 \\
0.8 & 1 & 0.8
\end{pmatrix}, 
B_d = \begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}.
\]

From Theorem 10, the linear system is positive. The \(I\)-minors of matrix \((A_d - I)\) are:

\[
M_1^{(1)} = \begin{pmatrix}
-0.1 & -0.2 & 0.5 & 1 & 0 & 0 \\
0.96 & -1.2 & 0.38 & -0.2 & 0 & 0
\end{pmatrix}^T,
M_1^{(2)} = \begin{pmatrix}
0.76 & 0.8 & 0 & 0 & 0 \\
1.16 & -0.2 & 0 & 0 & 0
\end{pmatrix}^T,
M_1^{(3)} = \begin{pmatrix}
0.38 & -0.5 & 1.08 & 0.6 & 0 & 0 \\
0.6 & -0.5 & 1.16 & -0.2 & 0 & 0
\end{pmatrix}^T.
\]

it is obvious that the \(I\)-rank of matrix \((I - A_d I)\) is 0. From Theorem 2, the autonomous discrete-time system is not asymptotically stable. The stability result can also be obtained from Theorem 6, the autonomous linear system is not asymptotically stable since the spectrum of \(\sigma(A_d) = (2.1458, 0.3542, -0.4)\), \(A_d\) has the unstable eigenvalue \(\lambda_1 = 2.1458\) we know that the linear system is not asymptotically stable. We obtained the gain matrix of the closed loop system (17) by Theorem 13 in this paper with \(e' = -(0.1, 0.1, 0.1)^T\), the solution \((d_1, d_2, d_3, z_1, z_2, z_3)^T = (580.5420, 394.1752, 392.3399, -300.3974, -266.4273, -266.7070)^T\), then \(K_d = (-0.5174, -0.6759, -0.6798)\). The closed-loop systems is positive and asymptotically stable since

\[
A_d + B_d K_d = \begin{pmatrix}
0.5000 & 0.0000 & 0.6000 \\
0.0826 & 0.1241 & 0.5202 \\
0.2826 & 0.3241 & 0.1202
\end{pmatrix},
\]

and the eigenvalues are \(\lambda_1 = 0.8674, \lambda_2 = 0.2517, \lambda_3 = -0.3748\).

**Remark 3.** Throughout the paper, output stabilization refers to output positively asymptotically stabilization. Note, the definition of stabilization ensures that stabilization is regardless of the control input (positive or negative), the plant will maintain nonnegativity of states for all initial conditions \(x_0 \in R^n\) of the plant (due to the Metzler property (Theorem 1 and Theorem 10)). It is worth pointing out that in “real life systems”, nonnegativity of states occurs quite often; however, the need for the input \(u\) to be also nonnegative, as in the original definition (Definition 5) may not always be a necessity, as was also pointed out in [16, 25]. Thus, throughout this paper, we do not restrict ourselves to nonnegative inputs, as it was shown in [26] that such a restriction breaks down the possibility of stabilization.

**Example 2.** Consider the continuous-time linear system

\[
\dot{x}(t) = A_c x(t) + B_c u(t),
\]

where

\[
A_c = \begin{pmatrix}
1 & 1 & 1 \\
1 & -2 & 2 \\
2 & 1 & 3
\end{pmatrix}, 
B_c = \begin{pmatrix}
1 & 1 & 1 \\
2 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
\]

From Theorem 9, the linear system is positive and the spectrum of \(A_c\) are \(\sigma(A_c) = (4.2974, -0.7148, -1.7913 + 0.8353i, -1.7913 - 0.8353i)\), the autonomous system is unstable since \(A_c\) has unstable eigenvalue \(\lambda = 4.2974\). As that in Example 1, also we can obtain the result from the \(I\)-rank of matrix \((A - I)\) according to Theorem 2(ii) and Theorem 6 in this paper, we have the same conclusion about the asymptotic stability since the \(I\)-rank is 0. By Theorem 14, take \(e' = -1.0 e - 01 * (1, 1, 1, 1)^T\), we obtained the solution

\[
(d_1, d_2, d_3, z_1, z_2, z_3) = (502.5335, 120.3769, 45.1729, 119.9020, -223.9135, 31.5827, -692.3870, -38.0045, -46.8862, 33.2171, -14.7687, 36.5199, -478.8018, 47.8386, -237.6350, 129.2268),
\]

compute the feedback control law

\[
K_c = \begin{pmatrix}
-0.4456 & -0.3157 & -0.3269 & 0.3990 \\
0.0628 & -0.3895 & 0.8084 & -1.9819 \\
-1.3778 & 0.2759 & -105993 & 1.0778
\end{pmatrix}
\]

and

\[
A + BK_c = \begin{pmatrix}
-0.3827 & 0.2948 & 1.4815 & 0.4171 \\
0.1089 & -2.6314 & 1.3461 & 0.7980 \\
0.2395 & 0.5707 & -7.1178 & 0.4948 \\
0.0628 & 1.6105 & 0.8084 & -2.9819
\end{pmatrix}
\]

with the spectrum

\[
\sigma(A_c + BK_c) = (-7.3582, -0.2334, -1.5541, -3.9681)
\]

which guarantee the positivity and asymptotic stability of the closed-loop system according to Theorem 14 in this paper.

**5. Concluding remarks**

New equivalent conditions of the asymptotic stability and stabilization of positive linear dynamical systems were studied in this paper. The problems of stability and the stabilization for the positive continuous-time and discrete-time linear systems are transformed into consistency conditions of the algebraic linear inequalities systems. We presented some conditions of
This paper is partly supported by the numerical examples are given in the end to show the effectiveness especially when the dimension of the state vector is large. A controller with compact form, which is easy to implement, approaches in the literature, our design approach constructs improvement of LP method in [15, 16]. Compared with existing LP approaches in the literature, our design approach constructs a controller with compact form, which is easy to implement, especially when the dimension of the state vector is large. Numerical examples are given in the end to show the effectiveness of our method.

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REFERENCES


