Theoretical Elements in Fourier Analysis of q-Gaussian Functions

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Abstract There is a consensus in signal processing that the Gaussian kernel and its partial derivatives enable the development of robust algorithms for feature detection. Fourier analysis and convolution theory have a central role in such development. In this paper, we collect theoretical elements to follow this avenue but using the q-Gaussian kernel that is a nonextensive generalization of the Gaussian one. Firstly, we review the one-dimensional q-Gaussian and its Fourier transform. Then, we consider the two-dimensional q-Gaussian and we highlight the issues behind its analytical Fourier transform computation. In the computational experiments, we analyze the q-Gaussian kernel in the space and Fourier domains using the concepts of space window, cut-off frequency, and the Heisenberg inequality.

Keywords q-Gaussian kernel, signal processing

1 Introduction

Feature extraction is an essential step for image analysis and computer vision tasks such as image matching and object recognition [1]. The specific case of edge detection has been extensively considered in the image processing literature [2]. In this subject, the Gaussian kernel and its partial derivatives have inspired a wide range of works in the image analysis community for the development of multiscale approaches [3].

These works had established the background for multiscale representation based on the viewpoint of the functional structure of digital images [4, 5]. Basically, the grayscale of the observed image is realized as a general function f of the space of square integrable functions on $\mathbb{R}^2$, denoted

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by $L^2 \left(\mathbb{R}^2 \right)$. The linear scale-space is generated by the convolution with scaled Gaussian kernels with filtering properties analyzed in the frequency space given by the Fourier transform. Such approach can be seen from the isotropic diffusion equation viewpoint, which opens the possibility of generating more general multiscale representations based on the anisotropic diffusion equation [6].

The Gaussian kernel also plays a fundamental role in statistical physics due to the fact that an enormous amount of phenomena in nature follow the Gaussian distribution and the extensive thermostatistics. The latter is governed by the Boltzmann-Gibbs entropy and the standard central limit theorem [7]. More recently, Tsallis nonextensive entropy and generalizations of the central limit theorem give the foundations for nonextensive counterparts of the Boltzmann-Gibbs statistical mechanics [8,9]. In this context, a generalization of Gaussian kernel within the Tsallis nonextensive scenario, named q-Gaussian, has been proposed [10] and applied for smoothing and edge detection [11,12]. These works demonstrate the potential of the q-Gaussian based methods by comparing their results with the ones obtained with traditional techniques that rely on the Gaussian function. However, the behaviour of the q-Gaussian in the frequency domain has been ignored.

In this paper, we collect theoretical elements, published in the references [13–15], to perform such analysis. We review the q-exponential function and the q-Gaussian distribution. Then, we offer details of the Fourier transform computation of the 1D q-Gaussian. The two-dimensional q-Gaussian and its Fourier transform are also considered from the analytical viewpoint. In the experimental results, we analyze the q-Gaussian in the frequency domain and compare its profile with the Gaussian one. In fact, we consider the Fourier transform of the one-dimensional q-Gaussian which emphasizes the fact that a q-Gaussian is a low-pass filter. We study the size of the space window and analyze the influence of the parameter q in the cut-off frequency and the Heisenberg inequality.

The paper is organized as follows. Sec. 2 focuses on Tsallis entropy, the q-exponential, q-Gaussian, and summarizes some of their basic properties. The Fourier transform computation of the q-Gaussian is discussed in Sec. 3 and 4. The computational results are presented in Sec. 5. Next, Sec. 6 discusses challenges of q-Gaussian and its Fourier analysis for signal processing, as well as possible directions to address some drawbacks of existing works in this area. Then, the conclusions are offered in Sec. 7. The Appendices A–D complete the material with details about the q-Gaussian and its Fourier transform. For details about the special Gamma (Γ), Whittaker (W), Bessel ($\mathbb{J}$) and Beta (B) functions, used along the text, we direct the reader to the references [16,17].

2   Tsallis Entropy and q-Gaussian

In the last decades, Tsallis [9] has proposed the following generalized nonextensive entropic form

$$S_q = k \frac{1 - \sum_{i=1}^{\eta} p_i^q}{q - 1}, \tag{1}$$

where k is a positive constant, p_i is a probability distribution, $q \in \mathbb{R}$ is called the entropic index, or q-index also, and η is the number of possible states of the system. This expression recovers the Shannon entropy in the limit $q \to 1$. The Tsallis entropy offers a new formalism in which the

real parameter q quantifies the level of nonextensivity of a physical systems [18]. In particular a general principle of maximum entropy (PME) has been considered to find out the distribution p_i to describe such systems. In this PME, the goal is to find the maximum of S_q subjected to

$$\sum_{i=1}^{\eta} p_i = 1 \tag{2}$$

and

$$\frac{\sum_{i=1}^{\eta} e_i p_i^q}{\sum_{i=1}^{\eta} p_i^q} = U_q, \tag{3}$$

where U_q is a known application dependent value, e_i represent the possible states of the system (in image processing, the gray-level intensities), and η is the number of system states. Eq. (2) is just a necessary condition for p_i to be probability and Eq. (3) is a generalized expectation value of the e_i (if $q = 1$ we get the usual mean value). The proposed PME can be solved using Lagrange multipliers and the solution has the form [18, 19]

$$p_j = \frac{\left[1 - (1 - q)\widetilde{\beta} e_j\right]^{\frac{1}{1-q}}}{\widetilde{Z}_q}, \tag{4}$$

where $\widetilde{\beta}$ and $\widetilde{Z}_q$ are defined by the expressions

$$\widetilde{\beta} = \frac{\beta}{\sum_{j=1}^{\eta} p_j^q + (1 - q)\beta U_q}, \tag{5}$$

$$\widetilde{Z}_q = \sum_{j=1}^{\eta} \left[1 - (1 - q)\widetilde{\beta} e_j\right]^{\frac{1}{1-q}}, \tag{6}$$

with β being the Lagrange multiplier associated with the constraint given by Eq. (3) and, if $q < 1$, then $p_i = 0$ whenever $1 - (1 - q)\widetilde{\beta} e_j < 0$ (cut-off condition). The Eq. (1) and (4) inspire the definition of the q-exponential function [20]

$$\exp_q(x) = \begin{cases} [1 + (1 - q)x]^{\frac{1}{1-q}}, & \text{if } 1 + (1 - q)x > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

It can be shown that the traditional exponential function exp is given by the limit

$$\exp(x) = \lim_{q \to 1} \exp_q(x). \tag{8}$$

The Eq. (8) motivates the definition of the d-dimensional q-Gaussian as [15]

$$G_{d,q}(\mathbf{x}, \Sigma, \beta) = C_{d,q}(\Sigma, \beta) \exp_q\left(-\beta \mathbf{x}^{\mathbf{T}} \Sigma^{-1} \mathbf{x}\right), \tag{9}$$

where

$$C_{d,q}(\Sigma, \beta) = \left(\int_{\mathbb{R}^d} \exp_q\left(-\beta \mathbf{x}^{\mathbf{T}} \Sigma^{-1} \mathbf{x}\right) d\mathbf{x}\right)^{-1}, \tag{10}$$

Σ is the covariance matrix (symmetric and positive definite), and β is a positive real parameter; that means, $\beta \in \mathbb{R}_+^*$. Due to Eq. (8) it is straightforward to show that

$$\lim_{q \to 1} \exp_q \left(-\beta \mathbf{x^T} \Sigma^{-1} \mathbf{x} \right) = \exp \left(-\beta \mathbf{x^T} \Sigma^{-1} \mathbf{x} \right). \tag{11}$$

Consequently, depending on the $C_{d,q}(\beta)$ functional form, we can recover the d-dimensional Gaussian, represented in the nonextensive literature as [21, 22]

$$G_{d,1}(\mathbf{x}, \Sigma) = \left[(2\pi)^{d/2} |\Sigma|^{1/2} \right]^{-1} \exp \left(-\frac{1}{2} \mathbf{x^T} \Sigma^{-1} \mathbf{x} \right), \tag{12}$$

by taking the limit of $G_{d,q}(\mathbf{x}, \Sigma, \beta)$ at $q \to 1$. In the Appendices B and C we give details of that developments showing that, if $\beta = 1/2$, then we obtain the one and two-dimensional Gaussian functions in this way. The corresponding q-expressions are reproduced bellow.

1. One-Dimensional q-Gaussian (see Appendix B)

$$G_{1,q}(x, \sigma, \beta) = C_{1,q}(\sigma, \beta) \exp_q \left(-\frac{\beta}{\sigma^2} x^2 \right)$$
$$= C_{1,q}(\beta) \left[1 + (q-1) \frac{\beta}{\sigma^2} x^2 \right]^{\frac{1}{1-q}}, \tag{13}$$

where

$$C_{1,q}(\sigma, \beta) = \frac{\Gamma \left(\frac{1}{q-1} \right) \left[(q-1) \frac{\beta}{\sigma^2} \right]^{1/2}}{\sqrt{\pi} \Gamma \left(\frac{1}{q-1} - \frac{1}{2} \right)}, \quad 1 < q < 3, \tag{14}$$

and

$$C_{1,q}(\sigma, \beta) = \frac{\Gamma \left(\frac{1}{1-q} + \frac{3}{2} \right) \left((1-q) \frac{\beta}{\sigma^2} \right)^{1/2}}{\sqrt{\pi} \Gamma \left(\frac{1}{1-q} + 1 \right)}, \quad q < 1 \tag{15}$$

with Eq. (13) being subject to

$$|x| \leq \left((1-q) \frac{\beta}{\sigma^2} \right)^{-1/2}, \quad q < 1. \tag{16}$$

2. Two-Dimensional q-Gaussian (see Appendix C)

$$G_{2,q}(\mathbf{x}, \Sigma, \beta) = \frac{\beta(2-q)}{\pi \sqrt{|\Sigma|}} \left[1 + (1-q) \left(-\beta \mathbf{x^T} \Sigma^{-1} \mathbf{x} \right) \right]^{\frac{1}{1-q}}, \quad 1 < q < 2, \tag{17}$$

$$G_{2,q}(\mathbf{x}, \Sigma, \beta) = \frac{\beta(2-q)}{\pi \sqrt{|\Sigma|}} \left[1 + (1-q) \left(-\beta \mathbf{x^T} \Sigma^{-1} \mathbf{x} \right) \right]^{\frac{1}{1-q}}, \quad q < 1, \tag{18}$$

subject to the constraint

$$0 < \left(\mathbf{x}^T \Sigma^{-1} \mathbf{x} \right)^{1/2} < \frac{1}{\sqrt{\beta(1-q)}}, \quad q < 1. \tag{19}$$

The Fig. (1) shows the evolution of the one dimensional q-Gaussian profile respect to q. We must observe in Fig. (1).(a) that the profile of the q-Gaussian is far from the Gaussian one for $q \in \{-0.5, 0.0\}$ but for $q = 0.5$ (blue curve) the result is closer the Gaussian plot. The constraint given by Eq. (16) is responsible for this behavior. For $q > 1.0$, such constraint is not valid and the curve shapes look like the Gaussian profile, as we can visually check in Fig. (1).(b).

(a) (b)
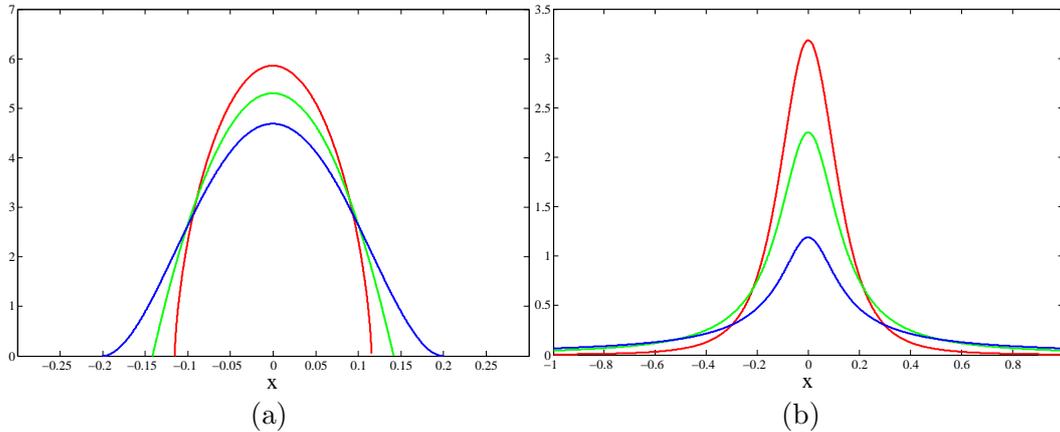
Figure 1 (a) One dimensional q-Gaussian (Eq. (13)) for $\sigma = 0.1$, $\beta = 0.5$ and $q = -0.5$ (red), $q = 0.0$ (green), and $q = 0.5$ (blue). (b) Behavior of Eq. (13) for $\sigma = 0.1$, $\beta = 0.5$ and $q = 1.5$ (red), $q = 2.0$ (green), and $q = 2.5$ (blue).

3 FOURIER TRANSFORM OF 1D q-GAUSSIAN

From Eq. (13) we can compute the Fourier transform of the one-dimensional q-Gaussian as

$$\mathcal{F}\left(C_{1,q}(\sigma, \beta) \exp_q\left(-\frac{\beta}{\sigma^2} x^2\right), y\right) = C_{1,q}(\sigma, \beta) \mathcal{F}\left(\exp_q\left(-\frac{\beta}{\sigma^2} x^2\right), y\right), \tag{20}$$

where $\mathcal{F}(g(x), y)$ means the Fourier transform of function g computed at the frequency y. In this paper, the Fourier transform is defined by

$$\mathcal{F}(g(x), y) = \int_{-\infty}^{+\infty} \exp(-2j\pi xy) \, g(x) \, \mathrm{d}x, \tag{21}$$

where $j^2 = -1$.

The Appendix D develops the computation of the Fourier transform for a q-exponential. So,

using Eq. (124) we obtain

$$\mathcal{F}\left(G_{1,q}\left(x,\sigma,\beta\right),y\right) = C_{1,q}\left(\sigma,\beta\right)\left[\left(q-1\right)\frac{\beta}{\sigma^2}\right]^{-1/2} \times$$

$$\times\left(-\operatorname{sign}\left(2\pi\left[\left(q-1\right)\frac{\beta}{\sigma^2}\right]^{-1/2}y\right)2\pi 2^{\frac{1}{1-q}}\frac{\left|2\pi\left[\left(q-1\right)\frac{\beta}{\sigma^2}\right]^{-1/2}y\right|^{\frac{1}{q-1}-1}}{\Gamma\left(\frac{1}{q-1}\right)}\right) \times \quad (22)$$

$$\times W_{0,\frac{1}{2}+\frac{1}{1-q}}\left(2\left|2\pi\left[\left(q-1\right)\frac{\beta}{\sigma^2}\right]^{-1/2}y\right|\right), \quad 1 < q < 3,$$

where $W_{0,\frac{1}{2}+\frac{1}{1-q}}$ is a special type of Whittaker functions, defined in the Appendix F of reference [22]. Consequently

$$\operatorname{abs}\left[\mathcal{F}\left(G_{1,q}\left(x,\sigma,\beta\right),y\right)\right] = \operatorname{abs}\left[C_{1,q}\left(\beta\right) \times\right.$$

$$\times \pi^{3/2}\frac{\left[\left(q-1\right)a\right]^{-1/2}}{\Gamma\left(\frac{1}{q-1}\right)}\left(j\exp\left(\frac{j\pi}{2}\left(-\frac{1}{2}-\frac{1}{1-q}\right)\right)\right) \times$$

$$\times \left(\frac{z}{2}\right)^{\frac{1}{q-1}-\frac{1}{2}}\mathbb{J}_{-\frac{1}{2}+\frac{1}{q-1}}\left(jz\right) + j\frac{\left[\left(\frac{z}{2}\right)^{\frac{1}{q-1}-\frac{1}{2}}\mathbb{J}_{-\frac{1}{2}+\frac{1}{q-1}}\left(jz\right)\cos\left(\pi\left(-\frac{1}{2}+\frac{1}{q-1}\right)\right) - j^{\frac{1}{2}-\frac{1}{q-1}}S\left(z\right)\right]}{\sin\left(\pi\left(-\frac{1}{2}+\frac{1}{q-1}\right)\right)}\right],$$

$$(23)$$

where $z = \left|2\pi\left[\left(q-1\right)a\right]^{-1/2}y\right|$, $a = \frac{\beta}{\sigma^2}$, $1 < q < 3$, with $\mathbb{J}_{-\frac{1}{2}+\frac{1}{q-1}}$ being the Bessel functions [16] and

$$S\left(z\right) = \sum_{k=0}^{+\infty}\frac{\left(\frac{z^2}{4}\right)^k}{k!\Gamma\left(\frac{1}{2}-\frac{1}{q-1}+k+1\right)}. \quad (24)$$

Analogously, through Eq. (135) and (140) we can show that

$$\mathcal{F}\left(G_{1,q}\left(x,\sigma,\beta\right),y\right) = C_{1,q}\left(\sigma,\beta\right)\frac{\sqrt{\pi}}{\left(\left(1-q\right)\frac{\beta}{\sigma^2}\right)^{1/2}} \times$$

$$\times \left(-\frac{\left(\left(1-q\right)\frac{\beta}{\sigma^2}\right)^{1/2}}{\pi y}\right)^{\frac{1}{1-q}+\frac{1}{2}}\Gamma\left(\frac{1}{1-q}+1\right)\mathbb{J}_{\frac{1}{1-q}+\frac{1}{2}}\left(-\frac{2\pi y}{\left(\left(1-q\right)\frac{\beta}{\sigma^2}\right)^{1/2}}\right), \quad q < 1,\ y \neq 0,$$

$$(25)$$

and

$$\mathcal{F}\left(G_{1,q}\left(x,\sigma,\beta\right),0\right)$$

$$= C_{1,q}\left(\sigma,\beta\right)\frac{2^{\frac{2}{1-q}+1}}{\left(\left(1-q\right)\frac{\beta}{\sigma^2}\right)^{1/2}}\frac{\Gamma\left(\frac{1}{(1-q)}+1\right)\Gamma\left(\frac{1}{(1-q)}+1\right)}{\Gamma\left(\frac{2}{1-q}+2\right)},\quad q<1,\ y=0 \tag{26}$$

respectively.

The Fig. (2), shows the Fourier transform for the 1D q-Gaussians represented in Fig. (1). For $q < 1.0$ we observe in Fig. (2).(a) some oscillations of the absolute value of the Fourier transform for higher frequencies (y value). This happens due to the constraint given by Eq. (16) once out of the corresponding interval the q-Gaussian is null, generating discontinuities in the first derivative, as observed in the red and green curves of Fig. (1). The plots for $q > 1$ in Fig. (2).(b) do not have such artifacts. However, we notice that while increasing q the shape of the q-Gaussian Fourier transform becomes more different from the shape of the Gaussian distribution.
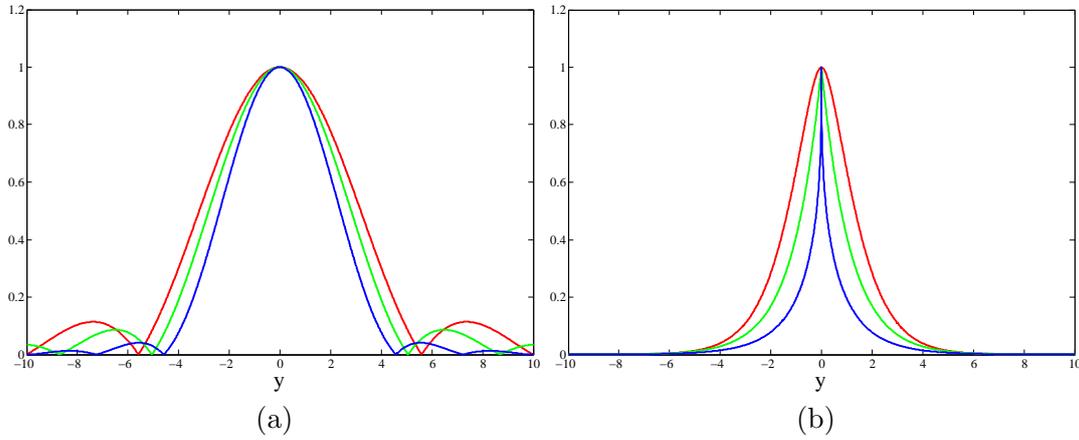
Figure 2 (a) Absolute value of the Fourier transform of q-Gaussian (Eq. (25)) for $\sigma = 0.1$, $\beta = 0.5$ and $q = -0.5$ (red), $q = 0.0$ (green), and $q = 0.5$ (blue). (b) Behavior of the Eq. (23) for $\sigma = 0.1$, $\beta = 0.5$ and $q = 1.5$ (red), $q = 2.0$ (green), and $q = 2.5$ (blue).

4 FOURIER TRANSFORM OF 2D q-GAUSSIAN

In this section we offer a sketch of the Fourier transform for the two-dimensional q-Gaussian, defined by Eq. (17)–(18), for a diagonal matrix $\Sigma = \text{diag}\left(\sigma_1^2,\sigma_2^2\right)$, which can be computed as

$$\mathcal{F}\left(G_{2,q}\left(\mathbf{z},\Sigma,\beta\right),\omega_1,\omega_2\right) = C_{2,q}\left(\Sigma,\beta\right)\sqrt{|\Sigma|}\times$$

$$\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left[-2\pi j\left(\omega_1 z_1+\omega_2 z_2\right)\right]\left[1+\left(1-q\right)\left(-\beta\left(z_1^2+z_2^2\right)\right)\right]^{\frac{1}{1-q}}\mathrm{d}z_1\mathrm{d}z_2, \tag{27}$$

where $z_1 = x/\sigma_1$, $z_2 = y/\sigma_2$ and $\mathbf{z} = (z_1, z_2)$. Considering just the double integral of Eq. (27) we can write

$$\int_{-\infty}^{+\infty} \exp\left[-2\pi j\omega_2 z_2\right] \left\{\int_{-\infty}^{+\infty} \exp\left[-2\pi j\omega_1 z_1\right] \left[1 + (q-1)\beta z_2^2 + (q-1)\beta z_1^2\right]^{\frac{1}{1-q}} \, dz_1\right\} dz_2 \quad (28)$$

$$= \int_{-\infty}^{+\infty} \exp\left[-2\pi j\omega_2 z_2\right] \times$$
$$\times \left\{\int_{-\infty}^{+\infty} \exp\left[-2\pi j\omega_1 z_1\right] \left[1 + (q-1)\beta z_2^2\right]^{\frac{1}{1-q}} \left[1 + \left(\frac{q-1}{1+(q-1)\beta z_2^2}\right)\beta z_1^2\right]^{\frac{1}{1-q}} \, dz_1\right\} dz_2 \quad (29)$$

$$= \int_{-\infty}^{+\infty} \exp\left[-2\pi j\omega_2 z_2\right] \left[1 + (q-1)\beta z_2^2\right]^{\frac{1}{1-q}} \times$$
$$\times \left\{\int_{-\infty}^{+\infty} \exp\left[-2\pi j\omega_1 z_1\right] \left[1 + (q-1)\zeta z_1^2\right]^{\frac{1}{1-q}} \, dz_1\right\} dz_2, \quad (30)$$

where

$$\zeta = \left(\frac{\beta}{1+(q-1)\beta z_2^2}\right). \quad (31)$$

Let

$$F_1(\omega_1, \zeta(z_2, q), q) = \int_{-\infty}^{+\infty} \exp\left[-2\pi j\omega_1 z_1\right] \left[1 + (q-1)\zeta z_1^2\right]^{\frac{1}{1-q}} \, dz_1. \quad (32)$$

Therefore, we can return to Eq. (27) and write

$$\mathcal{F}\left(G_{2,q}(\mathbf{z}, \Sigma, \beta), \omega_1, \omega_2\right) = C_{2,q}(\Sigma, \beta)\sqrt{|\Sigma|} \times$$
$$\times \int_{-\infty}^{+\infty} \exp\left[-2\pi j\omega_2 z_2\right] \left[1 + (q-1)\beta z_2^2\right]^{\frac{1}{1-q}} F_1(\omega_1, \zeta(z_2, q), q) \, dz_2. \quad (33)$$

By using the property that the Fourier transform of the product of two functions can be computed by the convolution in the Fourier transform domain, we can re-write Eq. (33) as

$$\mathcal{F}\left(G_{2,q}(\mathbf{z}, \beta), \omega_1, \omega_2\right)$$
$$= C_{2,q}(\Sigma, \beta)\sqrt{|\Sigma|} \left(\int_{-\infty}^{+\infty} \exp\left[-2\pi j\omega_2 z_2\right] \left[1 + (q-1)\beta z_2^2\right]^{\frac{1}{1-q}} \, dz_2\right) *$$
$$* \left(\int_{-\infty}^{+\infty} \exp\left[-2\pi j\omega_2 z_2\right] F_1(\omega_1, \zeta(z_2, q), q) \, dz_2\right), \quad (34)$$

where $*$ denotes the usual convolution operation. Unfortunately, this expression can not be analytically resolved to get the explicit representation of Eq. (27), like in the one-dimensional case. Therefore, we should apply numerical methods to calculate Eq. (27), as done in [22] using the isotropic setup ($\sigma_1 = \sigma_2 = \sigma$) and applying usual discretization methods to approximate Eq. (27) by the double summations.

5 Computational Experiments

Before proceeding, we shall notice that Eq. (22)–(26) need some considerations before their computations. Eq. (22) involves the modified Bessel functions (Eq. (8.407), reference [17]), which is not defined for $\mu \in \mathbb{Z}_-$. Also, the Gamma function is not valid for $z \in \mathbb{Z}_-$, which imposes constraints for the Bessel functions also [16, 22]. We must notice that the Gamma function occurs also in the normalization factor $C_{1,q}$ given in Eq. (14)–(15). Therefore, if we put all the constraints together, we conclude that these expression can be computed only if:

1. Case $1 < q < 3$

$$\left\{ \frac{1}{q-1} + \frac{1}{2} \right\} \notin \mathbb{Z} \quad \text{and} \quad \left\{ \frac{1}{q-1} \right\} \notin \mathbb{Z}_-. \tag{35}$$

2. Case $q < 1$

$$\left\{ \frac{1}{1-q} + 1 \right\} \notin \mathbb{Z}_- \quad \text{and} \quad \left\{ \frac{1}{1-q} + \frac{1}{2} \right\} \notin \mathbb{Z}_- \quad \text{and} \quad \left\{ \frac{1}{1-q} + \frac{3}{2} \right\} \notin \mathbb{Z}_-. \tag{36}$$

The 2-norm $\|\psi\|_2$, center x^*, and radius (width) Δ_ψ of a function $\psi = \psi(x)$ are defined to be

$$\|\psi\|_2 = \left\{ \int_{-\infty}^{+\infty} |\psi(x)|^2 \, \mathrm{d}x \right\}^{1/2}, \tag{37}$$

$$x^* = \frac{1}{\|\psi\|_2^2} \int_{-\infty}^{+\infty} x \, |\psi(x)|^2 \, \mathrm{d}x, \tag{38}$$

$$\Delta_\psi = \frac{1}{\|\psi\|_2} \|(x - x^*) \psi\|_2 = \frac{1}{\|\psi\|_2} \left\{ \int_{-\infty}^{+\infty} (x - x^*)^2 \, |\psi(x)|^2 \, \mathrm{d}x \right\}^{1/2}. \tag{39}$$

If $x^* < \infty$ and $\Delta_\psi < \infty$, we say that the signal ψ is *localized* about the point t^* with the space window $[x^* - \Delta_\psi, x^* + \Delta_\psi]$. The space window corresponding to the one-dimensional q-Gaussian, given by Eq. (13), can be computed by noticing that $x^* = 0$ due to the fact that $G_{1,q}(x, \sigma, \beta) = G_{1,q}(-x, \sigma, \beta)$. Besides, we can use a methodology that is analogous to the one applied in Appendix B to show that, for $q > 1$ we get

$$\|G_{1,q}(x, \sigma, \beta)\|_2 = \frac{C_{1,q}(\sigma, \beta)}{\left[(q-1) \frac{\beta}{\sigma^2} \right]^{1/4}} \sqrt{B\left(\frac{1}{2}, \frac{2}{q-1} - \frac{1}{2} \right)}, \quad 1 < q < 3 \tag{40}$$

and

$$\|x G_{1,q}(x, \sigma, \beta)\|_2 = \frac{C_{1,q}(\sigma, \beta)}{\left[(q-1) \frac{\beta}{\sigma^2} \right]^{3/4}} \sqrt{B\left(\frac{3}{2}, \frac{2}{q-1} - \frac{3}{2} \right)}, \quad 1 < q < \frac{7}{3}, \tag{41}$$

$$\Delta_{G_{1,q}}^{\sigma, \beta} = \frac{\sqrt{B\left(\frac{3}{2}, \frac{2}{q-1} - \frac{3}{2} \right)}}{\sqrt{\left[\left((q-1) \frac{\beta}{\sigma^2} \right) \right] B\left(\frac{1}{2}, \frac{2}{q-1} - \frac{1}{2} \right)}}, \quad 1 < q < \frac{7}{3}, \tag{42}$$

where $C_{1,q}(\sigma, \beta)$ is given by Eq. (14), and B is the Beta function [16]. In the Sec. B.3 we show that for $\beta = 1/2$ the q-Gaussian generates the traditional Gaussian kernel in the limit $q \to 1$. Therefore, to allow further comparisons, we set $\beta = 1/2$, and arbitrarily set $\sigma = 0.1$ in Eq. (13). The Fig. (3).(a) shows the behaviour of Eq. (42) in the corresponding q range.
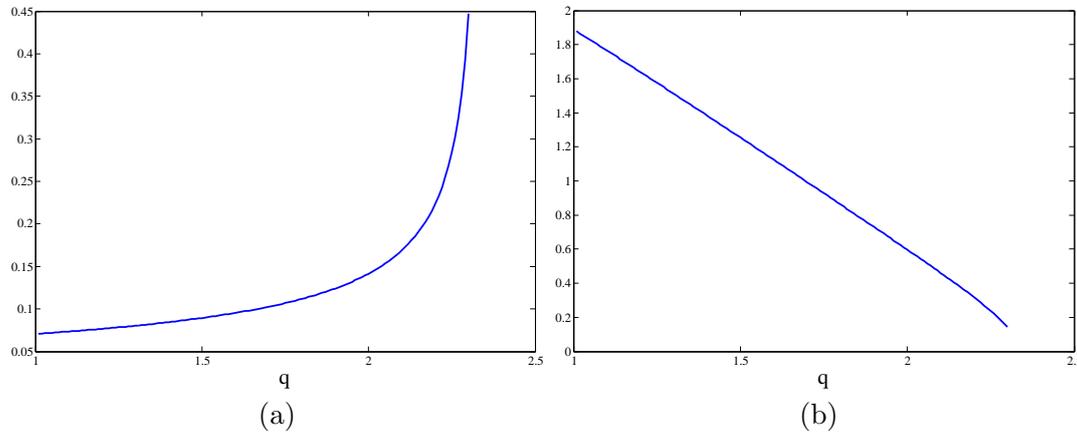
Figure 3 (a) Size of space window for q-Gaussian with parameters $\beta = 1/2$, $\sigma = 0.1$. (b) Behavior of Eq. (45) for q-Gaussian with $\beta = 1/2$, $\sigma = 0.1$.

We notice that the size of the space window of $G_{1,q}(x, 0.1, 0.5)$ about $x^* = 0$ is a monotone increasing function respect to q. Consequently, we expect the behavior shown in Fig. (4).(a). Also, due to the Heisenberg inequality

$$4\pi \left\| x G_{1,q}(x, \sigma, \beta) \right\|_2 \cdot \left\| y \mathcal{F}\left(G_{1,q}(x, \sigma, \beta), y\right) \right\|_2 \geq \left\| G_{1,q}(x, \sigma, \beta) \right\|_2^2 \tag{43}$$

we get that

$$\left\| y \mathcal{F}\left(G_{1,q}(x, \sigma, \beta), y\right) \right\|_2 \geq \Delta \mathcal{F}_{\sigma, \beta, q}, \tag{44}$$

where

$$\Delta \mathcal{F}_{1,q}^{\sigma, \beta} = \frac{C_{1,q} \left[(q-1) \frac{\beta}{\sigma^2}\right]^{1/4}}{4\pi} \frac{B\left(\frac{1}{2}, \frac{2}{q-1} - \frac{1}{2}\right)}{\sqrt{B\left(\frac{3}{2}, \frac{2}{q-1} - \frac{3}{2}\right)}}, \quad 1 < q < \frac{7}{3}, \tag{45}$$

$C_{1,q}$ is given by Eq. (14), and B is the Beta function. The Fig. (3).(b) indicates that the window size in the frequency domain is a monotone decreasing function respect to q. We can check this fact through Fig. (4).(b) which pictures the profile of the absolute value of the Fourier transform of a q-Gaussian for three values of the entropic index q. We can notice that when increasing q the q-Gaussian becomes more localized about $y = 0$, which agrees with the decreasing behavior pictured in Fig. (3).(b) for Eq. (45).

An analogous analysis can be performed for $q < 1$. In this case, we get also $x^* = 0$ from

(a)   (b)
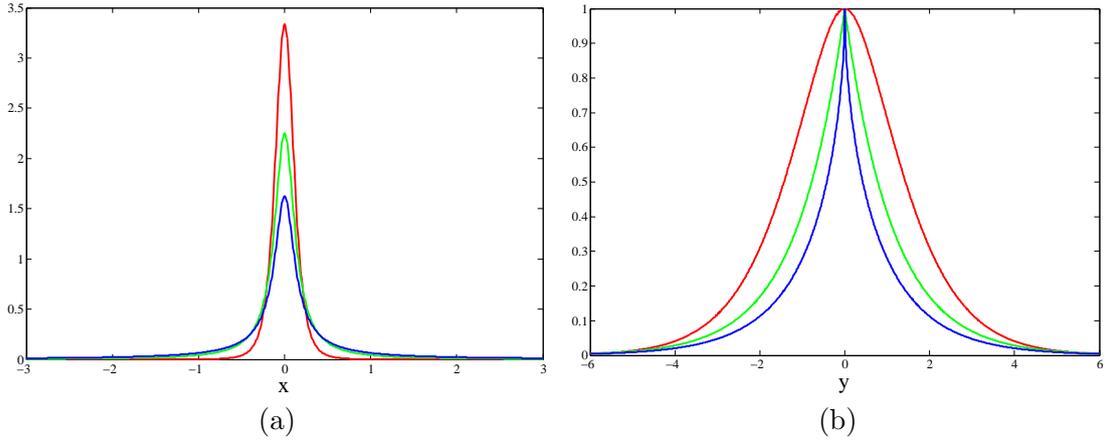
Figure 4 (a) Plot for q-Gaussian in the space domain with parameters $\beta = 1/2$, $\sigma = 0.1$ and $q = 1.41$ (red), $q = 2.0$ (green) and $q = 2.3$ (blue). (b) Absolute value of the Fourier transform of q-Gaussian (Eq. (23)) with parameters $\beta = 1/2$, $\sigma = 0.1$ and $q = 1.41$ (red), $q = 2.0$ (green) and $q = 2.3$ (blue).

Eq. (38) and we can use a the same methodology applied in the Appendix B to get that

$$\|G_{1,q}(x,\sigma,\beta)\|_2 = \frac{C_{1,q}}{\left[(1-q)\frac{\beta}{\sigma^2}\right]^{1/4}} \sqrt{B\left(\frac{1}{2}, \frac{2}{1-q}+1\right)}, \quad q < 1, \tag{46}$$

$$\|xG_{1,q}(x,\sigma,\beta)\|_2 = \frac{C_{1,q}}{\left[(1-q)\frac{\beta}{\sigma^2}\right]^{3/4}} \sqrt{B\left(\frac{3}{2}, \frac{2}{1-q}+1\right)}, \quad q < 1, \tag{47}$$

$$\Delta_{G_{1,q}}^{\sigma,\beta} = \frac{\sqrt{B\left(\frac{3}{2}, \frac{2}{1-q}+1\right)}}{\sqrt{\left[\left((1-q)\frac{\beta}{\sigma^2}\right)\right] B\left(\frac{1}{2}, \frac{2}{1-q}+1\right)}}, \quad q < 1, \tag{48}$$

where $|\mathbf{x}| \leq \left((1-q)\frac{\beta}{\sigma^2}\right)^{-1/2}$, $C_{1,q}$ is given by Eq. (15), and B is the Beta function [16]. By using Heisenberg inequality, given by Eq. (43), we can obtain

$$\|y\mathcal{F}(G_{1,q}(x,\sigma,\beta),y)\|_2 \geq \Delta \mathcal{F}_{\sigma,\beta,q} \tag{49}$$

with

$$\Delta \mathcal{F}_{1,q}^{\sigma,\beta} = \frac{C_{1,q}}{4\pi} \left[(1-q)\frac{\beta}{\sigma^2}\right]^{1/4} \frac{B\left(\frac{1}{2}, \frac{2}{1-q}+1\right)}{\sqrt{B\left(\frac{3}{2}, \frac{2}{1-q}+1\right)}}, \quad q < 1. \tag{50}$$

The Fig. (5).(a) shows the behaviour of Eq. (48) in the range $-2.0 < q < 0.99$ and the Fig. (5).(b) pictures the behavior of Eq. (50) in the same range. Likewise in the $q > 1$ case, we

observe that $\Delta_{G_{1,q}}^{0.5,0.1}$ and $\Delta\mathcal{F}_{1,q}^{0.5,0.1}$ are monotone increasing and monotone decreasing functions, respectively, in the considered q range.
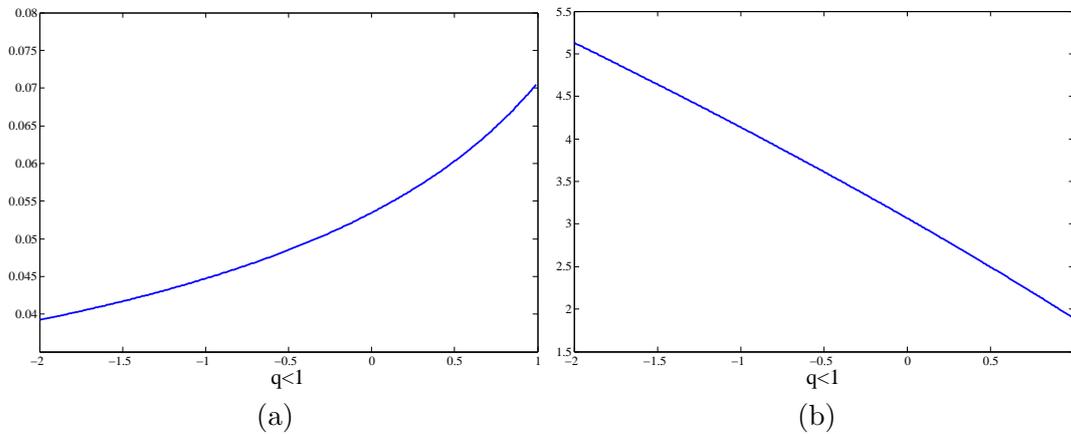
Figure 5 (a) Size of space window for q-Gaussian with parameters $\beta = 1/2$, $\sigma = 0.1$ for $q < 1$. (b) Behavior of Eq. (50)) for q-Gaussian with $\beta = 1/2$, $\sigma = 0.1$ and $q < 1$.

We can check these facts through Fig. (6) which pictures the q-Gaussian and its Fourier transform for $q = 0.1, 0.5, 0.99$. We can notice by Fig. (6).(a) that when increasing q the q-Gaussian becomes less localized about $y = 0$, which agrees with the increasing behavior of the window size pictured in Fig. (5).(a). On the other hand, the tendency for the Fourier transform localization given by Eq. (50), and represented in the Fig. (5).(b) is confirmed by the Fig. (6).(b).
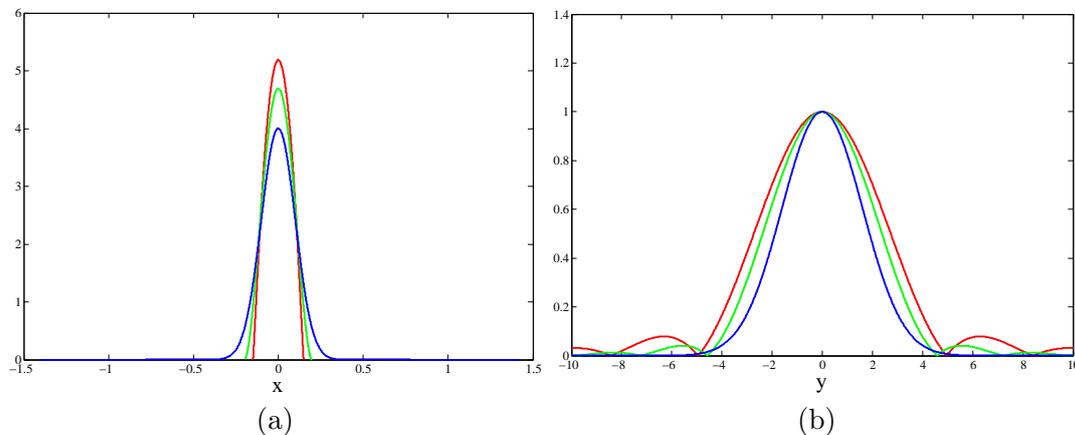
Figure 6 (a) Plot for q-Gaussian in the space domain with parameters $\beta = 1/2$, $\sigma = 0.1$ and $q = 0.1$ (red), $q = 0.5$ (green) and $q = 0.99$ (blue). (b) Fourier Transform of q-Gaussian (denoted by $\mathcal{F}(G_{1,q}(x,a),y)$) with parameters $\beta = 1/2$, $\sigma = 0.1$ and $q = 0.1$ (red), $q = 0.5$ (green) and $q = 0.99$ (blue).

The Fig. (4).(b) and (6).(b) shows that $\mathcal{F}(G_{1,q}(x,\sigma,\beta),0) = 1$ for the considered q values. In fact, from Eq. (23)–(26) we can prove this property for any $q \in \mathbb{R} - \{1\}$ and make comparisons

27

with the (normalized) Gaussian given by

$$G_{1,1}(x, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{51}$$

according to Eq. (12) for $d = 1$. The Fourier transform of the Gaussian distribution from Eq. (51) is the function

$$\mathcal{F}(G_{1,1}(x, \sigma), y) = \exp\left(-\sigma y^2\right). \tag{52}$$

So, it is clear that $\mathcal{F}(G_{1,1}(x, \sigma), 0) = 1$ also. Besides, Fig. (4) and (6) show that, if we fix the parameters β and σ in Eq. (13), we can change the localization and the profile of the q-Gaussian by changing the q value. Therefore, in terms of low-pass filtering properties, the main point is how close $G_{1,q}(x, \sigma, \beta)$ (and $\mathcal{F}(G_{1,q}(x, \sigma, \beta), y)$) is from $G_{1,1}(x, \sigma)$ (and $\mathcal{F}(G_{1,1}(x, \sigma), y)$) when changing the q value in Eq. (13)? We must perform further developments in order to answer this question.

On the other hand, Fig. (7) shows the cut-off frequency $\bar{y} = \bar{y}(q)$, such that

$$\text{abs}\left[\mathcal{F}(G_{1,q}(x, 0.1, 0.5), \bar{y})\right] < 0.1. \tag{53}$$

(a) (b)
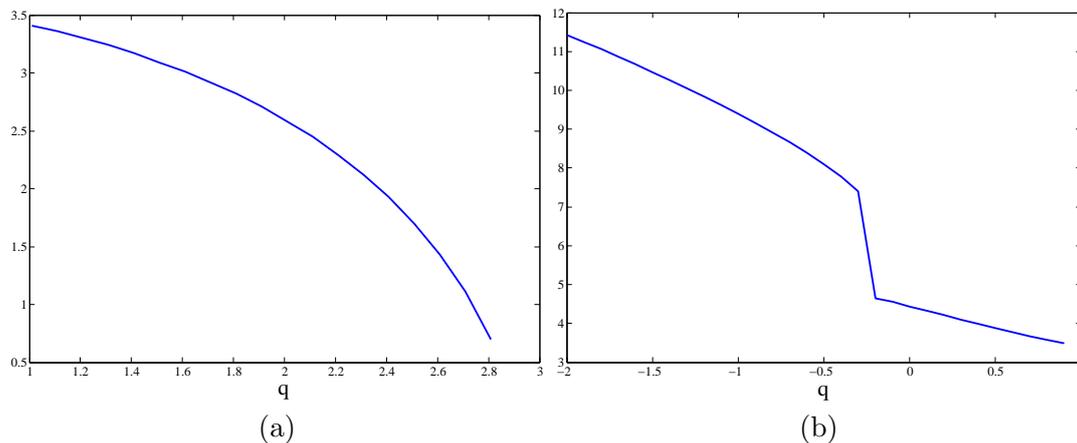
Figure 7 (a) Cut-off frequency $\bar{y}$ such that $\text{abs}\left[\mathcal{F}(G_{1,q}(x, 0.1, 0.5), \bar{y})\right] < 0.1$ for $1 < q < 3$. (b) Cut-off frequency $\bar{y}$ such that $\text{abs}\left[\mathcal{F}(G_{1,q}(x, 0.1, 0.5), \bar{y})\right] < 0.1$ for $-2 \leq q < 1$.

We notice that $\bar{y}$ is a decreasing function which is in accordance with the behaviour reported by Fig. (4).(a) and 6.(a).

Now, we consider the Fourier Transform of the q-Gaussian 2D defined by the parameters $\beta = 1$, $\sigma = \sqrt{8} \approx 2.8284$, $q = 0.5$. We apply the discretization approach presented in [22] to approximate the Eq. (27) as

$$\overline{\mathcal{F}}(G_{2,q}(\mathbf{x}, \beta), \omega_1, \omega_2) \approx \sum_{m=-M}^{M} \sum_{n=-M}^{M} \exp\left[-2\pi j(\omega_1 x_m + \omega_2 y_n)\right] G_{2,q}(x_m, y_n, \beta) T^2, \tag{54}$$

where $-T^{-1} < \omega_1, \omega_2 < T^{-1}$, $x_m = mT$, $y_n = nT$, with $-M \leq m, n \leq M$. In this case, we set $T = 0.25$, $M = 21$. The Fig. (8) pictures the obtained surface.
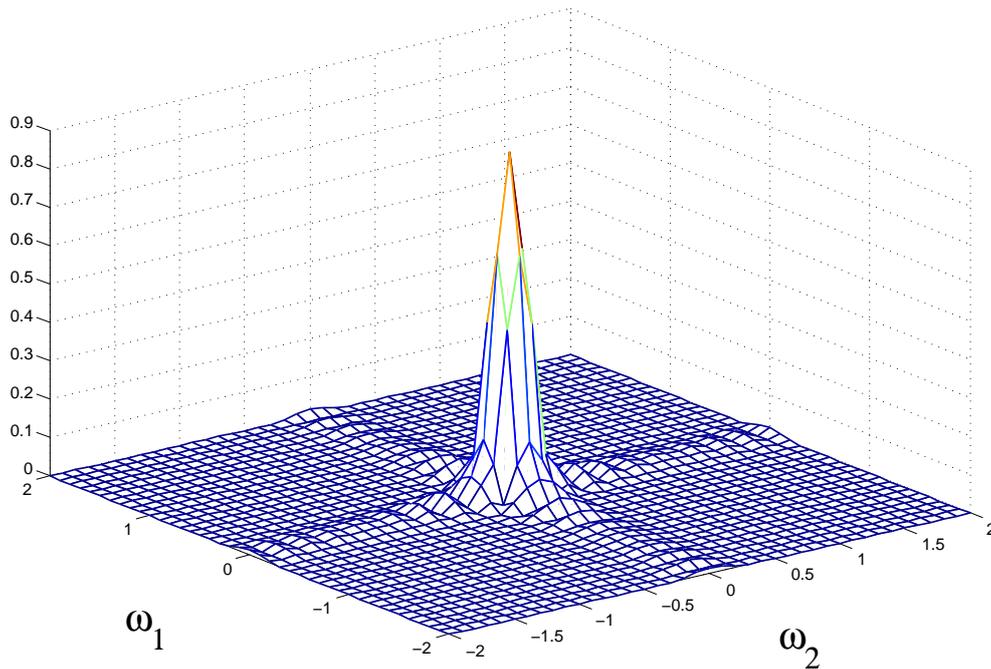
Figure 8 Plot of abs $\left[\mathcal{F} \left(G_{2,0.5} \left(x, y, \sqrt{8}, 1 \right), \omega_1, \omega_2 \right) \right]$ for $-2 \leq \omega_1, \omega_2 < 2$.

6 DISCUSSION

While applying the q-Gaussian function for signal processing, a fundamental issue is the choice of the q-index. Since it is not an intuitive idea, the general methodology to set this parameter is by trial and error. For instance, the work [12] presents an experimental study about the effect of q-Gaussian filtering of noise images for both edge detection and segmentation. The results are generated for $q \in \{0.1, 0.5, 0.99, 1.01, 1.5, 2.0, 2.5, 2.99\}$ and compared with the traditional Gaussian smoothing. Authors pointed out that the results obtained by q-Gaussian smoothing outperforms the Gaussian ones, particularly for $q = 0.1$. Also, the paper [11] introduces a method of extracting edges using convolutions between the input image and difference-of-q-Gaussian kernels instead of the traditional difference-of-Gaussian functions [23, 24]. Suitable results are obtained using $q \in \{-1.0, 2.5\}$ and $\sigma \in \{0.1, 0.2\}$, also obtained by computational experimentations.

The effect of the q value in the low-pass filtering characteristics of the q-Gaussian kernel can be visualized by considering the step function

$$ f(x) = \begin{cases} A, & 0 \leq x < L/2, \\ 0, & \text{otherwise.} \end{cases} \tag{55} $$

The Fig. (9) shows the result for the convolution between the q-Gaussian, for $q = 0.1$, $q = 1.0$ (Gaussian kernel), and $q = 2.5$, with the step function defined by $A = 1$ and $L = 2$.

We can visually check the fact that the results for $q = 0.1$ and $q = 1.0$ are very close one to each other although the former preserves the input signal more than the other two outputs.
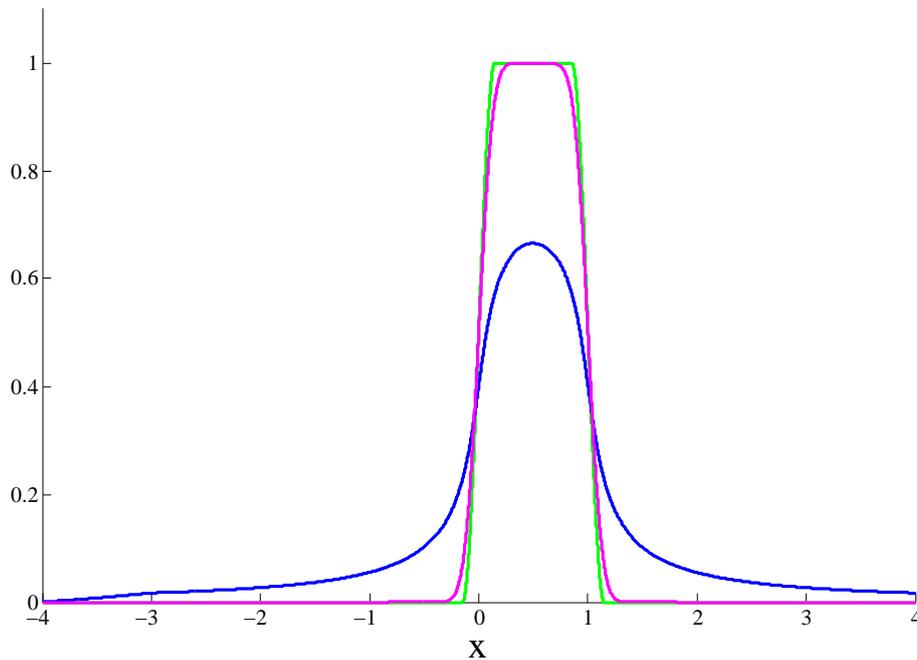
Figure 9 Convolution of step function, Eq. (55), and the q-Gaussian (Eq. (13)), for $\beta = 0.5$, $\sigma = 0.1$, $q = 0.1$ (green), $q = 1.0$ (magenta), and $q = 2.5$ (blue).

However, in the case of noisy images, we may need a lower cut-off frequency. Hence, according to the Fig. (7), we need to increase the q value. The result for $q = 2.5$ in Fig. (9) shows the effect of increasing the q value for the step function: the lost of the localization of the edges of the step, which is an undesirable effect.

In the case of thresholding techniques for image segmentation, we postulate in [25] that the optimum q value for a given database is the one that minimizes Tsallis entropy computed by Eq. (1). However, to the best of our knowledge, there is no automatic method already proposed to seek for a suitable q value for signal processing tasks represented by convolution kernels. Such linear filtering techniques are represented in the frequency domain by the product of the Fourier transform of the kernel of the filter and the Fourier transform of the input signal [26]

$$\mathcal{F}\left(G_{1,q}\left(x, \sigma, \beta\right) * f\left(x\right), y\right) = \mathcal{F}\left(G_{1,q}\left(x, \sigma, \beta\right), y\right)\mathcal{F}\left(f\left(x\right), y\right), \tag{56}$$

a result known as the *convolution theorem* [26]. Eq. (22), (25) and (26) summarizes the one-dimensional Fourier transform of the q-Gaussian function while for the specific case of the Gaussian kernel, given by Eq. (51), we get Eq. (52) as its Fourier transform.

The simplicity of the Gaussian expression, in both the space and Fourier domains, makes the analysis and implementation of filtering methods using Gaussian kernel easier if compared with the q-Gaussian function. We also pointed out in Sec. 5 the constraints that must be satisfied to allow the numerical computation of Eq. (22), (25) and (26). Besides, differently from Eq. (52), the shape of the q-Gaussian kernel in the frequency domain is far from of obvious. So, the application

of the convolution theorem (Eq. (56)) makes the low-pass filtering properties of the q-Gaussian not so evident as we notice in the Gaussian case.

In order to address these issues, we shall consider the q-Fourier analysis. If $q > 1$, then the q-Fourier transform of a non-negative integrable function $f : \mathbb{R} \rightarrow \mathbb{R}^+$ is defined as [27]

$$\mathcal{F}_q\left(f\left(x\right),y\right) = \int_{-\infty}^{+\infty} f\left(x\right) \exp_q\left(jxy\left[f\left(x\right)\right]^{q-1}\right)\mathrm{d}x, \quad 1 \leq q < 3. \tag{57}$$

With such definition it can be demonstrate that the q-Fourier transform of the q-Gaussian function (Eq. (13)) has a q-exponential shape given by expression [21]

$$\mathcal{F}_q\left(G_{1,q}\left(x,\sigma,\beta\right),y\right) = \left[\exp_q\left(-\frac{\widehat{C}\left(q\right)}{4\beta^{2-q}}\frac{y^2}{\sigma^2}\right)\right]^{\frac{3-q}{2}}, \tag{58}$$

where $\widehat{C}\left(q\right)$ is a normalization factor. Besides, for $1 \leq q < 2$ the inversion of the q-Fourier transform is well defined (see [27] for details). All these equations reduce to the usual ones as $q \rightarrow 1$.

The Eq. (57)–(58), together with the inverse q-Fourier transform and the q-convolution defined in [28], open the possibility of using q-Fourier analysis for signal processing. Specifically, we could characterize linear filtering (low-pass, high-pass and band-pass) in the space domain using q-convolutions between the input signal $f\left(x\right)$ and an appropriate kernel filter. In the frequency domain, we could analyse these processes through the q-Fourier transform and its inverse. Such approach should be considered in further works.

The kernel of Fourier transform in Eq. (21), given by

$$k_y(x) = \exp\left(-2j\pi xy\right) \tag{59}$$

is an infinitely extended kernel in the space domain. However, all physical signals are spatially limited. As a consequence, the Fourier transform is unable to satisfactorily represent analog signals in nature [26]. Besides, the Fourier transform kernel does not involve smoothing or scale-space operations. The first alternative to overcome these limitations of Fourier analysis is the Gabor transform [26, 29] that uses a modulation (window) function in order to achieved time-frequency localization. Also, the q-deformation of known functions, introduced in the q-calculus, allows considering some modifications in known functional transforms in order to discuss the performance of frequency domain analysis on noisy and blurred objects [13]. Inspired in these facts and in the Gabor transform theory, we propose here the q-Gabor transform, defined by

$$\mathcal{G}_{q,b}^{\sigma}\left(f\left(x\right),y\right) = \int_{-\infty}^{+\infty} f\left(x\right) G_{q,b}^{\sigma}\left(x\right) \exp\left(-2j\pi xy\right)\mathrm{d}x, \tag{60}$$

where

$$G_{q,b}^{\sigma}\left(x\right) = G_{1,q}\left(x - b, \sigma, 0.5\right) = C_{1,q}\left(0.5\right)\exp_q\left(-\frac{1}{2\sigma^2}\left(x - b\right)^2\right), \tag{61}$$

with $\exp_q(z)$ defined in Eq. (7). We shall observe that

$$\int_{-\infty}^{+\infty} \mathcal{G}_{q,b}^{\sigma}(f(x),y)\,\mathrm{d}b = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(x)\,G_{1,q}(x-b,\sigma,0.5)\exp(-2j\pi xy)\,\mathrm{d}x\right]\mathrm{d}b$$
$$= \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} G_{1,q}(x-b,\sigma,0.5)\,\mathrm{d}b\right] f(x)\exp(-2j\pi xy)\,\mathrm{d}x. \tag{62}$$

However, from Eq. (61) it is straightforward to show that

$$\int_{-\infty}^{+\infty} G_{1,q}(x-b,\sigma,0.5)\,\mathrm{d}b = 1. \tag{63}$$

Therefore, by substituting this result in Eq. (62) we obtain

$$\int_{-\infty}^{+\infty} \mathcal{G}_{q,b}^{\sigma}(f(x),y)\,\mathrm{d}b = \int_{-\infty}^{+\infty} f(x)\exp(-2j\pi xy)\,\mathrm{d}x = \mathcal{F}(f(x),y). \tag{64}$$

That is, likewise for the traditional Gabor transform (obtained in the limit $q \to 1$), Eq. (64) can be interpreted by saying that the q-Gabor transform of f allows to decompose the Fourier transform $\mathcal{F}(f(x),y)$ to obtain its local spectral information inside the space window $[-\Delta_{G_{q,b}^{\sigma}}, \Delta_{G_{q,b}^{\sigma}}]$, computed through Eq. (39). Besides, we can compare q-Gabor and the Fourier transforms respect to the sensitivity to noise. To perform this task we add to the step function (Fig. (10).(a)) an uniform noise in the range $[-0.1, 0.1]$ to get the function $\tilde{f}$, represented in Fig. (10).(b). The Fig. (10).(c)–(d) picture the absolute value of the Fourier transform of the step functions shown in Fig. (10).(a)–(b), respectively. Also, we set $\sigma = 0.1$, $q = 2.5$ and compute the q-Gabor transform of f and $\tilde{f}$ for $b = 0.5$, which are shown (in absolute value) in Fig. (10).(e)–(f), respectively. By comparing Fig. (10).(d) and (10).(f), it becomes clear that the Fourier transform is more sensitive respect to the noise of the input signal than the q-Gabor defined in expression Eq. (60). This fact represents an advantage of q-Gabor against the Fourier counterpart because it points forward to a more robust signal analysis framework based on the q-Gabor transform.

However, a problem still remains due to the use of a fixed scale window in q-Gabor: signal details much smaller than the window width $\Delta_{G_{q,b}^{\sigma}}$, computed through Eq. (39), are detected but not localized. Wavelet analysis was developed to solve this problem. The wavelet transform allows using small windows to identify high-frequency components of a signal, and large windows for low-frequency components. Although such development is far from our goal in this paper, it is worthwhile to point out that recent works have extended the application of q-deformed functions (q-mexican hat and q-trigonometric functions, for instance) into wavelet analysis presenting significant different behaviours, when compared with the original $q = 1.0$ cases.

7   Conclusions

In this paper we collect theoretical elements to analyze the q-Gaussian kernel for signal processing. We review some theoretical elements behind the q-Gaussian and its Fourier transform. We analyze the q-Gaussian kernel in the space and Fourier domains using the concepts of space window,
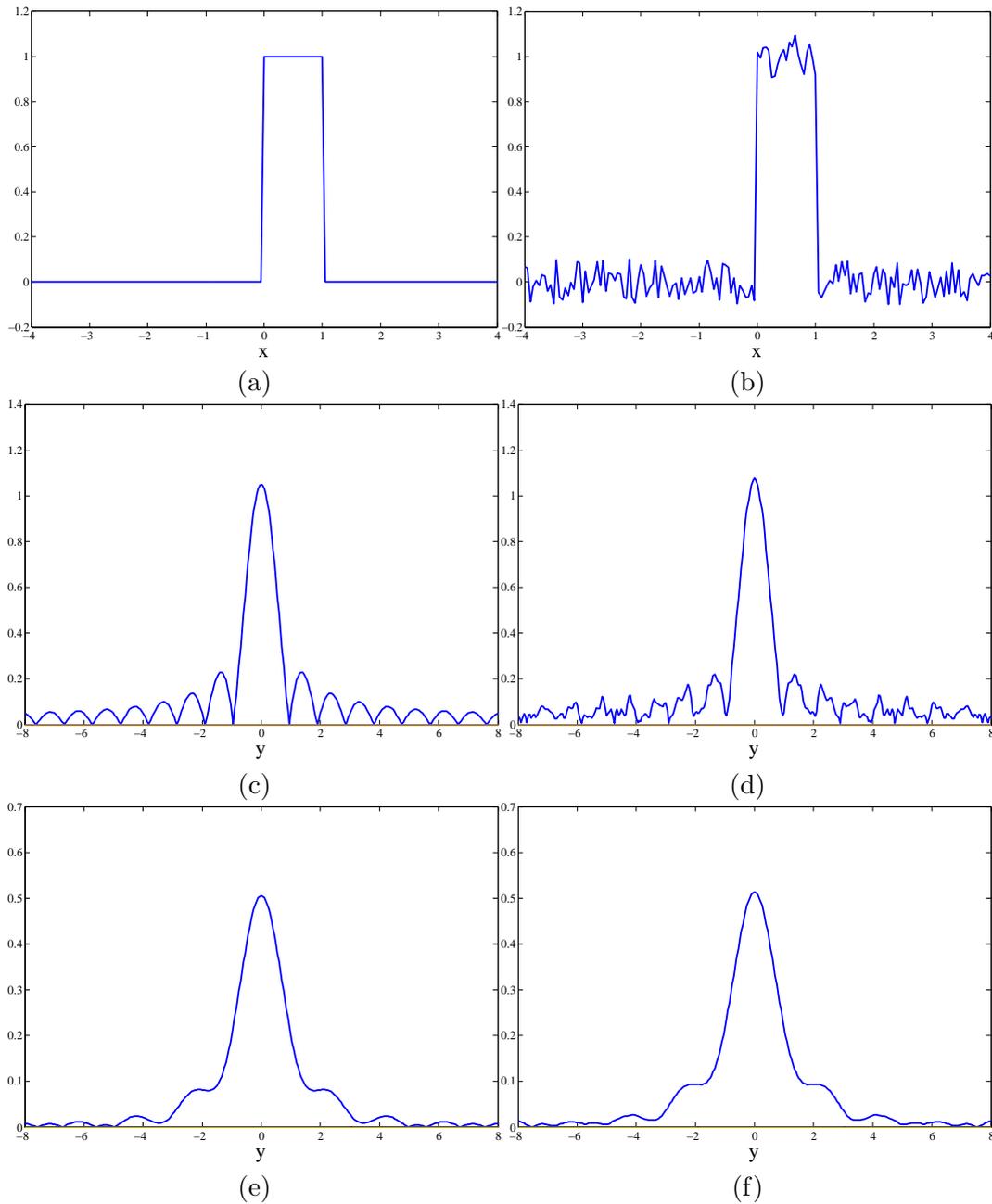
Figure 10 (a) Step function, computed by Eq. (55). (b) Noisy step function. (c) Fourier transform of step function. (d) Fourier transform of noisy step function. (e) q-Gabor transform of step function. (f) q-Gabor transform of noisy step function with $q = 2.5$, $b = 0.5$, $\beta = 0.5$.

cut-off frequency, and the Heisenberg inequality. We postulate that the comparison between the q-Gaussian and Gaussian kernels in the Fourier/space domains may allow explaining the observed

smoothing capabilities of the q-Gaussian kernel for $q < 1$.

Acknowledgement

Appendix A q-Gaussian in $\mathbb{R}^d$

The d-dimensional q-Gaussian is defined by

$$G_{d,q}\left(\mathbf{x}, \Sigma, \beta\right) = C_{d,q}\left(\Sigma, \beta\right) \exp_q\left(-\beta \mathbf{x}^{\mathbf{T}} \Sigma^{-1} \mathbf{x}\right), \tag{65}$$

where $C_{d,q}\left(\Sigma, \beta\right)$ is a normalization factor. In the following development we are supposing $\beta > 0$.

Once the covariance matrix Σ is symmetric and positive definite, there exists an orthogonal matrix U and a diagonal matrix D such that $\Sigma = U^T D U$, and, consequently, $D^{-1} = U \Sigma^{-1} U^T$. If

$$\mathbf{y} = U\mathbf{x}, \tag{66}$$

then, $\mathbf{x} = U^T \mathbf{y}$ and, using these results we can write the factor $C_{d,q}$ in Eq. (65) as

$$C_{d,q}\left(\Sigma, \beta\right) = \left(\int_{\mathbb{R}^d} \exp_q\left(-\beta \mathbf{y}^{\mathbf{T}} D^{-1} \mathbf{y}\right) d\mathbf{y}\right)^{-1}. \tag{67}$$

If D^{-1} is a diagonal matrix

$$D^{-1} = \operatorname{diag}\left(\sigma_1^{-2}, \sigma_2^{-2}, \ldots, \sigma_d^{-2}\right), \tag{68}$$

and, if we define

$$\sqrt{D^{-1}} = \operatorname{diag}\left(\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_d^{-1}\right), \tag{69}$$

then it is straightforward that $D^{-1} = \sqrt{D^{-1}}\sqrt{D^{-1}}$ and, consequently

$$\det\left(\sqrt{D^{-1}}\right) = \frac{1}{\sqrt{\det\left(U\Sigma U^T\right)}} = \frac{1}{\sqrt{|\Sigma|}}, \tag{70}$$

once $U^T U = UU^T = I$. Therefore, by performing the variable change

$$\mathbf{z} = \sqrt{D^{-1}}\mathbf{y}, \tag{71}$$

and using the result from Eq. (70) we find

$$d\mathbf{y} = \sqrt{|\Sigma|}d\mathbf{z}. \tag{72}$$

Consequently, Eq. (67) becomes

$$C_{d,q}\left(\Sigma, \beta\right) = \frac{1}{\sqrt{|\Sigma|}\int_{\mathbb{R}^d} \exp_q\left(-\beta|\mathbf{z}|^2\right) d\mathbf{z}}. \tag{73}$$

APPENDIX B   ONE-DIMENSIONAL q-GAUSSIAN

If $d = 1$ in Eq. (73) we get

$$C_{1,q}(\sigma, \beta) = \left(\int_{-\infty}^{\infty} \exp_q \left(-\beta y \sigma^{-2} y \right) \mathrm{d}y \right)^{-1} = \frac{1}{\int_{-\infty}^{\infty} \exp_q \left(-\frac{\beta}{\sigma^2} y^2 \right) \mathrm{d}y}. \tag{74}$$

Therefore, by using the q-exponential definition from Eq. (7), we can write Eq. (74) as

$$C_{1,q}(\sigma, \beta) = \frac{1}{\int_{-\infty}^{\infty} \left[1 + (q-1) \left(\frac{\beta}{\sigma^2} y^2 \right) \right]^{\frac{1}{1-q}} \mathrm{d}y}. \tag{75}$$

B.1   CASE $d = 1$ AND $q > 1$

If we perform the variable change

$$\tilde{y} = [(q-1)a]^{1/2} y, \tag{76}$$

with $a = \frac{\beta}{\sigma^2}$, then we can write the denominator of Eq. (75) as

$$\int_{-\infty}^{\infty} \left[1 + (q-1) \left(\frac{\beta}{\sigma^2} y^2 \right) \right]^{\frac{1}{1-q}} \mathrm{d}y = \frac{2}{[(q-1)a]^{1/2}} \int_{0}^{+\infty} \left[1 + \tilde{y}^2 \right]^{\frac{1}{1-q}} \mathrm{d}\tilde{y}. \tag{77}$$

From Eq. (3.251-2) and (8.384) of reference [17], we obtain

$$\int_{0}^{+\infty} x^{\mu-1} \left[1 + x^2 \right]^{\nu-1} \mathrm{d}x = \frac{1}{2} B \left(\frac{\mu}{2}, 1 - \nu - \frac{\mu}{2} \right), \quad \Re(\mu) > 0, \ \Re\left(\nu + \frac{1}{2}\mu \right) < 1, \tag{78}$$

where B is the Beta function [16]. Therefore, we can cast Eq. (77) in the above form by setting

$$\mu = 1, \quad \nu - 1 = \frac{1}{1-q}. \tag{79}$$

By inserting these values in the conditions for Eq. (78) we get

$$1 < q < 3. \tag{80}$$

So, we can insert the above results in Eq. (78) to obtain

$$\int_{0}^{+\infty} \left[1 + x^2 \right]^{\frac{1}{1-q}} \mathrm{d}x = \frac{1}{2} B \left(\frac{1}{2}, \frac{1}{q-1} - \frac{1}{2} \right). \tag{81}$$

By inserting in Eq. (81) the Beta function representation in terms of the Gamma function Γ [16], we can compute Eq. (75) as

$$C_{1,q}(\sigma, \beta) = \frac{1}{\frac{2}{[(q-1)a]^{1/2}} \frac{1}{2} \frac{\Gamma(\frac{1}{2})\Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{q-1}\right)}} = \frac{\Gamma\left(\frac{1}{q-1}\right)[(q-1)a]^{1/2}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)}, \quad q > 1, \tag{82}$$

once $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (see [30], page 37).

B.2   CASE $d = 1$ AND $q < 1$

In this case, Eq. (75) is well defined only if

$$1 + (q - 1) a y^2 \geq 0, \tag{83}$$

where $a = \frac{\beta}{\sigma^2}$, which implies

$$|y| \leq \frac{1}{\sqrt{(1 - q) a}} = ((1 - q) a)^{-1/2}. \tag{84}$$

Hence, the denominator of Eq. (75) becomes

$$\int_{-\infty}^{\infty} \left[1 + (q - 1) \left(a y^2 \right) \right]^{\frac{1}{1-q}} dy = \int_{-((1-q)a)^{-1/2}}^{((1-q)a)^{-1/2}} \left[1 + (q - 1) \left(a y^2 \right) \right]^{\frac{1}{1-q}} dy. \tag{85}$$

By using the variable change

$$\widetilde{y} = ((1 - q) a)^{1/2} y, \tag{86}$$

we get

$$\int_{-\infty}^{\infty} \left[1 + (q - 1) \left(a y^2 \right) \right]^{\frac{1}{1-q}} dy = \int_{-1}^{1} \left[1 - \widetilde{y}^2 \right]^{\frac{1}{1-q}} \frac{d\widetilde{y}}{((1 - q) a)^{1/2}}. \tag{87}$$

From Eq. (3.251-1) of reference [17], we see that

$$\int_{0}^{1} x^{\mu - 1} \left(1 - x^\lambda \right)^{\nu - 1} dx = \frac{1}{\lambda} B \left(\frac{\mu}{\lambda}, \nu \right), \quad \Re (\mu) > 0, \ \Re (\nu) > 0, \ \lambda > 0, \tag{88}$$

where B is the Beta function [16].

Therefore, by setting

$$\mu = 1, \ \lambda = 2, \ \nu = \frac{1}{1 - q} + 1, \tag{89}$$

in Eq. (88) we obtain

$$\int_{0}^{1} \left(1 - x^2 \right)^{\frac{1}{1-q}} dx = \frac{1}{2} B \left(\frac{1}{2}, \frac{1}{1 - q} + 1 \right). \tag{90}$$

By inserting this result in Eq. (75) and using the Beta function representation in terms of the Gamma function [16], we obtain

$$C_{1,q} (\sigma, \beta) = \frac{\Gamma \left(\frac{1}{1-q} + \frac{3}{2} \right) ((1 - q) a)^{1/2}}{\Gamma \left(\frac{1}{2} \right) \Gamma \left(\frac{1}{1-q} + 1 \right)}, \tag{91}$$

for $q < 1$.

B.3 One-Dimensional q-Gaussian Expression

By inserting Eq. (82) and (91) into Eq. (65) with $d = 1$ we get

$$G_{1,q}\left(x, \sigma, \beta\right) = C_{1,q}\left(\sigma, \beta\right) \exp_q\left(-\frac{\beta}{\sigma^2} x^2\right), \tag{92}$$

where

$$C_{1,q}\left(\sigma, \beta\right) = \frac{\Gamma\left(\frac{1}{q-1}\right)\left[(q-1)\frac{\beta}{\sigma^2}\right]^{1/2}}{\sqrt{\pi}\Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)}, \quad 1 < q < 3, \tag{93}$$

$$C_{1,q}\left(\sigma, \beta\right) = \frac{\Gamma\left(\frac{1}{1-q} + \frac{3}{2}\right)\left((1-q)\frac{\beta}{\sigma^2}\right)^{1/2}}{\sqrt{\pi}\Gamma\left(\frac{1}{1-q} + 1\right)}, \quad q < 1, \ |x| \le \left((1-q)\frac{\beta}{\sigma^2}\right)^{-1/2}. \tag{94}$$

Consequently, using Eq. (A7) of reference [31], which states that

$$\lim_{q \to 1^+} \frac{\Gamma\left(\frac{1}{q-1} - \alpha\right)}{(q-1)^\alpha \Gamma\left(\frac{1}{q-1}\right)} = 1, \quad 1 < q < 1 + \frac{1}{\alpha} \tag{95}$$

and

$$\lim_{q \to 1^-} \frac{\Gamma\left(\frac{1}{1-q} + 1\right)}{(1-q)^\alpha \Gamma\left(\frac{1}{1-q} + \alpha + 1\right)} = 1, \quad q < 1 \tag{96}$$

we can show that

$$\lim_{q \to 1} G_{1,q}\left(\mathbf{x}, \sigma, \beta\right) = \frac{\sqrt{\beta}}{\sigma\sqrt{\pi}} \exp\left(-\frac{\beta}{\sigma^2} x^2\right). \tag{97}$$

If $\beta = 0.5$, we obtain

$$\lim_{q \to 1} G_{1,q}\left(\mathbf{x}, \sigma, 0.5\right) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{98}$$

which is the traditional Gaussian given by Eq. (12) for $d = 1$.

Appendix C Two-Dimensional q-Gaussian

If $d = 2$ then $\mathbf{z} = (z_1, z_2)$ and, using polar coordinates

$$z_1 = r\cos\left(\theta\right), \ z_2 = r\sin\left(\theta\right), \ \det\left(J\right) = r, \tag{99}$$

we can write the integral in Eq. (73) as

$$\int_{\mathbb{R}^2} \exp_q\left(-\beta|\mathbf{z}|^2\right) \mathrm{d}\mathbf{z} = \int_0^{2\pi} \int_0^{+\infty} r \exp_q\left(-\beta r^2\right) \mathrm{d}r\mathrm{d}\theta. \tag{100}$$

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C.1 CASE $q > 1$

Then

$$\int_0^{2\pi} \int_0^{+\infty} r \exp_q\left(-\beta r^2\right) dr d\theta = 2\pi \int_0^{+\infty} r \left[1 - (1-q)\beta r^2\right]^{\frac{1}{1-q}} dr. \tag{101}$$

Variable change

$$u = 1 - (1-q)\beta r^2, \tag{102}$$

$$du = -2(1-q)\beta r dr = 2(q-1)\beta r dr. \tag{103}$$

Besides, once $q > 1$ and $\beta > 0$, we have

$$r \to +\infty \implies u \to +\infty, \quad \text{and} \quad u(0) = 1. \tag{104}$$

Therefore, using the variable change defined by Eq. (102)–(103) we obtain

$$2\pi \int_0^{+\infty} r \left[1 - (1-q)\beta r^2\right]^{\frac{1}{1-q}} dr = \frac{2\pi}{2(q-1)\beta} \left[\frac{u^{\frac{2-q}{1-q}}}{\frac{2-q}{1-q}}\right]_1^{+\infty}. \tag{105}$$

The above integral converges only if

$$\frac{1}{1-q} + 1 = \frac{2-q}{1-q} < 0 \implies q < 2. \tag{106}$$

Once we are considering $q > 1$, we get that, if $1 < q < 2$ then Eq. (105) becomes

$$2\pi \int_0^{+\infty} r \left[1 - (1-q)\beta r^2\right]^{\frac{1}{1-q}} dr = \frac{\pi}{\beta(2-q)}. \tag{107}$$

C.2 CASE $q < 1$

In this case, in order to get a real value in the integral given by Eq. (101) we need

$$1 - (1-q)\beta r^2 > 0 \implies 0 < r < \frac{1}{\sqrt{\beta(1-q)}}. \tag{108}$$

Then, with the constraints $q < 1$ and $0 < r < (\beta(1-q))^{-1/2}$, we can use the same variable change given by Eq. (102)–(103) and insert it in the integral in Eq. (101) to obtain

$$2\pi \int_0^{+\frac{1}{\sqrt{1-q}}} r \left[1 - (1-q)r^2\right]^{\frac{1}{1-q}} dr = \left(\frac{2\pi}{2(q-1)\beta}\right) \left[\frac{u^{\frac{2-q}{1-q}}}{\frac{2-q}{1-q}}\right]_1^0 = \frac{\pi}{\beta(2-q)}. \tag{109}$$

C.3 q-Gaussian 2D: Putting All Together

If we set $d = 2$, in Eq. (73) then, for $\beta > 0$, Eq. (107) renders

$$\int_{\mathbb{R}^2} \exp_q \left(-\beta |\mathbf{z}|^2 \right) d\mathbf{z} = \frac{\pi}{\beta (2-q)}, \quad if \quad 1 < q < 2, \tag{110}$$

If $q < 1$, we have the restriction

$$\Omega = \left(0, \frac{1}{\sqrt{\beta(1-q)}} \right), \tag{111}$$

due to Eq. (108). Therefore, according to Eq. (109), the integral in Eq. (100) becomes

$$\int_{\mathbb{R}^2} \exp_q \left(-\beta |\mathbf{z}|^2 \right) d\mathbf{z} = \frac{\pi}{\beta (2-q)}, \quad q < 1. \tag{112}$$

By assembling all the above results we obtain that, for $d = 2$, Eq. (65) gives

$$G_{2,q}(\mathbf{x}, \Sigma, \beta) = \frac{\beta(2-q)}{\pi \sqrt{|\Sigma|}} \left[1 + (1-q) \left(-\beta \mathbf{x}^{\mathbf{T}} \Sigma^{-1} \mathbf{x} \right) \right]^{\frac{1}{1-q}}, \quad q < 1 \quad \text{or} \quad 1 < q < 2, \tag{113}$$

subject to the constraint

$$0 < \left(\mathbf{x}^T \Sigma^{-1} \mathbf{x} \right)^{1/2} < \frac{1}{\sqrt{\beta(1-q)}}, \quad q < 1. \tag{114}$$

Consequently

$$\lim_{q \to 1} G_{2,q}(\mathbf{x}, \Sigma, \beta) = \frac{\beta}{\pi \sqrt{|\Sigma|}} \exp \left(-\beta \mathbf{x}^{\mathbf{T}} \Sigma^{-1} \mathbf{x} \right). \tag{115}$$

So, by setting $\beta = 1/2$, we get

$$\lim_{q \to 1} G_{2,q}\left(\mathbf{x}, \Sigma, \frac{1}{2} \right) = \frac{1}{2\pi \sqrt{|\Sigma|}} \exp \left(-\frac{1}{2} \mathbf{x}^{\mathbf{T}} \Sigma^{-1} \mathbf{x} \right), \tag{116}$$

which is the traditional two-dimensional Gaussian $G_{2,1}(\mathbf{x}, \Sigma)$, obtained from Eq. (12) for $d = 2$.

Appendix D   Fourier Transform of q-Exponential

Let

$$\mathcal{F} \left(\exp_q \left(-ax^2 \right), y \right) = \int_{-\infty}^{+\infty} \exp \left(-2j\pi xy \right) \left[1 + (q-1) ax^2 \right]^{\frac{1}{1-q}} dx. \tag{117}$$

If we perform the variable change

$$\tilde{x} = [(q-1)a]^{1/2} x, \quad \tilde{y} = [(q-1)a]^{-1/2} y. \tag{118}$$

Then, we can rewrite Eq. (117) as

$$\mathcal{F}\left(\exp_q\left(-ax^2\right), y\right) = \left[(q-1)\,a\right]^{-1/2} \int_{-\infty}^{+\infty} \exp\left(-2j\pi\tilde{x}\tilde{y}\right) \left[1+\tilde{x}^2\right]^{\frac{1}{1-q}} \mathrm{d}\tilde{x}. \tag{119}$$

We are going to use the fact that

$$\int_{-\infty}^{+\infty} (\beta+jx)^{-2\mu}\,(\gamma-jx)^{-2\nu}\,e^{-jpx}\mathrm{d}x$$

$$= \begin{cases} -2\pi\,(\beta+\gamma)^{-\mu-\nu}\,\dfrac{p^{\mu+\nu-1}}{\Gamma(2\nu)} \exp\left(\dfrac{\gamma-\beta}{2}p\right) W_{\mu-\nu,\frac{1}{2}-\nu-\mu}\,(\beta p+\gamma p)\,, & p>0 \\ 2\pi\,(\beta+\gamma)^{-\mu-\nu}\,\dfrac{(-p)^{\mu+\nu-1}}{\Gamma(2\mu)} \exp\left(\dfrac{\beta-\gamma}{2}p\right) W_{\mu-\nu,\frac{1}{2}-\nu-\mu}\,(-\beta p-\gamma p)\,, & p<0, \end{cases} \tag{120}$$

if $\Re(\beta) > 0$, $\Re(\gamma) > 0$ and $\Re(\mu+\nu) > 1/2$, where W denotes the Whittaker functions (see Eq. (9)–(220) of reference [17]).

Therefore, if $\beta = \gamma = 1$ and $\mu = \nu$ in the above expressions we obtain

$$\int_{-\infty}^{+\infty} \left(1+x^2\right)^{-2\mu}\,e^{-jpx}\mathrm{d}x = \begin{cases} -2\pi 2^{-2\mu}\dfrac{p^{2\mu-1}}{\Gamma(2\mu)}W_{0,\frac{1}{2}-2\mu}\,(2p)\,, & p>0, \\ 2\pi 2^{-2\mu}\dfrac{(-p)^{2\mu-1}}{\Gamma(2\mu)}W_{0,\frac{1}{2}-2\mu}\,(-2p)\,, & p<0, \end{cases} \tag{121}$$

where the Whittaker function $W_{0,\frac{1}{2}-2\mu}$ is defined in the Appendix F of reference [22].

So, we can put Eq. (121) to obtain

$$\int_{-\infty}^{+\infty} \left(1+x^2\right)^{-2\mu}\,e^{-jpx}\mathrm{d}x = -\operatorname{sign}(p)\,\pi\frac{1}{\Gamma(2\mu)}\left(\frac{|p|}{2}\right)^{2\mu-1}W_{0,\frac{1}{2}-2\mu}\,(2|p|)\,, \tag{122}$$

if $\Re(2\mu) > 1/2$.

By setting: $p = 2\pi\tilde{y}$ and $2\mu = (q-1)^{-1}$, with the constraint

$$\Re(2\mu) > 1/2 \implies (q-1)^{-1} > \frac{1}{2}, \tag{123}$$

we get the following cases to compute Eq. (119).

D.1 CASE $q > 1$

Due to the restriction from Eq. (123), we have $1 < q < 3$. By inserting Eq. (122) into Eq. (119) and returning to the original variable y through Eq. (118), we obtain

$$\mathcal{F}\left(\exp_q\left(-ax^2\right), y\right) = \left[(q-1)\,a\right]^{-1/2}\left(-\operatorname{sign}\left(2\pi\left[(q-1)\,a\right]^{-1/2}y\right)\right) \times$$

$$\times 2\pi\left(2^{\frac{1}{1-q}}\right)\frac{\left|2\pi\left[(q-1)\,a\right]^{-1/2}y\right|^{\frac{1}{q-1}-1}}{\Gamma\left(\frac{1}{q-1}\right)}W_{0,\frac{1}{2}-\frac{1}{q-1}}\left(2\left|2\pi\left[(q-1)\,a\right]^{-1/2}y\right|\right), \tag{124}$$

where the Whittaker function $W_{0,\frac{1}{2}-\frac{1}{q-1}}$ is defined in [22].

D.2 CASE $q < 1$

In this case, Eq. (117) is well defined only if

$$1 + (q - 1) ax^2 \geq 0, \tag{125}$$

which implies

$$|x| \leq \frac{1}{\sqrt{(1 - q) a}} = ((1 - q) a)^{-1/2}. \tag{126}$$

Hence, Eq. (117) becomes

$$\mathcal{F}\left(\exp_q\left(-ax^2\right), y\right) = \int_{-((1-q)a)^{-1/2}}^{((1-q)a)^{-1/2}} \exp\left(-2j\pi xy\right) \left[1 + (q - 1) ax^2\right]^{\frac{1}{1-q}} \mathrm{d}x. \tag{127}$$

By using the variable change

$$\begin{aligned}
\widetilde{x} &= ((1 - q) a)^{1/2} x, \\
\widetilde{y} &= ((1 - q) a)^{-1/2} y,
\end{aligned} \tag{128}$$

we can rewrite Eq. (127) as

$$\mathcal{F}\left(\exp_q\left(-ax^2\right), y\right) = \int_{-1}^{1} \exp\left(-2j\pi\widetilde{x}\widetilde{y}\right) \left[1 - \widetilde{x}^2\right]^{\frac{1}{1-q}} \frac{\mathrm{d}\widetilde{x}}{((1 - q) a)^{1/2}}. \tag{129}$$

However, it is known that (Eq. (3.387-2) of [17])

$$\int_{-1}^{1} \left(1 - x^2\right)^{\nu-1} \exp\left(j\mu x\right) \mathrm{d}x = \sqrt{\pi} \left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma\left(\nu\right) \mathbb{J}_{\nu-\frac{1}{2}}\left(\mu\right), \tag{130}$$

if $\Re\left(\nu\right) > 0$.

Therefore, if we set

$$\nu = \frac{1}{1 - q} + 1, \quad \mu = -2\pi\widetilde{y}, \quad x = \widetilde{x}, \tag{131}$$

in Eq. (130), we obtain $\Re\left(\nu\right) > 0$, and

$$\int_{-1}^{1} \left(1 - \widetilde{x}^2\right)^{\frac{1}{1-q}} \exp\left(-2j\pi\widetilde{x}\widetilde{y}\right) \mathrm{d}\widetilde{x} \tag{132}$$

$$= \sqrt{\pi} \left(-\frac{1}{\pi\widetilde{y}}\right)^{\frac{1}{1-q}+\frac{1}{2}} \Gamma\left(\frac{1}{1 - q} + 1\right) \mathbb{J}_{\frac{1}{1-q}+\frac{1}{2}}\left(-2\pi\widetilde{y}\right), \ y \neq 0, \tag{133}$$

where $\mathbb{J}$ denotes the Bessel functions [16]. Consequently, we can use the result from Eq. (133) to rewrite Eq. (127) as

$$\mathcal{F}\left(\exp_q\left(-ax^2\right), y\right) = \tag{134}$$

$$= \frac{\sqrt{\pi}}{((1 - q) a)^{1/2}} \Gamma\left(\frac{2 - q}{1 - q}\right) \left(-\frac{((1 - q) a)^{1/2}}{\pi y}\right)^{\frac{1}{1-q}+\frac{1}{2}} \mathbb{J}_{\frac{1}{1-q}+\frac{1}{2}}\left(-\frac{2\pi y}{((1 - q) a)^{1/2}}\right), \tag{135}$$

if $q < 1$ and $y \neq 0$.

If $y = 0$ in Eq. (127) we get

$$
\begin{aligned}
\mathcal{F}\left(\exp_q\left(-ax^2\right),0\right) &= \int_{-((1-q)a)^{-1/2}}^{((1-q)a)^{-1/2}} \left[1+(q-1)\,ax^2\right]^{\frac{1}{1-q}} \mathrm{d}x \\
&= \int_{-1}^{1} \left[1-\widetilde{x}^2\right]^{\frac{1}{1-q}} \frac{\mathrm{d}\widetilde{x}}{((1-q)\,a)^{1/2}}.
\end{aligned}
\tag{136}
$$

However, from Eq. (3.214), reference [17], we have

$$
\int_0^1 \left[(1+x)^{\mu-1}\,(1-x)^{\nu-1} + (1+x)^{\nu-1}\,(1-x)^{\mu-1}\right] \mathrm{d}x = 2^{\mu+\nu-1} B\left(\mu,\nu\right),
\tag{137}
$$

if $\Re\left(\mu\right) > 0$ and $\Re\left(\nu\right) > 0$. Therefore, if

$$
\mu = \nu = \frac{1}{(1-q)} + 1,
\tag{138}
$$

we satisfy $\Re\left(\mu\right) > 0, \Re\left(\nu\right) > 0$ and

$$
\begin{aligned}
2\int_0^1 \left[(1+x)^{\frac{1}{(1-q)}}\,(1-x)^{\frac{1}{(1-q)}}\right] \mathrm{d}x &= 2\int_0^1 \left(1-x^2\right)^{\frac{1}{(1-q)}} \mathrm{d}x \\
&= 2^{\frac{2}{1-q}+1} B\left(\frac{1}{(1-q)}+1, \frac{1}{(1-q)}+1\right).
\end{aligned}
\tag{139}
$$

So, using this result and the Bessel function definition through the Gamma function [16], we can compute Eq. (136) by

$$
\mathcal{F}\left(\exp_q\left(-ax^2\right),0\right) = \frac{2^{\frac{2}{1-q}+1}}{((1-q)\,a)^{1/2}} \frac{\Gamma\left(\frac{1}{(1-q)}+1\right)\Gamma\left(\frac{1}{(1-q)}+1\right)}{\Gamma\left(\frac{2}{1-q}+2\right)}.
\tag{140}
$$

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