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Global stability of discrete-time nonlinear systems with descriptor standard and fractional positive linear parts and scalar feedbacks

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The global stability of discrete-time nonlinear systems with descriptor positive linear parts and positive scalar feedbacks is addressed. Sufficient conditions for the global stability of standard and fractional nonlinear systems are established. The effectiveness of these conditions is illustrated on numerical examples.

Key words: stability, fractional, positive, nonlinear, discrete-time, feedback, system

1. Introduction

In positive systems inputs, state variables and outputs take only nonnegative values for any nonnegative inputs and nonnegative initial conditions, see [1, 4, 5]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollutions models. A variety of models having positive behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive systems theory is given in the monographs [1, 4, 5, 8, 17].

Mathematical fundamentals of the fractional calculus are given in the monographs [8, 17, 21, 22]. The fractional linear systems have been investigated in [3, 6–9, 23–26]. Positive linear systems with different fractional orders have been addressed in [6, 7, 26]. Descriptor positive systems have been analyzed in [2, 26]. Linear positive electrical circuits with state feedbacks have been addressed in [2].

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The superstabilization of positive linear electrical circuits by state feedbacks have been analyzed in [13] and the stability of nonlinear systems in [10, 11, 12, 14, 18]. The global stability of nonlinear systems with negative feedbacks and positive not necessary asymptotically stable linear parts has been investigated in [15]. The global stability of nonlinear continuous-time standard and fractional positive systems have been analyzed in [16].

In this paper the global stability of discrete-time nonlinear systems with positive descriptor linear parts and positive scalar feedbacks will be addressed.

The paper is organized as follows. In section 2 the basic definitions and theorems concerning the positive discrete-time descriptor linear systems are recalled. New sufficient conditions for the global stability of positive standard discrete-time nonlinear systems are established in section 3. Similar sufficient conditions for fractional positive nonlinear systems are given in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\mathfrak{R}$ – the set of real numbers, $\mathfrak{R}^{n\times m}$ – the set of $n\times m$ real matrices, $\mathfrak{R}_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\mathfrak{R}_+^n = \mathfrak{R}_+^{n\times 1}$, I_n – the $n\times n$ identity matrix.

2. Positive discrete-time descriptor linear systems

Consider the descriptor discrete-time standard linear system

$$Ex_{i+1} = Ax_i + Bu_i, \qquad i = 0, 1, \ldots, \tag{1}$$

$$y_i = Cx_i, \tag{2}$$

where $x_i \in \mathfrak{R}^n$, $u_i \in \mathfrak{R}^m$, $y_i \in \mathfrak{R}^p$ are the state, input and output vectors and $E, A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{n\times m}$, $C \in \mathfrak{R}^{p\times n}$. It is assumed that the pencil (E, A) of (1) is regular, i.e.

$$\det[Ez - A] \neq 0, \qquad z \in \mathbb{C}, \tag{3}$$

where $\mathbb{C}$ is the field of complex numbers.

Definition 1 *The descriptor system (1), (2) is called (internally) positive if* $x_i \in \mathfrak{R}_+^n$, $y_i \in \mathfrak{R}_+^p$, $i = 0, 1, \ldots$ *for every consistent nonnegative initial conditions* $x_0 \in \mathfrak{R}_+^n$ *and all inputs* $u_i \in \mathfrak{R}_+^m$.

The transfer matrix of the system (1), (2) is given by

$$T(z) = C[Ez - A]^{-1}B \in \mathfrak{R}^{p\times m}(z), \tag{4}$$

where $\mathfrak{R}^{p\times m}(z)$ is the set of $p\times m$ rational matrices in z. The transfer matrix (4) can be always decomposed into the strictly proper transfer matrix

$$T_{sp}(z) = C_1\left[I_{n_1}z - A_1\right]^{-1}B_1 \tag{5}$$

and the polynomial matrix

$$P(z) = D_0 + D_1 z + \ldots + D_q z^q \in \Re^{p\times m}(z), \tag{6}$$

where $\Re^{p\times m}(z)$ is the set of $p \times m$ polynomial matrices in z and q is the index of E.

Theorem 1 [17] *The descriptor system (1), (2) is positive if and only if*

$$A_1 \in \Re_+^{n_1\times n_1}, \qquad B_1 \in \Re_+^{n_1\times m}, \qquad C_1 \in \Re_+^{p\times n_1} \tag{7}$$

and

$$D_k \in \Re_+^{p\times m} \qquad \textit{for} \qquad k = 0, 1, \ldots, q. \tag{8}$$

It is assumed that the singular matrix E has only $n_1 < n$ linearly independent columns and the pencil (E, A) is regular. In this case by Weierstrass-Kronecker theorem [17] there exist nonsingular matrices $P \in \Re^{n\times n}$ and $Q \in \Re^{n\times n}$ monomial (in each row and in each column only one entry is positive and the remaining entries are zero) such that

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \qquad PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \qquad n = n_1 + n_2, \tag{9}$$

where $N \in \Re^{n_2\times n_2}$ is the nilpotent matrix such that $N^\mu = 0$, $N^{\mu-1} \neq 0$, μ is the nilpotency index, $A_1 \in \Re^{n_1\times n_1}$ and $n_1 = \deg\det[Ez - A]$.

Premultiplying the equation (1) by the matrix $P \in \Re^{n\times n}$ and defining the new state vector

$$\begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} = Q^{-1} x_i, \qquad x_{1i} \in \Re^{n_1}, \quad x_{2i} \in \Re^{n_2}, \quad i = 0, 1, \ldots \tag{10}$$

we obtain

$$x_{1,i+1} = A_1 x_{1i} + B_1 u_i, \tag{11}$$

$$N x_{2,i+1} = x_{2i} + B_2 u_i, \tag{12}$$

where $A_1 \in \Re^{n_1\times n_1}$, $B_1 \in \Re^{n_1\times m}$, $B_2 \in \Re^{n_2\times m}$ and $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB$.

Note that if $Q \in \Re_+^{n\times n}$ is monomial then $Q^{-1} \in \Re_+^{n\times n}$ and $x_{1i} \in \Re_+^{n_1}$ and $x_{2i} \in \Re_+^{n_2}$ for $i = 0, 1, \ldots$ if $x_i \in \Re_+^n$, $i = 0, 1, \ldots$. Defining $CQ = [\, C_1 \;\; C_2 \,]$, $C_1 \in \Re_+^{p\times n_1}$, $C_2 \in \Re^{p\times n_2}$ for any $C \in \Re_+^{p\times n}$ from (2) we have

$$y_i = C_1 x_{1i} + C_2 x_{2i}. \tag{13}$$

It is easy to verify that

$$\begin{aligned} T(z) &= C[Ez - A]^{-1}B = CQ[P(Ez - A)Q]^{-1}PB \\ &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} I_{n_1}z - A_1 & 0 \\ 0 & Nz - I_{n_2} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ &= C_1 \left[I_{n_1}z - A_1\right]^{-1} B_1 - C_2 \left[I_{n_2} + Nz + \ldots + N^{\mu-1}z^{\mu-1}\right] B_2 \,. \end{aligned} \tag{14}$$

From (11), (12) and (13) we have the following theorem.

Theorem 2 [17] *The descriptor discret-time system (1), (2) is positive if and only if*

$$\begin{gathered} A_1 \in \Re_+^{n_1\times n_1}, \qquad B_1 \in \Re_+^{n_1\times m}, \qquad -B_2 \in \Re_+^{n_2\times m}, \\ C_1 \in \Re_+^{p\times n_1}, \qquad C_2 \in \Re_+^{p\times n_2}. \end{gathered} \tag{15}$$

Theorem 3 [8, 17] *The positive linear discrete-time system (11) is asymptotically stable (the matrix A_1 is Schur) if and only if one of the following equivalent conditions is satisfied:*

1. *All coefficient of the characteristic polynomial*

$$p_n(z) = \det\left[I_n(z+1) - A_1\right] = z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0 \tag{16}$$

   are positive, i.e. $a_i > 0$ for $i = 0, 1, \ldots, n-1$.

2. *There exists strictly positive vector $\lambda^T = [\,\lambda_1 \ \cdots \ \lambda_n\,]^T$, $\lambda_k > 0$, $k = 1, \ldots, n$ such that*

$$(A_1 - I_{n_1})\lambda < 0 \qquad or \qquad \lambda^T(A_1 - I_{n_1}) < 0. \tag{17}$$

3. Global stability of standard descriptor nonlinear feedback systems

Consider the nonlinear feedback system shown in Fig. 1 which consists of the descriptor positive linear part, the nonlinear element with characteristic $u = f(e)$ and positive scalar gain feedback h. The descriptor linear part is described by the equations

$$Ex_{i+1} = Ax_i + Bu_i\,, \qquad i = 0, 1, \ldots, \tag{18}$$

$$y_i = Cx_i\,, \tag{19}$$

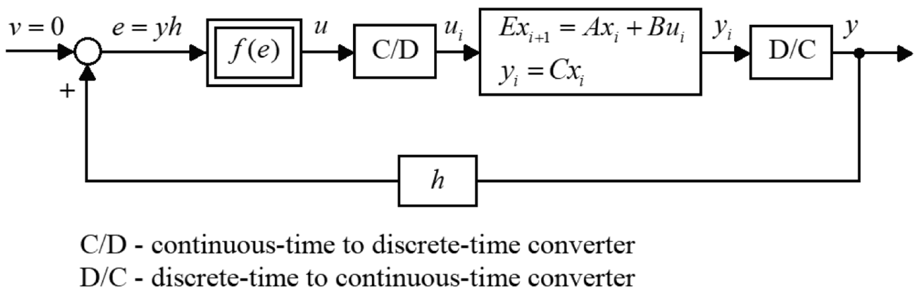

Figure 1: The nonlinear feedback system

where $x_i \in \Re^n$, $u_i \in \Re^1$, $y_i \in \Re^1$ are the state vector, input and output of the system $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times 1}$, $C \in \Re^{1\times n}$. The characteristic $f(e)$ of the nonlinear element (Fig. 2) satisfies the condition

$$0 < f(e) < ke, \qquad 0 < k < \infty. \tag{20}$$

It is assumed that:

1) the pencil (E, A) is regular (the condition (3) is satisfied),
2) the matrix E has n_1 linearly independent columns,
3) $rankE = \deg\ \det[Ez - A] = n_1$.

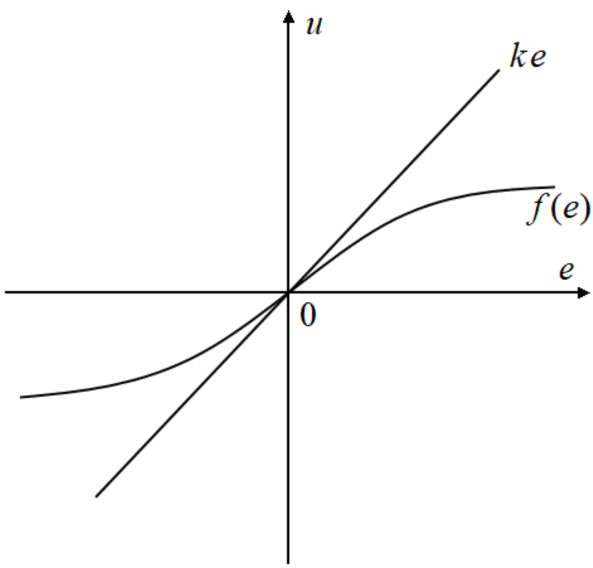

Figure 2: The characteristic of nonlinear element

If the assumptions are satisfied then by Weirstrass-Kronecker theorem there exist nonsingular matrices $P \in \Re^{n\times n}$ and $Q \in \Re^{n\times n}$ monomial such that

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \qquad n = n_1 + n_2, \tag{21}$$

where $A_1 \in \Re^{n_1\times n_1}$ and $n_1 = \deg\det[Es - A]$.

Premultiplying the equation (18) by the matrix $P \in \Re^{n\times n}$ and defining new state vector

$$\begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} = Q^{-1}x_i, \qquad x_{1i} \in \Re^{n_1}, \qquad x_{2i} \in \Re^{n_2} \tag{22}$$

we obtain

$$x_{1,i+1} = A_1x_{1i} + B_1u_i, \tag{23}$$

$$0 = x_{2i} + B_2u_i, \tag{24}$$

where $A_1 \in \Re^{n_1\times n_1}$, $B_1 \in \Re^{n_1\times 1}$, $B_2 \in \Re^{n_2\times 1}$ and

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB. \tag{25}$$

Note that if $Q \in \Re_+^{n\times n}$ is monomial then $Q^{-1} \in \Re_+^{n\times n}$ and $x_{1i} \in \Re_+^{n_1}$ and $x_{2i} = 0$ for $i = 0, 1, \ldots$, $B_2 = 0$ since $N = 0$. In this case defining $CQ = [\, C_1 \;\; C_2 \,]$, $C_1 \in \Re_+^{1\times n_1}$, $C_2 \in \Re^{1\times n_2}$ for any $C \in \Re_+^{1\times n}$ from (19) we have

$$y_i = C_1x_{1i}\,. \tag{26}$$

Definition 2 *The nonlinear positive system is called globally stable if it is asymptotically stable for all nonnegative initial conditions* $x_0 \in \Re_+$.

The following theorem gives sufficient conditions for the global stability of the descriptor positive nonlinear system.

Theorem 4 *The nonlinear system consisting of the positive linear part satisfying the assumptions 1), 2), 3), the nonlinear element satisfying the condition (20) and the gain feedback h is globally stable if the matrix*

$$A_1 + khB_1C_1 \in \Re_+^{n_1} \tag{27}$$

is Schur.

Proof. The proof will be accomplished by the use of the Lyapunov method [19, 20]. As the Lyapunov function $V(x_{1i})$ we choose

$$V(x_{1i}) = \lambda^T x_{1i} \geqslant 0 \qquad \text{for} \quad x_{i1} \in \Re_+^{n_1}, \tag{28}$$

where λ is strictly positive vector, i.e. $\lambda_k > 0$, $k = 1, \ldots, n_1$.

Using (28) and (23) we obtain

$$\begin{aligned} \Delta V(x_{1i}) &= V(x_{1,i+1}) - V(x_{1i}) = \lambda^T(x_{1,i+1} - x_{1i}) \\ &= \lambda^T(A_1 - I_{n_1})x_{1i} + B_1 h f(e) \leqslant \lambda^T(A_1 + khB_1C_1)x_{1i}, \end{aligned} \tag{29}$$

since $\lambda^T(A_1 - I_{n_1}) < 0$ and $(I_{n_1} - A_1)x_{1i} > B_1 h f(e)$.

From (29) it follows that $\Delta V(x_{1i}) < 0$ if the matrix (27) is Schur and the nonlinear system is globally stable. □

Example 1. Consider the nonlinear system with the descriptor positive linear part with the matrices

$$\begin{gathered} E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 0.1 & 0 & 0.2 \\ 1 & 0.5 & 0 & 0.5 \\ 1 & 0.5 & 1 & 0.5 \\ 1 & 0.1 & 0 & 0.2 \end{bmatrix}, \\ B = \begin{bmatrix} 0.5 \\ 1 \\ 1 \\ 0.5 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0.6 & 0 & 0.2 \end{bmatrix}, \end{gathered} \tag{30}$$

the nonlinear element satisfying the condition (20) and the gain feedback $h = 0.5$. Find k satisfying (20) for which the nonlinear system is globally stable.

In this case the matrices P and Q have the forms

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{31}$$

Using (21), (30) and (31) we obtain

$$PEQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \tag{32}$$

$$PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.1 & 0 & 0 \\ 0.3 & 0.4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{33}$$

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}, \tag{34}$$

$$CQ = \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.6 & 0 & 0 \end{bmatrix}. \tag{35}$$

Taking into account (27), (32)–(35) and $h = 0.5$ we obtain

$$\begin{aligned} A_1 + khB_1C_1 &= \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.4 \end{bmatrix} + k \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix} \begin{bmatrix} 0.2 & 0.6 \end{bmatrix} \\ &= \begin{bmatrix} 0.2 + 0.05k & 0.1 + 0.15k \\ 0.3 + 0.05k & 0.4 + 0.15k \end{bmatrix} \end{aligned} \tag{36}$$

and

$$\begin{aligned} \det\left[I_2(z+1) - A_1 - khB_1C_1\right] &= \det \begin{bmatrix} z + 0.8 - 0.05k & -0.1 - 0.15k \\ -0.3 - 0.05k & z + 0.6 - 0.15k \end{bmatrix} \\ &= z^2 + (1.4 - 0.2k)z + 0.45 - 0.2k. \end{aligned} \tag{37}$$

The matrix (36) is Schur if the coefficients of the polynomial (37) are positive, i.e. if $1.4 - 0.2k > 0$ and $0.45 - 0.2k > 0$ and this implies that $k < 2.25$. Therefore, the nonlinear positive system is globally stable if the characteristic $u = f(e)$ of the nonlinear element satisfies the condition (20) for $k < 2.25$.

4. Global stability of fractional descriptor nonlinear feedback systems

Consider the fractional discrete-time linear system, described by the state-space equations

$$E\Delta^\alpha x_{i+1} = Ax_i + Bu_i, \qquad i = 0, 1, \ldots, \tag{38}$$

$$y_i = Cx_i, \tag{39}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $E,\ A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

In this paper the following definition of the fractional discrete-time derivative will be used [8]

$$\Delta^\alpha x_i = \sum_{k=0}^{\infty} c_k x_{i-k}, \tag{40}$$

where $\alpha \in \Re$ is the order of the fractional difference and

$$c_k = (-1)^k \binom{\alpha}{k} = \begin{cases} 1 & \text{for } k = 0, \\ (-1)^k \dfrac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} & \text{for } k = 1, 2, \ldots . \end{cases} \tag{41}$$

Note that values of coefficients c_k for $\alpha \in (0,\ 1)$ are negative and $\sum_{k=1}^{\infty} c_k = -1$.

Theorem 5 [8, 17] *Let* $0 < \alpha < 1$. *Then the fractional system (38), (39) is positive if and only if*

$$A + E\alpha \in \Re_+^{n\times n}, \qquad B \in \Re_+^{n\times m}, \qquad C \in \Re_+^{p\times n}. \tag{42}$$

Consider the fractional nonlinear feedback system shown in Fig. 3 which consists of the positive linear part, the nonlinear element with characteristic $u = f(e)$ and scalar feedbacks. The characteristic of the nonlinear element is shown in Fig. 2 and it satisfies the condition (20).

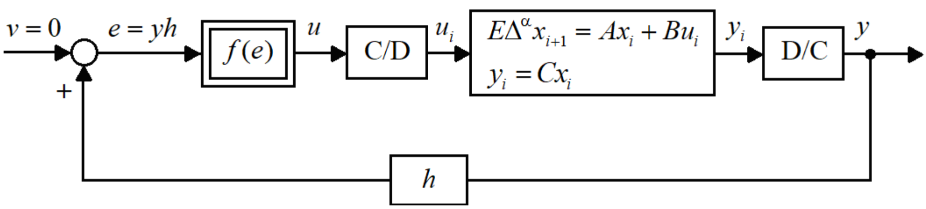

Figure 3: The fractional nonlinear system

The positive linear part is described by the equations

$$E\Delta^\alpha x_{i+1} = Ax_i + Bu_i, \qquad i = 0, 1, \ldots, \qquad \alpha \in (0,\ 1), \tag{43}$$

$$y_i = Cx_i, \tag{44}$$

where $x_i \in \Re_+^n$, $u_i \in \Re_+^1$, $y_i \in \Re_+^1$ are the state, input and output vectors of the system E, $A \in \Re^{n\times n}$, $B \in \Re^{n\times 1}$, $C \in \Re^{1\times n}$.

Using the definition (40) we may write the equations (43), (44) in the form

$$Ex_{i+1} = A_\alpha x_i - E\sum_{k=2}^{\infty} c_k x_{i-k+1} + Bu_i, \tag{45}$$

$$y_i = Cx_i, \tag{46}$$

where $A_\alpha = A + E\alpha$.

It is assumed that:

1) the pencil (E, A) is regular (the condition (3) is satisfied),

2) the matrix E has n_1 linearly independent columns,

3) $rank\, E = \deg \det[Ez - A] = n_1$.

If the assumptions are satisfied then by Weirstrass-Kronecker theorem from (45) we have

$$PEQQ^{-1}x_{i+1} = PAQ^{-1}x_i - PEQ\sum_{k=2}^{\infty} c_k Q^{-1}x_{i-k+1} + PBu_i\,, \tag{47}$$

hence

$$\begin{aligned}\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}\begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} \\ &\quad - \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}\sum_{k=2}^{\infty} c_k \begin{bmatrix} x_{1,i-k+1} \\ x_{2,i-k+1} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_i\end{aligned} \tag{48}$$

and

$$\begin{aligned} x_{1,i+1} &= A_1 x_{1i} - \sum_{k=2}^{\infty} c_k x_{i-k+1} + B_1 u_i\,, \\ B_2 &= 0, \qquad x_{2i} = 0. \end{aligned} \tag{49}$$

where $A_1 \in \Re^{n_1\times n_1}$, $B_1 \in \Re^{n_1\times 1}$, $B_2 \in \Re^{n_2\times 1}$, $PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$.

From (46) we have

$$y_i = CQQ^{-1}x_i\,, \tag{50}$$

hence

$$y_i = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} \tag{51}$$

and

$$\begin{aligned} y_i &= C_1 x_{1i}\,, \\ C_2 &= 0, \end{aligned} \tag{52}$$

where $C_1 \in \Re_+^{1\times n_1}$, $C_2 \in \Re^{1\times n_2}$, $CQ = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$.

The following theorem gives sufficient conditions for the global stability of the fractional descriptor positive nonlinear system.

Theorem 6 *The fractional nonlinear system consisting of the positive linear part satisfying the assumptions 1), 2), 3), the nonlinear element with characteristics satisfying the condition (20) and the gain feedback h is globally stable if the matrix*

$$A_{1\alpha} + khB_1C_1 \in \Re_+^{n_1} \tag{53}$$

is Schur, where $A_{1\alpha} = A_1 - I_{n_1}\alpha$.

Proof. We choose

$$V(x_{1i}) = \lambda^T x_{1i} \geqslant 0, \qquad i = 0, 1, \ldots, \tag{54}$$

where λ is strictly positive vector. Then

$$\begin{aligned} \Delta V(x_{1i}) &= V(x_{1,i+1}) - V(x_{1,i}) = \lambda^T (x_{1,i+1} - x_{1,i}) \\ &= \lambda^T \left[(A_1 - I_{n_1}) x_{1,i} - \sum_{k=2}^{\infty} c_k x_{1,i-k+1} \right] + B_1 u_i \\ &= \lambda^T (A_1 - I_{n_1}) x_{1,i} + khB_1C_1 x_{1,i} - \sum_{k=2}^{\infty} c_k x_{1,i-k+1} \\ &\leqslant \lambda^T (A_{1\alpha} + khB_1C_1) x_{1,i} \end{aligned} \tag{55}$$

since $\sum_{k=2}^{\infty} c_k = \alpha - 1$ and $-\sum_{k=2}^{\infty} c_k x_{1,i-k+1} > 0$.

From (55) it follows that $\Delta V(x_{1i}) < 0$ if the matrix (53) is Schur and the fractional nonlinear system is globally stable. □

Example 2. Consider the nonlinear system with the fractional descriptor positive linear part with the matrices

$$\begin{gathered} E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.7 & 1 & 0 & 0.3 \\ 0.25 & 0 & 1 & 1.25 \\ 0 & 1 & 0 & 0 \\ 0.25 & 0 & 2 & 1.25 \end{bmatrix}, \\ B = \begin{bmatrix} 0.6 \\ 0.5 \\ 0 \\ 0.5 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.5 & 0 & 0 & 1 \end{bmatrix}, \end{gathered} \tag{56}$$

the nonlinear element satisfying the condition (20) and the gain feedback $h = 0.5$. Find k satisfying (20) for which the nonlinear system is globally stable for $\alpha = 0.2$.

In this case the matrices P and Q have the forms

$$P = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0.8 & 0 & -0.4 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{57}$$

Using (21), (56) and (57) we obtain

$$PEQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \tag{58}$$

$$PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 & 0 & 0 \\ 0.1 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{59}$$

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.2 \\ 0 \\ 0 \end{bmatrix}, \tag{60}$$

$$CQ = \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} 0.5 & 1 & 0 & 0 \end{bmatrix}. \tag{61}$$

Taking into account (53), (58)–(61) $h = 0.5$ and $\alpha = 0.2$ we obtain

$$\begin{aligned} A_{1\alpha} + khB_1C_1 &= \begin{bmatrix} 0.5 & 0.3 \\ 0.1 & 0.3 \end{bmatrix} + k \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 + 0.15k & 0.3 + 0.3k \\ 0.1 + 0.05k & 0.3 + 0.1k \end{bmatrix} \end{aligned} \tag{62}$$

and

$$\begin{aligned} \det\left[I_2(z+1) - A_{1\alpha} - khB_1C_1\right] &= \det \begin{bmatrix} z + 0.5 - 0.15k & -0.3 - 0.3k \\ -0.1 - 0.005k & z + 0.7 - 0.1k \end{bmatrix} \\ &= z^2 + (1.2 - 0.25k)z + 0.32 - 0.2k. \end{aligned} \tag{63}$$

The matrix (62) is Schur if the coefficients of the polynomial (63) are positive, i.e. if $1.2 - 0.25k > 0$ and $0.32 - 0.2k > 0$ and this implies that $k < 1.6$. Therefore, the fractional nonlinear positive system is globally stable if the characteristic $u = f(e)$ of the nonlinear element satisfies the condition (20) for $k < 1.6$.

5. Concluding remarks

The global stability of discrete-time nonlinear systems with descriptor positive linear parts and positive scalar feedbacks has been investigated. Sufficient conditions for the global stability of this class of nonlinear systems have been established (Theorem 4). Similar results have been obtained for fractional nonlinear discrete-time systems (Theorem 6). The effectiveness of these conditions has been illustrated on numerical examples. An open problem is an extension of the considerations to nonlinear systems of fractional orders with descriptor positive linear parts and interval state matrices.

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