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Fuzzy goal programming technique for multi-objective indefinite quadratic bilevel programming problem

R. ARORA and K. GUPTA

Bilevel programming problem is a non-convex two stage decision making process in which the constraint region of upper level is determined by the lower level problem. In this paper, a multi-objective indefinite quadratic bilevel programming problem (MOIQBP) is presented. The defined problem (MOIQBP) has multi-objective functions at both the levels. The followers are independent at the lower level. A fuzzy goal programming methodology is employed which minimizes the sum of the negative deviational variables of both the levels to obtain highest membership value of each of the fuzzy goal. The membership function for the objective functions at each level is defined. As these membership functions are quadratic they are linearized by Taylor series approximation. The membership function for the decision variables at both levels is also determined. The individual optimal solution of objective functions at each level is used for formulating an integrated pay-off matrix. The aspiration levels for the decision makers are ascertained from this matrix. An algorithm is developed to obtain a compromise optimal solution for (MOIQBP). A numerical example is exhibited to evince the algorithm. The computing software LINGO 17.0 has been used for solving this problem.

Key words: bilevel programming, indefinite quadratic programming, multi-objective programming, pay-off matrix, Taylor series approximation, LINGO 17.0

1. Introduction

Fuzzy programming is a tool to find compromise optimal solution for multi-objective mathematical problem. Fuzzy mathematical programming was developed by Tanaka et al. [9]. Thereafter, Zimmermann in 1978 [8] elaborated the method of fuzzy programming for linear programming problem with several ob-

R. Arora (corresponding Author), E-mail: ritunarora@yahoo.in is with Department of Mathematics, Keshav Mahavidyalaya, University of Delhi, Delhi, India.

K. Gupta is with Department of Mathematics, Kirori Mal College, University of Delhi, Delhi, India. E-mail: gupta_kavita31@yahoo.com

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jectives. Mohanty and Vijayaraghavan [2] in 1995 converted a multi-objective programming problem into its corresponding goal programming problem by fuzzy approach. In 2006, Waiel et al. [4] presented fuzzy goal programming approach to multi-objective transportation problems. In 2007, Pramanik and Roy [17] suggested a solution methodology for a hierarchical consortium. They solved multilevel programming problem by applying linear fuzzy goal programming. Zhang and Lu [7] contemplated the fuzzy programming strategies for bilevel programming problems.

In this paper, a bilevel programming problem is considered which has multi-objective indefinite quadratic functions at both the levels. The compromise optimal solution for the problem (MOIQBP) is obtained by Fuzzy programming method. The membership functions are defined for the decision makers as well as for the decision variables at both levels. The approach in this paper is novel to the extent that it proposes method which linearizes the quadratic membership functions using Taylor series approximation. The paper has been organized into the following sections: Literature review of the paper is presented in section 2, section 3 exhibits the mathematical formulation of the problem (MOIQBP). In section 4, membership functions for the objective functions and decision variables have been defined. Section 5 describes the method of solving the problem by fuzzy goal programming approach. The algorithm is presented in section 6 followed by an example elaborating its viable usage.

2. Literature review

A bilevel programming problem (BLPP) is a hierarchical optimization problem. This problem has two levels namely, the upper level called the leader and the lower level, called the follower. BLPP is mathematically defined as,

$$\begin{aligned} \text{(BLPP)}: \quad & \underset{X}{\text{Max}}\ Z_1(X,Y) \\ \text{where} \quad & Y \ \text{solves} \\ & \underset{Y}{\text{Max}}\ Z_2(X,Y), \quad \text{for a given } X \\ & \text{subject to } (X,Y) \in S_0, \end{aligned}$$

where $S_0 = \{(X,Y)\colon AX + BY \leqslant b; X, Y \geqslant 0\}$.

Here, S_0 is the feasible region for the problem (BLPP). It is assumed to be non-empty and bounded. The leader and the follower controls the vector of decision variables $X \in R^{n_1}$ and $Y \in R^{n_2}$ respectively. $Z_1(X,Y)$ and $Z_2(X,Y)$ are the objective functions of the leader and the follower respectively. They can be linear or non-linear. There are various significant techniques for solving BLPP such as cutting plane method [5], branch and bound method [14] and the ranking method [1, 12], to name a few. BLPP has been extensively utilized by researchers in distinctive fields like economics, transportation engineering etc. In 1992, Ben-Ayed et al. [15] contemplated the highway network design problem. In 2003, Cote et al. [10] came out with a bilevel modelling approach in the airline industry. This procedure provided the solution for various problems confronted by North American airline.

There are varying scenarios among decision makers that are suitably represented by Multi-objective Programming Problem (MOPP). MOPP is mathematically defined as,

$$\text{(MOPP)}: \quad \begin{array}{l} \text{Max } \{f_1(X) = Z_1\} \\ \text{Max } \{f_2(X) = Z_2\} \\ \dots\dots\dots\dots\dots\dots \\ \text{Max } \{f_k(X) = Z_k\} \\ \text{subject to } \ X \in S_M, \end{array}$$

where S_M is a feasible set and $f_j(X)$; $\{j = 1, 2, \dots, k\}$ be linear/non-linear.

Emam [16] in 2013 presented a solution procedure for bilevel integer multi-objective fractional programming problem. Abo Sinna and Baky [13] in 2007, proposed an algorithm for solving three-level multi-objective decision making model.

MOPP has a range of applications in real life situations. In 2007, Abdelaziz et al. [6] undertook the approach of multi-objective optimization for portfolio selection. Wari and Zhu [3] applied the mode of metaheuristics in food manufacturing industry. In 2017, Cui et al. [18] applied multi-objective optimization in the field of environment protection such as energy saving. Wang et al. [11] proposed a multi-objective optimization model which speculated the exactness of wind speed.

The research work of the authors referred herein has significant contribution in their respective research fields. The methodologies defined above have distinct applications in practical life. In this paper, a multi-objective bilevel programming problem is formulated where each objective function is indefinite quadratic. Mathematically, the indefinite quadratic programming problem (IQPP) is,

$$\text{(IQPP)}: \quad \begin{array}{l} \text{Max } Z(X) = Z_1(X).\ Z_2(X) = (C^T X + \alpha)(D^T X + \beta) \\ \text{subject to } \ AX \leqslant b \\ \qquad\qquad\quad X \geqslant 0 \end{array}$$

where $X \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $C,\ D \in \mathbb{R}^n$, $\alpha,\ \beta \in \mathbb{R}$ and $A \in \mathbb{R}^{m\times n}$.

The feasible region $S_I = \{X : AX \leqslant b; X \geqslant 0\}$ is both non-empty and compact. The objective function $Z(X)$ is a product of two linear functions. It is assumed that both $Z_1(X)$ and $Z_2(X)$ are positive for all $X \in S_I$. Thus, the function

$Z(X)$ is quasi-concave and quasi-convex on S_I. Therefore, the optimal solution to the problem (IQPP) occurs at an extreme point of S_I. Many applications of quadratic programming have important implications on decision making problems in practical aspects of life. These can be segregated in various categories such as marketing, economics, finance, agriculture, to name a few.

3. Mathematical formulation

The multi-objective indefinite quadratic bilevel programming problem (MOIQBP) is defined as

$$\begin{aligned}
\text{(MOIQBP)}: \quad & \underset{X,X_1,X_2,\ldots,X_n}{\text{Max}} \left(Z_1(X,X_1,X_2,\ldots,X_n),\ Z_2(X,X_1,X_2,\ldots,X_n),\ \ldots,\right. \\
& \qquad\qquad\qquad\qquad\qquad \left. Z_k(X,X_1,X_2,\ldots,X_n)\right) \\
& \text{subject to } \ AX + \sum_{\ell=1}^{n} D_\ell X_\ell \leqslant b \\
& \text{where } \ X_\ell \ \ (\ell = 1,2,\ldots,n) \ \text{ solves} \\
& \underset{X_\ell}{\text{Max}} \ \ z_{\ell w}(X, X_\ell) \quad (w = 1,2,\ldots,t_\ell) \\
& \text{subject to } \ B_\ell X + C_\ell X_\ell \leqslant b_\ell \\
& \qquad\qquad X \geqslant 0, \quad X_\ell \geqslant 0 \quad (\ell = 1,2,\ldots,n).
\end{aligned}$$

Denote $(X, X_1, \ldots, X_n)$ as $\bar{X}$ and $(X_1, X_2, \ldots, X_n)$ as $\bar{\bar{X}}$.
Here, $Z_j(\bar{X}) = Z_{j1}(\bar{X}).Z_{j2}(\bar{X})$; $j = 1,2,\ldots,k$
where $Z_{j1}(\bar{X}) = (a_{j1}X + b_{j1}X_1 + \ldots + p_{j1}X_n + \gamma_j)$; $j = 1,2,\ldots,k$
$Z_{j2}(\bar{X}) = (a_{j2}X + b_{j2}X_1 + \ldots + p_{j2}X_n + \delta_j)$; $j = 1,2,\ldots,k$.
Also, we have,

$$\begin{aligned}
\underset{X_\ell}{\text{Max}}\, z_{\ell w}(X, X_\ell) &= \underset{X_\ell}{\text{Max}}\, (h_{\ell w}(X, X_\ell)), \quad \ell = 1,2,\ldots,n; \ \ w = 1,2,\ldots,t_\ell \\
h_{1w}(X, X_1) &= (c_{w_1}X + c_{w_2}X_1 + \alpha_{w_1})(d_{w_1}X + d_{w_2}X_1 + \alpha_{w_2}); \quad w = 1,2,\ldots,t_1 \\
h_{2w}(X, X_2) &= (e_{w_1}X + e_{w_2}X_2 + \beta_{w_1})(f_{w_1}X + f_{w_2}X_2 + \beta_{w_2}); \quad w = 1,2,\ldots,t_2 \\
\ldots\ldots \quad & \ldots\ldots\ldots\ldots\ldots\ldots \\
h_{nw}(X, X_n) &= (u_{w_1}X + u_{w_2}X_n + \phi_{w_n})(v_{w_1}X + v_{w_2}X_n + \psi_{w_n}); \quad w = 1,2,\ldots,t_n
\end{aligned}$$

$$\begin{aligned}
c_{w_1},\ d_{w_1},\ X &\in \mathbb{R}^{n_0}; \quad w = 1,2,\ldots,t_1 \\
e_{w_1},\ f_{w_1},\ X &\in \mathbb{R}^{n_0}; \quad w = 1,2,\ldots,t_2 \\
\ldots\ldots\ldots\ldots \quad & \ldots\ldots\ldots\ldots \\
u_{w_1},\ v_{w_1},\ X &\in \mathbb{R}^{n_0}; \quad w = 1,2,\ldots,t_n \\
c_{w_2},\ d_{w_2},\ X_1 &\in \mathbb{R}^{n_1}; \quad w = 1,2,\ldots,t_1
\end{aligned}$$

$$
\begin{array}{ll}
e_{w_2},\ f_{w_2},\ X_2 \in \mathbb{R}^{n_2}; & w = 1,2,\ldots,t_2 \\
\ldots\ldots\ldots\ldots & \\
u_{w_2},\ v_{w_2},\ X_n \in \mathbb{R}^{n_n}; & w = 1,2,\ldots,t_n \\
\alpha_{w_1},\ \alpha_{w_2} \in \mathbb{R}; & w = 1,2,\ldots,t_1 \\
\ldots\ldots\ldots\ldots & \\
\phi_{w_n},\ \psi_{w_n} \in \mathbb{R}; & w = 1,2,\ldots,t_n \\
a_{j_1},\ a_{j_2},\ X \in \mathbb{R}^{n_0}; & j = 1,2,\ldots,k \\
b_{j_1},\ b_{j_2},\ X_1 \in \mathbb{R}^{n_1}; & j = 1,2,\ldots,k \\
\ldots\ldots\ldots\ldots & \\
p_{j_1},\ p_{j_2},\ X_n \in \mathbb{R}^{n_n}; & j = 1,2,\ldots,k \\
\gamma_j,\ \delta_j \in \mathbb{R}; & j = 1,2,\ldots,k \\
A \in \mathbb{R}^{m\times n_0}; \quad D_\ell \in \mathbb{R}^{m\times n_\ell}; & \ell = 1,2,\ldots,n \\
B_\ell \in \mathbb{R}^{m_\ell\times n_0}; \quad C_\ell \in \mathbb{R}^{m_\ell\times n_\ell}; & b \in \mathbb{R}^{m\times 1}, \quad b_\ell \in \mathbb{R}^{m_\ell\times 1}.
\end{array}
$$

The objective functions defined at each level in the problem (MOIQBP) are indefinite quadratic. These functions are the product of two positive valued affine functions, hence they are quasi-concave. Thus, the optimal solution for each objective function (Z_j; $j = 1,2,\ldots,k$; $z_{\ell w}$; $\ell = 1,2\ldots,n$, $w = 1,2,\ldots,t_\ell$) exists. The feasible region of the problem (MOIQBP) is defined as

$$
S(\bar{X}) = \Big\{(X, X_1,\ldots,X_n)\colon AX + \sum_{\ell=1}^{n} D_\ell X_\ell \leqslant b;\ \ B_\ell X + C_\ell X_\ell \leqslant b_\ell; \quad (\ell = 1,2,\ldots,n), \quad X,\ X_\ell \geqslant 0\Big\}.
$$

The feasible region S is assumed to be non-empty and compact.

For each value of X, the feasible set for the ℓ-th follower is defined as

$$
S_\ell(X) = \{X_\ell\colon C_\ell X_\ell \leqslant b_\ell - B_\ell X\}; \quad \ell = 1,\ldots,n.
$$

The constraints $AX + \sum_{\ell=1}^{n} D_\ell X_\ell \leqslant b$ are not included in the set $S_\ell(X)$, since these constraints influence the decision makers at the upper level. Projection of S onto the decision space of upper level problem is defined as

$$
S^* = \Big\{X \in \mathbb{R}^{n_0}\colon \exists(X_1, X_2,\ldots,X_n),\ \ AX + \sum_{\ell=1}^{n} D_\ell X_\ell \leqslant b,\ \ B_\ell X + C_\ell X_\ell \leqslant b_\ell; \quad \ell = 1,\ldots,n\Big\}.
$$

The inducible region for the upper level problem is defined as

IR $= \{(X, X_1, \ldots, X_n) \in S(\bar{X});\ X \geqslant 0;\ X_\ell$ solves Max$(z_{\ell w}(X, X_\ell))$, for a given X such that $B_\ell X + C_\ell X_\ell \leqslant b_\ell;\ X_\ell \geqslant 0,\ \ell = 1, \ldots, n\}$.

It is assumed that the inducible region IR of the problem (MOIQBP) is non-empty.

It establishes the fact that the solution of the problem (MOIQBP) exists.

Definition 1 Efficient Solution

A solution $(X^*,\ X_1^*,\ \ldots,\ X_n^*) \in S$ *is said to be an efficient solution for the problem (MOIQBP) if there is no* $(X, X_1, \ldots, X_n) \in S$*, such that for the upper level problem* $Z_j(X^*,\ X_1^*,\ \ldots,\ X_n^*) \leqslant Z_j(X, X_1,\ \ldots,\ X_n)$ *for* $j = 1, 2, \ldots, k$ *and* $Z_j(X^*, X_1^*, \ldots, X_n^*) < Z_j(X, X_1, \ldots, X_n)$ *for some* $j \in \{1, 2, \ldots, k\}$.

Also, for the lower level problem, $z_{\ell w}(X^*, X_\ell^*) \leqslant z_{\ell w}(X, X_\ell)$ *for* $\ell = 1, 2, \ldots, n$; $w = 1, 2, \ldots.\ t_\ell$ *and* $z_{\ell w}(X^*, X_\ell^*) < z_{\ell w}(X, X_\ell)$ *for some* $\ell \in \{1, 2, \ldots, n\}$; $w \in \{1, 2, \ldots, t_\ell\}$.

Definition 2 Compromise optimal solution

The best efficient solution mutually chosen by the decision makers at both the levels for the problem (MOIQBP) is the compromise optimal solution.

4. Methodology for solving (MOIQBP): fuzzy goal programming approach

In this paper, fuzzy programming approach is adopted to solve (MOIQBP) problem. Each objective function at both the levels is converted into fuzzy goals. The objective functions $Z_j(\bar{X})$; $(j = 1, 2, \ldots, k)$ and $z_{\ell w}(X, X_\ell)$; $(\ell = 1, \ldots, n$, $w = 1, 2, \ldots, t_\ell)$ at each level is solved discretely and its optimal solution is calculated. The aspiration level of each objective function both at upper and lower level is defined by allocating their respective maximum optimal values to them. This is because the objective functions when evaluated exclusively yield the best solution.

4.1. Membership function for the objective functions of the problem (MOIQBP)

The initial step for constructing membership functions would be to ascertain fuzzy goals and their respective aspiration levels. For this, calculate each objective function separately at both the levels. Let $(x^1, x_1^1, x_2^1, \ldots, x_n^1), \ldots, (x^k, x_1^k, \ldots, x_n^k)$ be the optimal solution of each objective at upper level. Let $(x^{1*}, x_1^{1*}), \ldots, (x^{1*}, x_1^{t_1*}), \ldots, (x^{1*}, x_n^{1*}), \ldots, (x^{1*}, x_n^{t_n*})$ be the optimal solution of each objective at lower level. These independent best solutions pave the formulation of an integrated pay-off matrix. This is defined as follows:

$$
\begin{bmatrix}
Z_1(x^1,x_1^1,...,x_n^1) & Z_2(x^1,x_1^1,...,x_n^1) & \dots & Z_k(x^1,x_1^1,...,x_n^1) & z_{11}(x^1,x_1^1,...,x_n^1) & \dots & z_{1t_1}(x^1,x_1^1,...,x_n^1) & \dots & z_{n_1}(x^1,x_1^1,...,x_n^1) & \dots & z_{nt_n}(x^1,x_1^1,...,x_n^1) \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
Z_1(x^k,x_1^k,...,x_n^k) & Z_2(x^k,x_1^k,...,x_n^k) & \dots & Z_k(x^k,x_1^k,...,x_n^k) & z_{11}(x^k,x_1^k,...,x_n^k) & \dots & z_{1t_1}(x^k,x_1^k,...,x_n^k) & \dots & z_{n_1}(x^k,x_1^k,..,x_n^k) & \dots & z_{nt_n}(x^k,x_1^k,...,x_n^k) \\
Z_1(x^{1*},x_1^{1*}) & Z_2(x^{1*},x_1^{1*}) & \dots & Z_k(x^{1*},x_1^{1*}) & z_{11}(x^{1*},x_1^{1*}) & \dots & z_{1t_1}(x^{1*},x_1^{1*}) & \dots & z_{n_1}(x^{1*},x_1^{1*}) & \dots & z_{nt_n}(x^{1*},x_1^{1*}) \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
Z_1(x^{1*},x_1^{t_1^*}) & Z_2(x^{1*},x_1^{t_1^*}) & \dots & Z_k(x^{1*},x_1^{t_1^*}) & z_{11}(x^{1*},x_1^{t_1^*}) & \dots & z_{1t_1}(x^{1*},x_1^{t_1^*}) & \dots & z_{n_1}(x^{1*}x_1^{t_1^*}) & \dots & z_{nt_n}(x^{1*},x_1^{t_1^*}) \\
Z_1(x^{1*},x_2^{1*}) & Z_2(x^{1*},x_2^{1*}) & \dots & Z_k(x^{1*},x_2^{1*}) & z_{11}(x^{1*},x_2^{1*}) & \dots & z_{1t_1}(x^{1*},x_2^{1*}) & \dots & z_{n_1}(x^{1*},x_2^{1*}) & \dots & z_{nt_n}(x^{1*},x_2^{1*}) \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
Z_1(x^{1*},x_2^{t_2^*}) & Z_2(x^{1*},x_2^{t_2^*}) & \dots & Z_k(x^{1*},x_2^{t_2^*}) & z_{11}(x^{1*},x_2^{t_2^*)} & \dots & z_{1t_1}(x^{1*},x_2^{t_2^*}) & \dots & z_{n_1}(x^{1*},x_2^{t_2^*}) & \dots & z_{nt_n}(x^{1*},x_2^{t_2^*}) \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
Z_1(x^{1*},x_n^{1*}) & Z_2(x^{1*},x_n^{1*}) & \dots & Z_k(x^{1*},x_n^{1*}) & z_{11}(x^{1*},x_n^{1*}) & \dots & z_{1t_1}(x^{1*},x_n^{1*}) & \dots & z_{n_1}(x^{1*},x_n^{1*}) & \dots & z_{nt_n}(x^{1*},x_n^{1*}) \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
Z_1(x^{1*},x_n^{t_n^*}) & Z_2(x^{1*},x_n^{t_n^*}) & \dots & Z_k(x^{1*},x_n^{t_n^*}) & z_{11}(x^{1*},x_n^{t_n^*}) & \dots & z_{1t_1}(x^{1*},x_n^{t_n^*}) & & z_{n_1}(x^{1*},x_n^{t_n^*}) & & z_{nt_n}(x^{1*},x_n^{t_n^*})
\end{bmatrix}.
$$

Each column of the pay-off matrix provides maximum and minimum value for each objective function. These values allocate their corresponding aspiration level of achievement. Let Z_j^U and Z_j^L $(j = 1, 2, \ldots, k)$ be the maximum and minimum values in the pay-off matrix for objective functions at the upper level. Similarly, $z_{\ell w}^U$ and $z_{\ell w}^L$ $(\ell = 1, \ldots, n,\ w = 1, 2, \ldots, t_\ell)$ be the maximum and minimum values for objective functions at the lower level. These maximum and minimum values yield the bounds of the objective functions Z_j;$(j = 1, 2, \ldots, k)$ and $z_{\ell w}$; $(\ell = 1, \ldots, n,\ w = 1, 2, \ldots, t_\ell)$.

For each $j = 1, 2, \ldots, k$, define the membership function for the objective functions at the upper level as:

$$\mu_j(Z_j) = \begin{cases} 1 & \text{if } Z_j(\bar{X}) \geqslant Z_j^U \\ \dfrac{Z_j - Z_j^L}{Z_j^U - Z_j^L} & \text{if } Z_j^L \leqslant Z_j \leqslant Z_j^U \\ 0 & \text{if } Z_j(\bar{X}) \leqslant Z_j^L \end{cases}. \tag{1}$$

Likewise, for each $\ell = 1, \ldots, n$, $w = 1, 2, \ldots, t_\ell$, define the membership function for the objective functions at the lower level as:

$$\nu_{\ell w}(z_{\ell w}) = \begin{cases} 1 & \text{if } z_{\ell w}(X, X_\ell) \geqslant z_{\ell w}^U \\ \dfrac{z_{\ell w} - z_{\ell w}^L}{z_{\ell w}^U - z_{\ell w}^L} & \text{if } z_{\ell w}^L \leqslant z_{\ell w} \leqslant z_{\ell w}^U \\ 0 & \text{if } z_{\ell w} \leqslant z_{\ell w}^L \end{cases}. \tag{2}$$

The membership function elucidated in equations (1) and (2) are quadratic. Therefore, these are linearized by applying Taylor series approximation.

4.2. Taylor series Approximation for Quadratic Membership functions

Let $x_j^* = (x^j, x_1^j, \ldots, x_n^j)$ $(j = 1, 2, \ldots, k)$ be the optimal solution obtained from each objective function at the upper level when calculated separately. Then, the Taylor series approximation for the upper level decision maker is represented as,

$$\mu_1(Z_1) = \mu_1(x_1^*) + (x - x^1)\frac{\partial}{\partial x}\mu_1(x_1^*) + (x_1 - x_1^1)\frac{\partial}{\partial x_1}\mu_1(x_1^*)$$
$$+ (x_2 - x_2^1)\frac{\partial}{\partial x_2}\mu_1(x_1^*) + \ldots + (x_n - x_n^1)\frac{\partial}{\partial x_n}\mu_1(x_1^*),$$

$$\mu_2(Z_2) = \mu_2(x_2^*) + (x - x^2)\frac{\partial}{\partial x}\mu_2(x_2^*) + (x_1 - x_1^2)\frac{\partial}{\partial x_1}\mu_2(x_2^*)$$
$$+ (x_2 - x_2^2)\frac{\partial}{\partial x_2}\mu_2(x_2^*) + \ldots + (x_n - x_n^2)\frac{\partial}{\partial x_n}\mu_2(x_2^*)$$
$$\ldots \quad \ldots$$
$$\mu_k(Z_k) = \mu_k(x_k^*) + (x - x^k)\frac{\partial}{\partial x}\mu_k(x_k^*) + (x_1 - x_1^k)\frac{\partial}{\partial x_1}\mu_k(x_k^*)$$
$$+ (x_2 - x_2^k)\frac{\partial}{\partial x_2}\mu_k(x_k^*) + \ldots + (x_n - x_n^k)\frac{\partial}{\partial x_n}\mu_k(x_k^*).$$

Let these Taylor series approximations be denoted by $\delta_1(Z_1), \delta_2(Z_2), \ldots, \delta_k(Z_k)$.

In the same way, Taylor series approximation for the decision makers at the lower level can be defined. Let these be denoted by $\eta_{11}(z_{11})$, $\eta_{12}(z_{12})$, $\ldots$, $\eta_{1t_1}(z_{1t_1})$, $\eta_{21}(z_{21})$, $\ldots$, $\eta_{2t_2}(z_{2t_2})$, $\ldots$, $\eta_{n_1}(z_{n_1})$, $\ldots$, $\eta_{nt_n}(z_{nt_n})$. The advantage of defining Taylor series approximation for the objective functions at both the levels is that it transforms the quadratic membership function to linear membership functions. The concurrence between the decision makers at both the levels is essential to obtain a compromise optimal solution. In order to define fuzzy goal programming model, bounds need to be imposed on the decision variables controlled by the upper and the lower level objective functions.

The optimal solution obtained from each objective function at both the levels specifies the preference bounds on the decision variables controlled by the respective decision makers.

4.3. Membership function for the decision variables of the problem (MOIQBP)

From section 3.1, Z_j^L and Z_j^U are the minimum and maximum objective function values for Z_j; $(j = 1, 2, \ldots, k)$ at the upper level. It accedes to the bounds on those decision variables which are controlled by upper level. Let these bounds be denoted by X^L and X^U. The membership function for the decision variables at the upper level is defined as

$$\mu(X) = \begin{cases} 1 & \text{if } X \geqslant X^U \\ \dfrac{X - X^L}{X^U - X^L} & \text{if } X^L \leqslant X \leqslant X^U \\ 0 & \text{if } X \leqslant X^L \end{cases}. \tag{3}$$

Similarly, membership function for the decision variables X_ℓ; $(\ell = 1, 2, \ldots, n)$ at the lower level can be defined as

$$\mu(X_\ell) = \begin{cases} 1 & \text{if } X_\ell \geqslant X_\ell^U \\ \dfrac{X_\ell - X_\ell^L}{X_\ell^U - X_\ell^L} & \text{if } X_\ell^L \leqslant X_\ell \leqslant X_\ell^U \\ 0 & \text{if } X_\ell \leqslant X_\ell^L \end{cases}. \tag{4}$$

5. Technique for solving (MOIQBP) by fuzzy goal programming

The objective of fuzzy goal programing approach is to obtain the satisfactory solution for the decision makers at both the levels. The membership functions for the objective functions and the decision variables at both the upper level and lower level are defined. The highest degree a membership function could attain is unity. Thus, the negative deviational variables are to be minimized so that each decision maker could maximize his membership function nearest to unity. Accordingly, for the above defined linear membership functions, the flexile membership goals with aspiration level unity are presented as,

$$\begin{array}{ll} \delta_j(Z_j) + d_j^- - d_j^+ = 1, & j = 1,2,\ldots,k \\ \eta_{\ell w}(z_{\ell w}) + d_{\ell w}^- - d_{\ell w}^+ = 1, & \ell = 1,2,\ldots,n; \quad w = 1,2,\ldots,t_\ell \\ \mu(X) + d_X^- - d_X^+ = 1 & \\ \mu(X_\ell) + d_{X_\ell}^- - d_{X_\ell}^+ = 1, & \ell = 1,2,\ldots,n \end{array} . \tag{5}$$

Here, d_j^-, $d_{\ell w}^-$, d_j^+, $d_{\ell w}^+$ ($\geqslant 0$); ($j = 1,2,\ldots,k$; $\ell = 1,2,\ldots,n$; $w = 1,2,\ldots,t_\ell$) are under and over deviational variables for the decision makers at the upper level and lower level respectively. Similarly, d_X^-, $d_{X_\ell}^-$ ($\geqslant 0$) and d_X^+, $d_{X_\ell}^+$ ($\geqslant 0$); ($\ell = 1,2,\ldots,n$; $w = 1,2,\ldots,t_\ell$) are under and over deviational variables for the decision variables at the upper level and lower level respectively. Thus, the problem (MOIQBP) develops into the following fuzzy goal programming model, defined as

(MOIQFGP):

$$\text{Min}\,\xi = \left(\sum_{j=1}^{k} d_j^- + \sum_{j=1}^{k} d_j^+\right) + \left(\sum_{w=1}^{t_\ell}\sum_{\ell=1}^{n} d_{\ell w}^- + \sum_{w=1}^{t_\ell}\sum_{\ell=1}^{n} d_{\ell w}^+\right) + \left(d_X^- + d_X^+\right) + \left(\sum_{\ell=1}^{n} d_{X_\ell}^- + \sum_{\ell=1}^{n} d_{X_\ell}^+\right)$$

$$\text{subject to}\ \ AX + \sum_{\ell=1}^{n} D_\ell X_\ell \leqslant b$$

$$B_\ell X + C_\ell X_\ell \leqslant b_\ell, \quad \ell = 1,2,\ldots,n$$

$$\delta_j(Z_j) + d_j^- - d_j^+ = 1; \quad j = 1,2,\ldots,k$$

$$\eta_{\ell w}(z_{\ell w}) + d_{\ell w}^- - d_{\ell w}^+ = 1; \quad \ell = 1,2,\ldots,n; \quad w = 1,2,\ldots,t_\ell$$

$$\mu(X) + d_X^- - d_X^+ = 1$$

$$\mu(X_\ell) + d^-_{X_\ell} - d^+_{X_\ell} = 1, \qquad \ell = 1, 2, \ldots, n,$$
$$0 \leqslant d^-_j \leqslant 1; \quad 0 \leqslant d^+_j \leqslant 1, \qquad j = 1, 2, \ldots, k; \quad 0 \leqslant d^-_X \leqslant 1; \quad 0 \leqslant d^+_X \leqslant 1,$$
$$0 \leqslant d^-_{\ell w} \leqslant 1; \quad 0 \leqslant d^+_{\ell w} \leqslant 1, \qquad \ell = 1, 2, \ldots, n; \quad w = 1, 2, \ldots, t_\ell,$$
$$0 \leqslant d^-_{X_\ell} \leqslant 1; \quad 0 \leqslant d^+_{X_\ell} \leqslant 1, \qquad \ell = 1, 2, \ldots, n.$$

It is pertinent to the note that negative deviational variable becomes zero as the membership goal is achieved absolutely whereas; the negative deviational variable takes the value 1 in the solution if the membership goal is not achieved. Thus, the above model can be rewritten as,

(MOIQFGP1):

$$\text{Min}\,\xi = \sum_{j=1}^{k} d^-_j + \sum_{w=1}^{t_\ell} \sum_{\ell=1}^{n} d^-_{\ell w} + d^-_X + \sum_{\ell=1}^{n} d^-_{X_\ell}$$

$$\text{subject to } \; AX + \sum_{\ell=1}^{n} D_\ell X_\ell \leqslant b,$$
$$\begin{aligned}
B_\ell X + C_\ell X_\ell &\leqslant b_\ell, && \ell = 1, 2, \ldots, n,\\
\delta_j(Z_j) + d^-_j &= 1; && j = 1, 2, \ldots, k,\\
\eta_{\ell w}(z_{\ell w}) + d^-_{\ell w} &= 1; && \ell = 1, 2, \ldots, n; \quad w = 1, 2, \ldots, t_\ell,\\
\mu(X) + d^-_X - d^+_X &= 1, &&\\
\mu(X_\ell) + d^-_{X_\ell} - d^+_{X_\ell} &= 1, && \ell = 1, 2, \ldots, n,\\
0 \leqslant d^-_j &\leqslant 1; && j = 1, 2, \ldots, k,\\
0 \leqslant d^-_X \leqslant 1; \quad 0 \leqslant d^-_{X_\ell} &\leqslant 1; && \ell = 1, 2, \ldots, n,\\
0 \leqslant d^-_{\ell w} &\leqslant 1; && \ell = 1, 2, \ldots, n; \quad w = 1, 2, \ldots, t_\ell.
\end{aligned} \tag{6}$$

6. Algorithm for solving multi-objective indefinite quadratic bilevel programming problem (MOIQBP)

Step 1. Consider a multi-objective indefinite quadratic bilevel programming problem (MOIQBP).

Step 2. Determine the optimal solution of each objective function at the upper level and at the lower level individually, subject to the given set of constraints.

Step 3. The optimal solutions of the decision makers at both the levels formulate the integrated pay-off matrix. The maximum and minimum values in each column provide the aspiration levels of achievement for the decision makers.

Step 4. Enumerate the quadratic membership goals $\mu_j(Z_j)$; $(j = 1, 2, \ldots, k)$ for the upper level decision makers and $\nu_{\ell w}(z_{\ell w})$ $(\ell = 1, 2, \ldots, n;\ w = 1, 2, \ldots, t_\ell)$ for the lower level decision makers.

Step 5. The quadratic membership functions are transformed into linear membership functions by applying Taylor series approximation to them. These are defined as $\delta_j(Z_j)$; $(j = 1, 2, \ldots, n)$ for the objective functions at the upper level and $\eta_{\ell w}(z_{\ell w})$; $(\ell = 1, 2, \ldots, n;\ w = 1, 2, \ldots, t_\ell))$ for the objective functions at the lower level respectively.

Step 6. Specify the lower and upper bounds on the decision variables at both the levels. Determine the membership function of the decision makers at two levels as $\mu(X)$ and $\mu(X_\ell)$.

Step 7. Convert the problem (MOIQBP) into fuzzy goal programming model (MOIQFGP1).

Step 8. The procured solution is a compromise optimal solution for (MOIQBP).

7. Numerical illustration

Suppose there are two milk industries I and II producing four different types of milk, namely, full cream milk, toned milk, double toned milk and cow milk. These industries have processing units say $g_1(X)$ and $g_2(X)$, respectively. Industry I serves areas $h_1(X)$, $h_2(X)$ and industry II serves areas $p_1(X)$ and $p_2(X)$. Let the milk be supplied from two production units to different areas through trucks T_1 and T_2. Let x_1 denote the quantity of full cream milk supplied together from I and II. Similarly, x_2, x_3 and x_4 denote the supplied quantity of toned milk, double toned and cow milk respectively. It is observed that if the distribution of x_3 and x_4 are enhanced in four areas, the industries can maximize their sales. Thus, the objective at upper level is to maximize the production of milk. At lower level, the objective is to maximize sales of x_3 and x_4 which in turn will maximize their profits. The constraints at upper level are the capacity constraints. The constraints at lower level are the number of households catered by I and II. The objective of the problem is to find a compromise optimal solution which would satisfy both the industries.

The data for the above problem is given below:

$$g_1(X) = (2x_1+x_3+15)(x_2+x_4+11); \quad g_2(X) = (2x_1+10)(x_2+x_3+2x_4+9),$$
$$h_1(X) = (3x_1+x_3+8)(x_2+x_3+3); \quad h_2(X) = (2x_1+x_3+7)(2x_2+6),$$
$$p_1(X) = (2x_1+x_4+2)(x_2+4); \quad p_2(X) = (3x_1+11)(x_2+2x_4+5).$$

The constraints at the upper level are

$$x_2 + x_3 + x_4 \leqslant 8; \qquad -x_1 + 3x_2 + x_3 + 2x_4 \leqslant 21;$$
$$2x_3 + x_4 \leqslant 7; \qquad x_1,\ x_2,\ x_3,\ x_4 \geqslant 0.$$

The constraints at the lower level are

$$2x_1 + x_2 + x_3 \leqslant 12; \qquad 4x_1 + x_2 + x_4 \leqslant 14; \qquad x_1,\ x_2,\ x_3,\ x_4 \geqslant 0.$$

It is assumed that quantity of milk produced by two processing units is wholly distributed in four areas. Moreover, there is no wastage of milk during supply.

Solution: The multi objective indefinite quadratic programming problem (MOIQBP) is defined as follows,

$$\max_{x_1,x_2,x_3,x_4} (g_1(X), g_2(X))$$

$$\text{subject to} \quad \begin{aligned} x_2 + x_3 + x_4 &\leqslant 8,\\ -x_1 + 3x_2 + x_3 + 2x_4 &\leqslant 21,\\ 2x_3 + x_4 &\leqslant 7, \end{aligned}$$

where $(x_3,\ x_4)$ solves

$$\max_{x_3}(h_1(X), h_2(X)) \text{ for a given } (x_1, x_2)$$
$$\max_{x_4}(p_1(X), p_2(X)) \text{ for a given } (x_1, x_2)$$

$$\text{subject to} \quad \begin{aligned} 2x_1 + x_2 + x_3 &\leqslant 12,\\ 4x_1 + x_2 + x_4 &\leqslant 14\\ x_1,\ x_2,\ x_3,\ x_4 &\geqslant 0. \end{aligned}$$

Here, $X = (x_1, x_2, x_3, x_4)$ are the set of decision variables.

$$g_1(X) = (2x_1+x_3+15)(x_2+x_4+11); \quad g_2(X) = (2x_1+10)(x_2+x_3+2x_4+9),$$
$$h_1(X) = (3x_1+x_3+8)(x_2+x_3+3); \quad h_2(X) = (2x_1+x_3+7)(2x_2+6),$$
$$p_1(X) = (2x_1+x_4+2)(x_2+4); \quad p_2(X) = (3x_1+11)(x_2+2x_4+5).$$

Solving the objective functions individually at both levels, their optimal solutions are given as follows,

$$\bar{g}_1(X) = 360.24 \ \text{at}\ (2.3, 4.2, 3.2, 0.6); \qquad \bar{g}_2(X) = 312 \ \text{at}\ (1.5, 1, 0, 7);$$
$$\bar{h}_1(X) = 192.5 \ \text{at}\ (2, 4.5, 3.5, 0); \qquad \bar{h}_2(X) = 234 \ \text{at}\ (2, 6, 2, 0);$$
$$\bar{p}_1(X) = 72.25 \ \text{at}\ (1.5, 4.5, 0, 3.5); \qquad \bar{p}_2(X) = 310 \ \text{at}\ (1.5, 1, 0, 7);$$

Formulating an integrated pay-off matrix,

$$\begin{bmatrix} 360.24 & 256.96 & 188.24 & 213.12 & 59.04 & 186.16 \\ 342 & 312 & 50 & 80 & 60 & 310 \\ 348.75 & 238 & 192.5 & 217.5 & 51 & 161.5 \\ 357 & 238 & 176 & 234 & 60 & 187 \\ 342 & 266.5 & 93.75 & 150 & 72.25 & 255.75 \\ 342 & 312 & 50 & 80 & 60 & 310 \end{bmatrix}.$$

Procure maximum and minimum values for each objective function from integrated pay-off matrix and define membership functions, as

$$\mu_1(Z_1) = \frac{g_1(X) - 342}{360.24 - 342}, \qquad \mu_2(Z_2) = \frac{g_2(X) - 238}{312 - 238},$$
$$\nu_{11}(z_{11}) = \frac{h_1(X) - 50}{192.5 - 50}, \qquad \nu_{12}(z_{12}) = \frac{h_2(X) - 80}{234 - 80},$$
$$\nu_{21}(z_{21}) = \frac{p_1(X) - 51}{72.25 - 51}, \qquad \nu_{22}(z_{22}) = \frac{p_2(X) - 161.5}{310 - 161.5}.$$

Linearizing these quadratic membership functions by Taylor series approximation approach, we get,

$$\delta_1(Z_1) = 1 + \frac{31.6}{18.24}x_1 + \frac{22.8}{18.24}x_2 + \frac{15.8}{18.24}x_3 + \frac{22.8}{18.24}x_4 - \frac{232.68}{18.24}, \tag{7}$$

$$\delta_2(Z_2) = 1 + \frac{48}{74}x_1 + \frac{13}{74}x_2 + \frac{13}{74}x_3 + \frac{26}{74}x_4 - \frac{267}{74}, \tag{8}$$

$$\eta_{11}(z_{11}) = 1 + \frac{33}{142.5}x_1 + \frac{17.5}{142.5}x_2 + \frac{28.5}{142.5}x_3 - \frac{244.5}{142.5}, \tag{9}$$

$$\eta_{12}(z_{12}) = 1 + \frac{36}{154}x_1 + \frac{26}{154}x_2 + \frac{18}{154}x_3 - \frac{264}{154}, \tag{10}$$

$$\eta_{21}(z_{21}) = 1 + \frac{17}{21.25}x_1 + \frac{8.5}{21.25}x_2 + \frac{8.5}{21.25}x_4 - \frac{93.5}{21.25}, \tag{11}$$

$$\eta_{22}(z_{22}) = 1 + \frac{60}{148.5}x_1 + \frac{15.5}{148.5}x_2 + \frac{31}{148.5}x_4 - \frac{322.5}{148.5}. \tag{12}$$

Using equations (7)–(12) and determining membership function for the decision variables x_1, x_2 at the upper level and x_3, x_4 at the lower level, (MOIQFGP1) model becomes

$$\text{Minimize } \xi = d_1^- + d_2^- + d_3^- + d_4^- + d_5^- + d_6^- + d_{x_1}^- + d_{x_2}^- + d_{x_3}^- + d_{x_4}^-$$

$$\begin{aligned}
\text{subject to } \; 31.6x_1 + 22.8x_2 + 15.8x_3 + 22.8x_4 + 18.24d_1^- &= 232.68,\\
48x_1 + 13x_2 + 13x_3 + 26x_4 + 74d_2^- &= 267,\\
33x_1 + 17.5x_2 + 28.5x_3 + 142.5d_3^- &= 244.5,\\
36x_1 + 26x_2 + 18x_3 + 154d_4^- &= 264,\\
17x_1 + 2.5x_2 + 8.5x_4 + 21.25d_5^- &= 93.5,\\
60x_1 + 15.5x_2 + 31x_4 + 148.5d_6^- &= 322.5,\\
x_1 + 0.8d_{x_1}^- &= 2.3,\\
x_2 + 3.2d_{x_2}^- &= 4.2,\\
x_3 + 3.5d_{x_3}^- &= 3.5,\\
x_4 + 7d_{x_4}^- &= 7,\\
x_2 + x_3 + x_4 &\leqslant 8,\\
-x_1 + 3x_2 + x_3 + 2x_4 &\leqslant 21,\\
2x_3 + x_4 &\leqslant 7,\\
2x_1 + x_2 + x_3 &\leqslant 12,\\
4x_1 + x_2 + x_4 &\leqslant 14,\\
0 \leqslant d_1^-, d_2^-, d_3^-, d_4^-, d_5^-, d_6^-, d_{x_1}^-, d_{x_2}^-, d_{x_3}^-, d_{x_4}^- &\leqslant 1.
\end{aligned}$$

Using LINGO 17.0, solution obtained is $d_1^- = 0.00$, $d_2^- = 0.605$, $d_3^- = 0.02736$, $d_4^- = 0.0935$, $d_5^- = 0.64$, $d_6^- = 0.678$, $d_{x_1}^- = 0.00$, $d_{x_2}^- = 0.00$, $d_{x_3}^- = 0.0857$, $d_{x_4}^- = 0.914$, $\xi = 3.04$, $x_1 = 2.3$, $x_2 = 4.2$, $x_3 = 3.2$, $x_4 = 0.6$.

Thus, compromise optimal solution for the problem is obtained. It is observed that if I increase its sales by 3.2 times and II by 0.6 times in their respective areas then I and II would maximize their profits.

8. Conclusion

In this paper, a methodology has been proposed to attain compromise optimal solution for the problem MOIQBP. Fuzzy programming is adopted to solve

MOIQBP. The aspiration values of each objective function at both the levels are ascertained by defining an integrated pay-off matrix. The membership functions so defined under these aspiration levels are quadratic. The quadratic membership functions are then linearized using Taylor series approximation. This step of linearization simplifies the calculations. This makes the computational approach more methodical. The significance of the method thus used has been elucidated with the help of an example.

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