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# Spectrum Aliasing Does not Occur in Case of Ideal Signal Sampling

Andrzej Borys

***Abstract*—A new model of ideal signal sampling operation is developed in this paper. This model does not use the Dirac comb in an analytical description of sampled signals in the continuous time domain. Instead, it utilizes functions of a continuous time variable, which are introduced in this paper: a basic Kronecker time function and a Kronecker comb (that exploits the first of them). But, a basic principle behind this model remains the same; that is it is also a multiplier which multiplies a signal of a continuous time by a comb. Using a concept of a signal object (or utilizing equivalent arguments) presented elsewhere, it has been possible to find a correct expression describing the spectrum of a sampled signal so modelled. Moreover, the analysis of this expression showed that aliases and folding effects cannot occur in the sampled signal spectrum, provided that the signal sampling is performed ideally.**

***Keywords*—Signal sampling, occurrence of spectrum aliasing and folding, modelling of signal sampling operation, Kronecker time function, Kronecker comb**

## I. Introduction

INHERENTLY, the notions of aliasing and folding are connected with the sampling operation of analog signals. However, they, specifically aliasing, are used in different contexts. But, here, we are must be precise. Therefore, yet at the beginning of our considerations presented in this paper, we define precisely the context in which the aforementioned notions are used. So, first of all, they refer to the effects or phenomena occurring in the spectra of sampled signals. In other words, they refer to what happens with images of the sampled signals viewed in the frequency domain. Second, aliasing and aliases regard repetitions of the signal spectra curves calculated in the range of frequencies from 0 to $0{,}5\,f_s$, where $f_s$ means the sampling frequency. That is we see their repetitions when we observe the whole frequency axis. And, we consider folding here as something similar in principle to aliasing, however, with focusing on the mirroring effect around the frequencies $0{,}5\,f_s$, $1{,}5\,f_s$, and so on.

Our understanding of the notions of sampling and folding, as described above, is illustrated in Fig. 1. And, note that this is the only and commonly used in the literature model of presenting that what happens in the spectrum of an analog signal after its sampling. In this paper, we show that this model is false, at least in the case of considering the sampling operation as being ideal.

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The effects of spectrum aliasing and folding as shown in Fig. 1 follow solely from the kind of modeling of the sampled signal in a continuous time domain as a series of the weighted Dirac deltas occurring on the time axis $t$ at regular intervals $T = 1/f_s$ – as shown by an upper curve of Fig. 2.

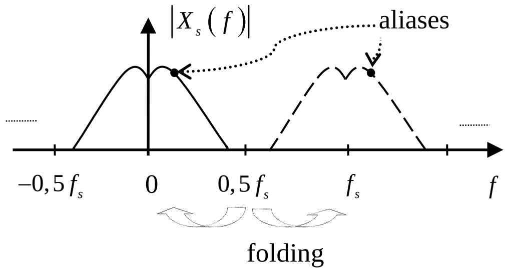


Fig. 1. Illustration to the notions of aliasing, aliases, and folding in an example spectrum $X_s(f)$ of a sampled bandlimited signal.

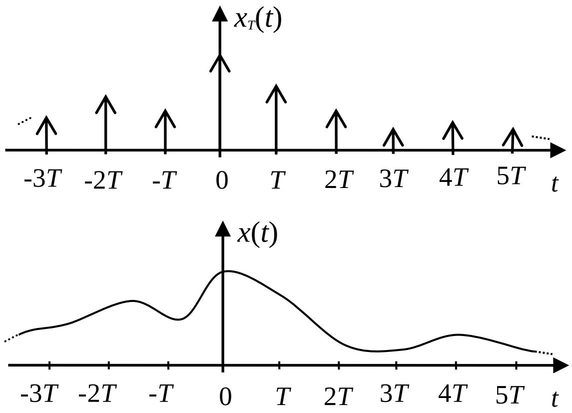


Fig. 2. Example sampled signal representation (upper curve) in form of a series of weighted Dirac deltas occurring uniformly on the continuous time axis in distance of $T$ from each other, and its un-sampled version (lower curve), where $t$ stands for a continuous time variable. Figure shows a signal discussed also in [1] and [2].

The modeled sampled signal $x_T(t)$ presented in Fig. 2 is described analytically as a signal $x(t)$ multiplied by the so-called Dirac comb $\delta_T(t)$. That is

$$x_T(t) = \delta_T(t) \cdot x(t)\ , \tag{1}$$

where the Dirac comb $\delta_T(t)$ is defined as

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \tag{2}$$

where $\delta(t-kT)$, $k=.,-1,0,1,.,$ mean the time-shifted Dirac deltas (called also Dirac distributions or Dirac impulses).

A graphical representation of the Dirac comb $\delta_T(t)$ given by (2) is presented in Fig. 3.

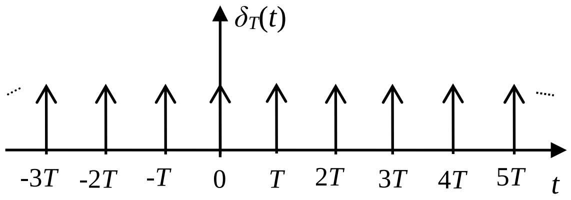


Fig. 3. Visualization of a Dirac comb signal given analytically by (2).

Observe now that the signal sampling model illustrated graphically by the upper curve of Fig. 2 and given analytically by (1) is rather not an adequate one. Why? Because the values of the signal $x_T(t)$ in Fig. 2 at the sampling points $t=kT$, $k=\ldots,-2,-1,0,1,2,\ldots$, are not simply real numbers, but some "strange" objects called the Dirac deltas. That is they are not physical quantities registered as the outputs of the signal sampling process. Note that as the outputs in any signal sampling process, we obtain sequences of real numbers. Therefore, because of this reason, we should conclude that a proper form of any ideally sampled signal is the one which is illustrated by an upper curve of Fig. 4.

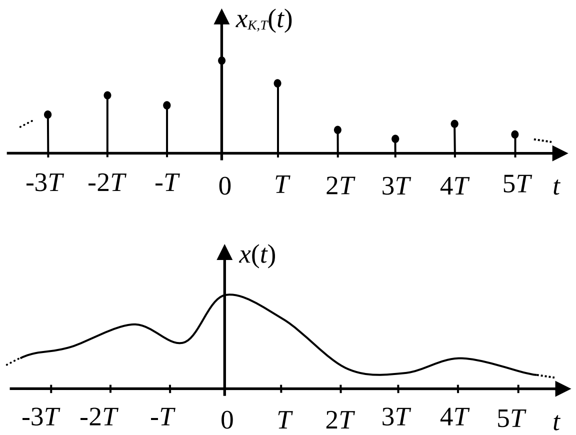


Fig. 4. Example sampled signal representation (upper curve) in form of a series of time-dependent signal samples occurring uniformly on the continuous time axis in distance of $T$ from each other, and its un-sampled version (lower curve), where $t$ stands for a continuous time variable. Figure shows a signal discussed also in [1] and [2].

The signals: $x_T(t)$ of Fig. 2 and $x_{K,T}(t)$ of Fig. 4, which evidently represent two different models of the sampling operation, are, however, related to each other. A difference between them can seem to be of only minor importance. This is so because, graphically, they differ from each other only slightly: the „posts" of $x_T(t)$ in Fig. 2 end with arrows, but those of $x_{K,T}(t)$ in Fig. 4 end with dots. Nevertheless, this has significant consequences as regards spectra of the above signals. They are considerably different – as we will see; and in the course of this paper, we will thoroughly explain why.

Furthermore, we will also show, in the next section, that analytical descriptions of the signals $x_T(t)$ and $x_{K,T}(t)$ differ from each other clearly.

In this paper, we present another model of the analog signal sampling operation that avoids the use of Dirac deltas. It is not an alternative to the one used nowadays everywhere in the literature. The model introduced here is basically a new one. And, we will show and prove throughout this paper that it is a more proper one because it avoids occurrence of such artifacts as the spectrum aliasing and folding illustrated in Fig. 1. Or, in other words, it will be utilized here to show that the artifacts mentioned above cannot appear in the case of an ideal modeling of the analog signal sampling process.

The remainder of this paper is organized as follows. In the next section, we introduce a basic Kronecker time function and a Kronecker comb, which utilizes it. Then, with the help of these tools, we develop an analytical description of a sampled signal. In Section III, an expression determining its spectrum is derived and afterwards analyzed. Here, a basic result of this paper is achieved. Namely, it is shown that aliases and folding effects cannot occur in the sampled signal spectrum The papers ends with two remarks.

## II. Analytical Description of a Sampled Signal in the Model Proposed

In our model, we will use the so-called Kronecker deltas to describe any sampled signal that can be displayed graphically as shown by the upper curve of Fig. 4. Generically, we will denote such a signal as $x_{K,T}(t)$ with the first subscript $K$ pointing to the use of the Kronecker delta and the second one $T$ indicating that the signal sampling period equals $T$. Furthermore, note that the usual Kronecker delta is defined as

$$\delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise}\end{cases}, \tag{3}$$

where the subscripts $i$ and $j$ at $\delta_{ij}$ belong to the set of integers. We denote this set here as $\mathrm{Z}$.

For the purposes of this paper, we must however modify slightly the Kronecker's delta definition given above. Namely, in what follows, we will extend it to

$$\delta_{i,r}=\begin{cases}1 & \text{if } \; i=r \text{ with } r \text{ meaning now a real number (or, in other words, when a real-valued } r \text{ assumes an integer value } i)\\ 0 & \text{otherwise,}\end{cases} \tag{4}$$

where it is assumed that the second subscript $r$ at $\delta_{i,r}$ stands, now, for a generalized one. That is its value belongs to the set of real numbers; $\mathrm{R}$ stands here for this set. Further, note also a slight modification in notation of $\delta_{i,r}$ compared to $\delta_{ij}$.

Because of the fact that one of the subscripts of $\delta_{i,r}$ is declared as an integer, but the second as a real number, to avoid any misunderstandings, they are separated from each other by a comma. Usefulness of this modified notation will prove in what follows.

By substituting $r = t/T$ in $\delta_{i,r}$ given by (4), we get a very useful function of a continuous time variable *t*, $\delta_{i,t/T}(t)$, for a given value of the integer subscript *i*.

By the way, note that $\delta_{i,t/T}$ with both *i* and *t* treated as variables represents a function of two variables, $\delta_{i,t/T}(i,t)$. In this paper, however, we use solely the form $\delta_{i,t/T}(t)$ – with *t* assumed to be a variable and *i* being a parameter of that function. The function $\delta_{i,t/T}(t)$ is illustrated in Fig. 5.

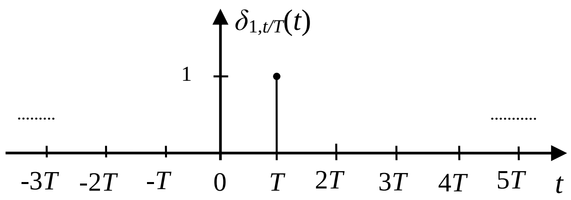


Fig. 5. Illustration of the function $\delta_{i,t/T}(t)$ for the parameter $i = 1$.

Consider now how to incorporate an operation of time shifting on the *t* axis of that single nonzero impulse seen in Fig. 5 – into the description of the function $\delta_{i,t/T}(t)$ introduced above. And, for this purpose, take into account a function $\delta_{i,(t-kT)/T}(t-kT) = \delta_{i,t/T-k}(t-kT)$, where $k \in \mathrm{Z}$. Observe that this function expresses a time shift of *k* time units *T* of the "one" occurring in the function $\delta_{i,t/T}(t)$ at $t = iT$ – to the right of the time axis *t* if $k > 0$, and to its left when $k < 0$.

In what follows, we will be interested in delays, or more generally, time shifts of a function $\delta_{0,t/T}(t)$ calculated for the parameter $i = 0$. This function will play a role of a "reference" in our further considerations because it positions the only nonzero value occurring in the function $\delta_{i,t/T}(t)$ just at the origin of the time axis (where $t = 0$). Let us call it here a basic Kronecker time function.

So, in view of what was said above, our basic time-shifted Kronecker time function $\delta_{0,t/T-k}(t)$ will mean the function $\delta_{0,t/T}(t)$ shifted *kT* units to the right, when $k > 0$, or $|kT|$ units to the left, when $k < 0$. This interpretation follows obviously from the definition of $\delta_{i,r}$ given by (4). That is for $\delta_{i,r}$ to be equal to 1 we need to have $i = r$. Or, in other words, the following: $0 = t/T - k$ must now hold. And, this results in $t = kT$.

Further, observe that multiplication of a signal of continuous time $x(t)$ by $\delta_{i,t/T}(t)$ gives

$$x(t)\cdot\delta_{i,t/T}(t) = \delta_{i,t/T}(t)\cdot x(t) = x(iT)\,\delta_{i,t/T}(t). \qquad (5)$$

That is this multiplication is a commutative operation and results in a modified function $\delta_{i,t/T}(t)$ that has its "one" occurring at $t = iT$ replaced by the value of the signal $x(t)$ calculated at $t = iT$. So, one might say that the function $\delta_{i,t/T}(t)$ just sifts the sample $x(iT)$ from $x(t)$.

Note also that a signal of continuous time $x(t)$ multiplied by our basic time-shifted Kronecker time function $\delta_{0,t/T-k}(t)$ results in

$$x(t)\cdot\delta_{0,t/T-k}(t) = \delta_{0,t/T-k}(t)\cdot x(t) = x(kT)\,\delta_{0,t/T-k}(t). \qquad (6)$$

Thus, we obtain here a similar result, namely, a resulting function that is a modified function $\delta_{0,t/T-k}(t)$ with its "one" occurring at $t = kT$ replaced by the value of the signal $x(t)$ calculated at $t = kT$. So, as just before, one might say that the function $\delta_{0,t/T-k}(t)$ sifts the sample $x(kT)$ from $x(t)$.

Furthermore, one can guess that putting $i = k$ in (5) results in the same functions on the right-hand sides of (5) and (6).

Really, the last observation is true; it follows from the fact that we can add the same integer to both the indices of $\delta_{0,t/T-k}(t)$ without changing this function. That is the following:

$$\delta_{0,t/T-k}(t) = \delta_{0+k,t/T-k+k}(t) = \delta_{k,t/T}(t) \qquad (7)$$

holds because the condition $0 = t/T - k$, when adding *k* on both sides of this equality, remains unchanged.

Note also that incidentally we arrived in (7) at a new notation for the „delayed" function $\delta_{0,t/T-k}(t)$, which was introduced before, and which incorporates the time shift of *kT* time units with reference to $\delta_{0,t/T}(t)$. Namely, it can be expressed in a shorter form as $\delta_{k,t/T}(t)$, where the first index *k* means now a normalized integer-valued time shift, $k = kT/T$. Because of this fact, in what follows, we will use rather this more compact form.

Finally, note also that the description of $\delta_{i,t/T}(t)$ given above and beneath Fig. 5 corresponds with that for $\delta_{k,t/T}(t)$ – as it should be.

In the next step that aims in finding an analytical description of such signals as the one illustrated by the upper curve in Fig. 4, we need to define an alternative for the Dirac comb. Intuitively, the best way will be by choosing a similar comb, however now with functions $\delta_{k,t/T}(t)$ in places of Dirac deltas. So, let us define it as

$$\delta_{K,T}(t) = \sum_{k=-\infty}^{\infty} \delta_{0,t/T-k}(t-kT) = \sum_{k=-\infty}^{\infty} \delta_{k,t/T}(t), \qquad (8)$$

where the first index $K$ at $\delta_{K,T}(t)$ stands for the name of Kronecker, but the second one, $T$, means a repetition period. Further, because of the reasons given above, let us call the function $\delta_{K,T}(t)$ a Kronecker comb. It is illustrated in Fig. 6.

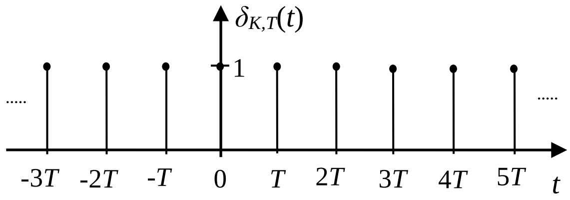


Fig. 6. Visualization of a Kronecker comb given analytically by (8).

Using (6), (7), and (8), we are now able to describe a sampled signal, $x_{K,T}(t)$, analytically in terms of our model. So, then, it will be given by

$$x_{K,T}(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta_{k,t/T}(t) \qquad (9)$$

where, similarly as before, the first index $K$ at $x_{K,T}(t)$ stands for the name of Kronecker, but the second one, $T$, means a repetition period.

Further, the following

$$\begin{aligned} x_{K,T}(t) &= \sum_{k=-\infty}^{\infty} x(kT)\delta_{k,t/T}(t) = \sum_{k=-\infty}^{\infty} \delta_{k,t/T}(t)\cdot x(t) = \\ &= \left(\sum_{k=-\infty}^{\infty} \delta_{0,t/T-k}(t-kT)\right)\cdot x(t) = \delta_{K,T}(t)\cdot x(t) \end{aligned} \qquad (10)$$

then also holds. So, concluding (10), it allows us to write

$$x_{K,T}(t) = \delta_{K,T}(t)\cdot x(t) \ . \qquad (11)$$

Finally, at the end of this section, it would be advisable as well as instructive to compare the common analytical description of sampled signals that exploits Dirac deltas with the one just derived using Kronecker time functions. So, to this end, observe first that the form of the corresponding describing equations (1) and (11) is the same. That is both the $x_T(t)$ in (1) and $x_{K,T}(t)$ in (11) are expressed by the signal $x(t)$ multiplied by a comb. However, the corresponding combs in these two cases are different, what we can symbolically express as $\delta_T(t) \neq \delta_{K,T}(t)$. Therefore, the expressions describing $x_T(t)$ and $x_{K,T}(t)$ differ from each other.

Second, $x_T(t)$ is not an ordinary function; it is strictly a distribution. Unlike this, $x_{K,T}(t)$ is an ordinary function.

Thirdly, it is possible to find a relation between these two representations of sampled signals. To this end, note first that the following:

$$x_{K,T}(nT) = \delta_{K,T}(nT)\cdot x(nT) = 1\cdot x(nT) = x(nT) \qquad (12)$$

holds. Next, see that using the well-known sifting property of the Dirac delta in the definition of the Dirac comb gives

$$\int_{-\infty}^{\infty} \delta_T(t-nT)\cdot x(t)\,dt = \sum_{n=-\infty}^{\infty} x(nT). \qquad (13)$$

And, finally, applying (12) in (13) results in

$$\begin{aligned} \int_{-\infty}^{\infty} \delta_T(t-nT)\cdot x(t)\,dt &= \sum_{n=-\infty}^{\infty} x_{K,T}(nT) = \\ &= \sum_{n=-\infty}^{\infty} \delta_{K,T}(nT)\cdot x(nT) \ . \end{aligned} \qquad (14)$$

Furthermore, note that we can show in a similar way that the following:

$$\int_{-\infty}^{\infty} \delta(t-nT)\cdot x(t)\,dt = \delta_{n,t/T}(t=nT)\cdot x(nT) \qquad (15)$$

holds, too. Moreover, some other interesting and useful properties, which hold within our model, like for example this one

$$\delta_{k,t/T}(t)\cdot x(t) = x(kT)\cdot\delta_{k,t/T}(t) \qquad (16)$$

can be also easily derived using the relationships already given.

## III. Spectrum of a Sampled Signal in the Model Proposed

In the previous section, it has been shown that in our model not only the graphical but also the analytical representation for sampled signals differs, evidently, from the description in the model that uses Dirac deltas. So, it is logical to suppose that the sampled signals in the models mentioned have also different representations in the frequency domain. In other words, that they have different spectra; specifically having in mind the fact that the Fourier transform used for calculation of spectra is a linear operation.

This section is devoted to discussion of differences in the spectra of sampled signals we obtain in these two different models mentioned above. And, our considerations presented here will aim in answering a natural question: which of these models is more proper in description of a real world?

We start with recalling a common result that is given in the literature, for example see [3], for a spectrum of a sampled signal, namely the following expression:

$$X_s(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(f-kf_s) \ , \qquad (17)$$

where $X(f)$ and $X_s(f)$ mean the spectra (Fourier transforms) of an un-sampled signal $x(t)$ and of its sampled version $x_s(t)$, respectively. The frequency variable is denoted by $f$ in (17). Furthermore, $X(f-kf_s)$, $k \in \mathrm{Z}$, in (17) stand for the frequency-shifted $X(f)$. Moreover, as before, $T$ and $f_s$ mean the sampling period and the sampling frequency, accordingly; $T = 1/f_s$.

As well known, the formula given by (17) is "responsible" for these effects, which are visualized in Fig. 1. That is for the occurrence of spectrum aliasing and folding in the spectrum of a sampled signal. The formula (17) is their analytical description; for more details regarding this, see, for example, [3].

The formula given by (17) has been derived with the use of the first model mentioned; that is with the application of the Dirac comb to describe analytically a sampled signal, as presented by (1). In other words, in this case, it has been assumed that the sampled signal $x_s(t)$, generally denoted so in this paper, is modelled by $x_T(t)$ given by (1). In what follows, we will show that the form of (17) obtained is solely due to the use of the modelling with application of the Dirac deltas. It will be absolutely impossible to get it or something similar within the second model considered in this paper.

Let us start with the following observation: the analytical descriptions of a sampled signal in our models, given by (1) and (11), respectively, have the same form. That is this is a multiplication of the corresponding comb by an un-sampled signal. Therefore, (1) and (11) must also possess the same form in the frequency domain. More precisely, this form in the frequency domain is a convolution of the corresponding Fourier transforms. That is

$$X_T(f) = \mathrm{F}\left(x_T(t)\right) = \mathrm{F}\left(\delta_T(t)\right) \otimes \mathrm{F}\left(x(t)\right) = \\ = \Delta_T(f) \otimes X(f) \tag{18a}$$

and

$$X_{K,T}(f) = \mathrm{F}\left(x_{K,T}(t)\right) = \\ = \mathrm{F}\left(\delta_{K,T}(t)\right) \otimes \mathrm{F}\left(x(t)\right) = \\ = \Delta_{K,T}(f) \otimes X(f)\ , \tag{18b}$$

respectively. In (18), $\mathrm{F}(\cdot)$ means a Fourier transform of an object or a function indicated. So, more precisely, $X_T(f) = \mathrm{F}\left(x_T(t)\right)$, $\Delta_T(f) = \mathrm{F}\left(\delta_T(t)\right)$, $X_{K,T}(f) = \mathrm{F}\left(x_{K,T}(t)\right)$, and $\Delta_{K,T}(f) = \mathrm{F}\left(\delta_{K,T}(t)\right)$. Furthermore, the symbol $\otimes$ in (18) means performing the operation of convolution.

Performing calculation of the convolution indicated in (18a) leads to (17), as shown, for example, in [3]. Shortly, it follows from the fact that $\Delta_T(f) = \mathrm{F}\left(\delta_T(t)\right)$ is itself a Dirac comb [3],

$$\Delta_T(f) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(2\pi\left(f - kf_s\right)\right) \tag{19}$$

and that

$$\int_{-\infty}^{\infty} \delta\left(2\pi\left(\upsilon - kf_s\right)\right) X\left(f-\upsilon\right) d\upsilon = \frac{1}{2\pi} X\left(f - kf_s\right) \tag{20}$$

holds.

Let us now calculate the Fourier transform $\Delta_{K,T}(f) = \mathrm{F}\left(\delta_{K,T}(t)\right)$. To this end, we will use a standard definition of this transform, (8), and (11). Applying this, we get

$$\begin{aligned} &\Delta_{K,T}(f) = \\ &= \int_{-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} \delta_{0,t/T-k}\left(t-kT\right) \right) \cdot x(t) \exp\left(-j2\pi ft\right) dt = \\ &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_{0,t/T-k}\left(t-kT\right) x(t) \exp\left(-j2\pi ft\right) dt = \\ &= \sum_{k=-\infty}^{\infty} \int_{kT}^{kT} 1 \cdot x(t) \exp\left(-j2\pi ft\right) dt\ . \end{aligned} \tag{21}$$

In the next step, observe that the integrals under the summation symbol in (21), if considered as the Riemann's integrals, do not exist. So, consequently, $X_{K,T}(f)$ after (18b) also does not exist.

However, note that when the integrals under the summation symbol in (21) are treated in the sense of Lebesgue, then they are correctly determined. But, all of them are then equal to zero leading to $\Delta_{K,T}(f) = 0$. Therefore, applying the latter in (18b) gives

$$\begin{aligned} &X_{K,T}(f) = \Delta_{K,T}(f) \otimes X(f) = \\ &= \int_{-\infty}^{\infty} 0 \cdot X\left(f-\upsilon\right) d\upsilon = \\ &= \int_{-\infty}^{\infty} 0 \cdot d\upsilon = \mathrm{const}\Big|_{-\infty}^{\infty} \equiv 0\ . \end{aligned} \tag{22}$$

So, in this case, the spectrum of the sampled signal does exist, however, it is identically equal to zero. But, we would await rather another outcome because it is really difficult to imagine that every nonzero sampled signal possesses the identically zero spectrum.

In this paper, we argue that both the results (17) and (22), obtained in the case of an ideal sampling, are not correct. The formula (17) and its interpretation as illustrated in Fig. 1 are not correct because they have been received in the model that uses a description of the sampled signal in the continuous time domain as visualized by an example in Fig. 2 (upper curve), $x_T(t)$. Obviously, this image of the sampled signal that utilizes non-physical objects, which are the Dirac impulses, is not true. The true image of the sampled signal in the continuous time domain is the one which is illustrated in Fig. 4 (upper curve), $x_{K,T}(t)$. However, there are problems with the latter when calculating its spectrum. Simply because the signal spectrum is defined as a Fourier integral (Fourier integral transform); and in this specific case either the integrals do not exist or have identically zero values. More precisely, these integrals considered as Riemann's ones do not exist, but assumed to be Lebesgue integrals provide zeros (see discussion of (21)).

The above problem of unsatisfactory expressions determining the spectra of sampled signals can be however solved with help of a concept of a signal object. This powerful idea was proposed for the first time by the author of this paper in [1], and successfully utilized in [2]. Note also that in principle the solution to the problem posed in [4] is based on the idea of a signal object, too.

According to the results already obtained in [2] and [4] (we do not want to repeat here their accompanying derivations), the spectrum of the sampled signal (let us use a special notation SPECT for denoting it in our model) is given by

$$\begin{aligned}&\mathrm{SPECT}\left(f \middle| x_s(t)=x_{K,T}(t)\right)= \\ &=\sum_{n=-\infty}^{\infty} x(nT)\, T\, \mathrm{rect}(fT)\exp(-j2\pi fnT)= \\ &\quad=\frac{1}{f_s}\sum_{n=-\infty}^{\infty} x(nT)\exp\left(-j2\pi\frac{f}{f_s}n\right) \\ &\quad\text{for } \left|f/f_s\right|\le 1/2 \text{ and} \\ &\quad\mathrm{SPECT}\left(f \middle| x_s(t)=x_{K,T}(t)\right)\equiv 0 \\ &\quad\text{for } \left|f/f_s\right|>1/2\ ,\end{aligned} \tag{23}$$

where the function $\mathrm{SPECT}\left(f \middle| x_s(t)=x_{K,T}(t)\right)$ of frequency $f$ stands for the spectrum of the sampled signal $x_s(t)=x_{K,T}(t)$. Furthermore, the function $\mathrm{rect}(x)$ used in (23) means the following:

$$\mathrm{rect}(x)=1 \text{ for } |x|\le\frac{1}{2} \text{ and } 0 \text{ for } |x|>\frac{1}{2}\ . \tag{24}$$

It has been shown in [2] and [4] that when the sampling is so performed that the so-called Nyquist frequency is larger or equal to the maximal frequency in the spectrum of a signal to be sampled the following:

$$\mathrm{SPECT}\left(f \middle| x_s(t)=x_{K,T}(t)\right)=\mathrm{F}\left(x(t)\right) \tag{25}$$

holds. That is in this case the spectrum of the sampled signal equals the spectrum of its un-sampled version. In other words, this means that the sampling operation does not introduce any distortion into the signal spectrum.

Also, it has been shown in [2] and [4] that when the sampling is carried out in such a way that the Nyquist frequency is smaller from the maximal frequency in the spectrum of a signal to be sampled (that is in the case of its under-sampling) the following:

$$\mathrm{SPECT}\left(f \middle| x_s(t)=x_{K,T}(t)\right)=\mathrm{F}\left(x_a(t)\right) \tag{26}$$

holds. In (26), $x_a(t)$ stands for a signal that is reconstructed from the samples of the signal $x(t)$ in case of under-sampling. Obviously, the signal $x_a(t)$ resembles in some way the signal $x(t)$; its spectrum can be viewed as a filtered and shaped at the same time spectrum of the signal $x(t)$. Moreover, the spectrum of the signal $x_a(t)$ remains a bandlimited one. For more details regarding these topics, see [2] and [4].

Observe now that because the spectra $\mathrm{F}\left(x(t)\right)$ and $\mathrm{F}\left(x_a(t)\right)$ on the right-hand sides of (25) and (26), respectively, are bandlimited ones this fact precludes occurrence of such effects as aliases and foldings (as defined graphically in Fig. 1) in the signal spectrum. Consequently, according to the equalities (25) and (26), the spectrum $\mathrm{SPECT}\left(f \middle| x_s(t)=x_{K,T}(t)\right)$ of the sampled signal, independently of whether it is sampled to enable a later perfect recovery or not, does not contain any aliases and foldings.

Note that the above finding follows, directly, also from (23) expressing the spectrum $\mathrm{SPECT}\left(f \middle| x_s(t)=x_{K,T}(t)\right)$. Simply, see that the second part of (23), which has the following form:

$$\mathrm{SPECT}\left(f \middle| x_s(t)=x_{K,T}(t)\right)\equiv 0 \text{ for } \left|f/f_s\right|>1/2\ , \tag{27}$$

says that the spectrum $\mathrm{SPECT}\left(f \middle| x_s(t)=x_{K,T}(t)\right)$ is identically zero for the frequencies $\left|f/f_s\right|>1/2$. Obviously, this fact precludes occurrence of any infinite series of aliases and foldings.

Therefore, finally, it follows clearly from the above that the spectrum aliasing and folding do not occur in case of ideal signal sampling.

## IV. Two Remarks

Finally in this paper, we want to remark, first, that many people believe that such topics like sampling of signals, sampling theorem, and reconstruction formula are fully developed. The results achieved and presented in this paper, however, as seen, contradict this believing. They show that the problem of modelling of the sampled signal should be treated rather in another way, without an undue use of Dirac deltas.

Second, we would like to draw here the reader's attention also to the fact that the tools developed in the theory of sampling of analog signals can be successfully used in other areas, as for example, to model – quite generally – the measuring process. The first results regarding this interesting approach have already appeared [5], [6].

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