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Mechanics of infinitesimal gyroscopes on helicoid-catenoid deformation family of minimal surfaces

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Abstract. In this paper we explore the mechanics of infinitesimal gyroscopes (test bodies with internal degrees of freedom) moving on an arbitrary member of the helicoid-catenoid family of minimal surfaces. As the configurational spaces within this family are far from being trivial manifolds, the problem of finding the geodesic and geodetic motions presents a real challenge. We have succeeded in finding the solutions to those motions in an explicit parametric form. It is shown that in both cases the solutions can be expressed through the elliptic integrals and elliptic functions, but in the geodetic case some appropriately chosen compatibility conditions for glueing together different branches of the solution are needed. Additionally, an action-angle analysis of the corresponding Hamilton-Jacobi equations is performed for external potentials that are well-suited to the geometry of the problem under consideration. As a result, five different sets of conditions between the three action variables and the total energy of the infinitesimal gyroscopes are obtained.

Key words: action-angle analysis; mechanics of infinitesimal gyroscopes; geodesic and geodetic equations of motion; helicoid-catenoid deformation family of minimal surfaces; elliptic integrals and elliptic functions.

1. Introduction

By definition, the minimal surface is a surface with zero mean curvature in all points [1, 2]. The mean curvature H at some point of the given surface is calculated as the average value of its two principal curvatures κ_1, κ_2 (the maximal and minimial ones), whereas the Gaussian curvature K is equal to their product, i.e., for any minimal surface we have that

$$\kappa_1 = -\kappa_2, \qquad H = \frac{\kappa_1+\kappa_2}{2} = 0, \qquad K = \kappa_1\kappa_2 = -\kappa_2^2 \leq 0. \tag{1}$$

In order to find a minimal surface specified by the boundary conditions, we need to solve the boundary value problem with the use of variational calculus. Sometimes it is also called the Plateau's problem, because this 19-th century Belgian physicist was the one, who performed a lot of experiments with soap films in order to illustrate the obtained solutions of this variational problem. For this he immersed wire frames of different shapes (accordingly to the given boundary conditions) in soap and obtained a soapy surface that realizes the variational solution to the corresponding boundary value problem and produces some minimal surfaces in this way.

Any minimal surface (or a part of a minimal surface) which is immersed into the three-dimensional Euclidean space and parameterized as $z = f(x,y)$ satisfies the Lagrange's equation

$$\left(1+f_x^2\right) f_{yy} - 2 f_x f_y f_{xy} + \left(1+f_y^2\right) f_{xx} = 0. \tag{2}$$

Contrary to the intuition, a sphere is not a minimal surface even though it minimizes the surface-to-volume ratio. It is an example of Delaunay surfaces, i.e., the surfaces of revolution that have constant mean curvature (1), but for a sphere $H = 1/r$, where r is the radius of the sphere. A plane is the trivial minimal surface, whereas the simplest nontrivial ones are so-called catenoid and helicoid surfaces which were found by Meusnier in 1776 [3]. A catenoid (the only minimal surface of revolution) can be visualized as a soap film spanned by two circular rings, whereas a helicoid (the only ruled minimal surface) comes from the similarity with a helix, i.e., at every point in helicoid there exists a helix that passes through it and is completely contained in the helicoid. It can be visualized as an Archimedes' screw or stairs in a high castle tower.

The organization of the present paper can be described as follows. In Section 2 we introduce the parametrization of the helicoid-catenoid family of minimal surfaces, construct its first fundamental form and solve the corresponding geodesic equations in the parametrical form, i.e., we obtain the solutions expressed through the elliptic integrals and elliptic functions.

In Section 3 we introduce the d'Alembert form of the kinetic energy for infinitesimal gyroscopes (test body with internal degrees of freedom) moving on some minimal surface from the helicoid-catenoid deformation family and with the help of the Legendre transformation obtain the corresponding form of the Hamiltonian.

In Section 4 we obtain the geodetic (without any external potential) solutions of the Euler–Lagrange equations of motion in

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the parametrical form with some appropriately chosen compatibility conditions for glueing together different branches of the solution.

Finally, in Section 5 we perform the action-angle analysis of the Hamilton–Jacobi equation obtained for the special kind of external potentials that are well-suited to the geometry of the problem. As a result, five different sets of conditions are obtained that connect three action variables and the total energy of the infinitesimal gyroscope.

2. Surface's parameterization and geodesics

The interesting fact about catenoids and helicoids is that by cutting one turn of a helicoid and reconnecting the edges one can obtain a catenoid. The deformation which defines the helicoid-catenoid family of surfaces can be described explicitly by the formulas

$$\begin{aligned} x(u,v;\theta) &= \cosh u \cos v \sin\theta + \sinh u \sin v \cos\theta\,, \\ y(u,v;\theta) &= \cosh u \sin v \sin\theta - \sinh u \cos v \cos\theta\,, \\ z(u,v;\theta) &= u\sin\theta + v\cos\theta \end{aligned} \tag{3}$$

where $u \in (-\infty,\infty)$ and $v \in [-\pi,\pi]$ are the coordinates on the surface and $\theta \in (-\pi,\pi]$ is the deformation parameter defining the family of isometric minimal surfaces with $\theta = \pm\pi/2$ corresponding to a catenoid and $\theta = 0$, $\theta = \pi$ corresponding to left- and right-handed helicoids respectively [4].

In particular, for any $u = \beta$ under the above deformation (3) a helix $\boldsymbol{\alpha}(v;\beta)$ which is contained in the helicoid is transformed into a circle $\boldsymbol{\gamma}(v;\beta)$ which is contained in the catenoid (see Fig. 1) and their respective parameterizations are given as

$$\begin{aligned} \boldsymbol{\alpha}(v;\beta) &= (\sinh\beta\sin v, -\sinh\beta\cos v, v)\,, \\ \boldsymbol{\gamma}(v;\beta) &= (\cosh\beta\cos v, \cosh\beta\sin v, \beta)\,. \end{aligned}$$

Moreover, this transformation is isometric because the lengths of both curves are equal, i.e.,

$$\int_{-\pi}^{\pi} \left| \frac{\mathrm{d}\boldsymbol{\alpha}(v;\beta)}{\mathrm{d}v} \right| \mathrm{d}v = \int_{-\pi}^{\pi} \left| \frac{\mathrm{d}\boldsymbol{\gamma}(v;\beta)}{\mathrm{d}v} \right| \mathrm{d}v = 2\pi\cosh\beta. \tag{4}$$

From (3) we can calculate the first fundamental form for the helicoid-catenoid deformation family of surfaces as

$$\mathrm{I} = g_{uu}\,\mathrm{d}u^2 + g_{vv}\,\mathrm{d}v^2 = \cosh^2 u\left(\mathrm{d}u^2 + \mathrm{d}v^2\right). \tag{5}$$

This means that parameterization (u,v) is isothermal. Moreover, the coefficients of the metric tensor g are functions of only one variable u, hence, $\dot{g}_{uu} = g_{uu,u}\dot{u}$ and $\dot{g}_{vv} = g_{vv,u}\dot{u}$.

Next we can calculate the Christoffel symbols (i.e., an affine connection Γ derived from the Bregman divergence function [5]) corresponding to the metric tensor g defined by (5), i.e.,

$$\Gamma^i{}_{jk} = \begin{Bmatrix} i \\ jk \end{Bmatrix} = \frac{1}{2} g^{im}\left(g_{mj,k} + g_{mk,j} - g_{jk,m}\right). \tag{6}$$

We see that the only non-zero components of Γ are

$$\Gamma^u{}_{uu} = -\Gamma^u{}_{vv} = \Gamma^v{}_{uv} = \Gamma^v{}_{vu} = \tanh u. \tag{7}$$

Therefore, from the geodesic equation

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \Gamma^i{}_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0 \tag{8}$$

for the helicoid-catenoid deformation family of surfaces we obtain that geodesics fulfill the following system of two second-order ordinary differential equations (with specified initial conditions)

$$\begin{aligned} \ddot{u} + \tanh u\left(\dot{u}^2 - \dot{v}^2\right) &= 0, \\ \ddot{v} + 2\tanh u\;\dot{u}\dot{v} &= 0. \end{aligned} \tag{9}$$

From (9) we can obtain the first integrals of the above system quite straightforwardly, i.e., multiplying the first equation by $2\cosh^2 u\;\dot{u}$, the second equation by $2\cosh^2 u\;\dot{v}$, and summing them up we obtain that

$$\cosh^2 u\left(\dot{u}^2 + \dot{v}^2\right)^{\cdot} + \left(\cosh^2 u\right)^{\cdot}\left(\dot{u}^2 + \dot{v}^2\right) = 0 \tag{10}$$

and therefore

$$\dot{u}^2 + \dot{v}^2 = A^2 \mathrm{sech}^2 u \tag{11}$$

Fig. 1. A helix contained in the helicoid is transformed into a circle contained in the catenoid under the helicoid-catenoid transformation (3)

whereas multiplying the second equation alone by $\cosh^2 u$ we obtain that

$$\cosh^2 u\,\ddot{v} + \left(\cosh^2 u\right)^{\cdot}\,\dot{v} = 0 \;\Rightarrow\; \dot{v} = B\,\mathrm{sech}^2 u \tag{12}$$

where A and B are denoting some constants of integration.

Finally, performing the integration of (11) and (12) we obtain the solutions of (9) in the parametric form given as

$$At(u) = \int \frac{\cosh^2 u\,\mathrm{d}u}{\sqrt{\cosh^2 u - (B/A)^2}}, \tag{13}$$

$$v(u) = \frac{B}{A}\int \frac{\mathrm{d}u}{\sqrt{\cosh^2 u - (B/A)^2}}. \tag{14}$$

We see that in the above expressions the constant A is simply rescaling the time t and the other integration constant B. Therefore, without any loss of generality, later on we can suppose that $A = 1$ with the possibility of reobtaining it in the final expressions with the substitution $t \mapsto At$ and $B \mapsto B/A$.

Let us rewrite (13) and (14) with $A = 1$ as

$$t(u) = \int \frac{\sqrt{1+\sinh^2 u}}{\sqrt{1-B^2+\sinh^2 u}}\,\mathrm{d}\sinh u, \tag{15}$$

$$v(u) = B\int \frac{\mathrm{d}\sinh u}{\sqrt{1+\sinh^2 u}\sqrt{1-B^2+\sinh^2 u}}. \tag{16}$$

Using the substitution $k^2 = 1 - B^2$, $kx = \mathrm{i}\sinh u$ as well as definitions in the Legendre normal forms of the incomplete elliptic integrals of second and first kinds of elliptic modulus k, i.e.,

$$E(z,k) = \int_0^z \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,\mathrm{d}x, \tag{17}$$

$$F(z,k) = \int_0^z \frac{\mathrm{d}x}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \tag{18}$$

we obtain that

$$t(u) = t_0 - \mathrm{i}E\left(\mathrm{i}\frac{\sinh u}{\sqrt{1-B^2}}, \sqrt{1-B^2}\right), \tag{19}$$

$$v(u) = v_0 - \mathrm{i}BF\left(\mathrm{i}\frac{\sinh u}{\sqrt{1-B^2}}, \sqrt{1-B^2}\right). \tag{20}$$

Finally, taking into account the formulas from the Gradstein and Ryzhik's book [6] that connect the incomplete elliptic integrals of first and second kinds with purely imaginary and real arguments we obtain that (13) and (14) can be integrated as

$$\begin{aligned} t(u) = t_0 + F\left(\frac{\sinh u}{\sqrt{\cosh^2 u - B^2}}, B\right) \\ - E\left(\frac{\sinh u}{\sqrt{\cosh^2 u - B^2}}, B\right) + \frac{\sinh u \cosh u}{\sqrt{\cosh^2 u - B^2}}, \end{aligned} \tag{21}$$

$$v(u) = v_0 + BF\left(\frac{\sinh u}{\sqrt{\cosh^2 u - B^2}}, B\right). \tag{22}$$

By the way, from (11) and (12) we can show that for the elliptic modulus k and complementary elliptic modulus $k' = \sqrt{1-k^2}$ there should be $k^2 = (1-B^2) \in [0,1]$ and $k'^2 = B^2 \in [0,1]$. In fact, when we substitute (12) into (11) we obtain that

$$1 = \cosh^2 u\left(\dot{u}^2 + B^2\mathrm{sech}^4 u\right) \geq B^2\mathrm{sech}^2 u \tag{23}$$

i.e., $\cosh^2 u \geq B^2 \geq 0$. This should be true for any value of the variable u, therefore, also for $u = 0$, then we obtain $B^2 \in [0,1]$.

Hence, the geodesics on the helicoid-catenoid family of surfaces can be drawn (see, e.g., Figs. 2, 3) using only the expression (22) that can be rewritten with the use of Jacobi's elliptic sine and cosine functions, i.e., supposing that $v_0 = 0$ we obtain that the connection between u and v variables is given as

$$\mathrm{sn}\left(\frac{v}{B}, B\right) = \frac{\sinh u}{\sqrt{\cosh^2 u - B^2}} \tag{24}$$

or otherwise

$$\sinh u = \sqrt{1-B^2}\,\frac{\mathrm{sn}\,(v/B,B)}{\mathrm{cn}\,(v/B,B)}. \tag{25}$$

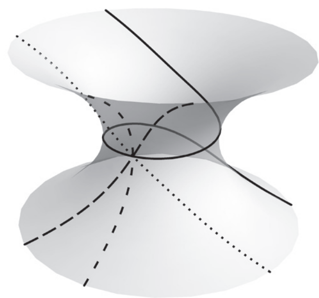

Fig. 2. Geodesics given by (22) on the catenoid ($\vartheta = \pi/2$) with the following values of parameters: $v_0 = 0$, $B = -0.8$ (dotted line), $B = 0$ (short-dashed line), $B = 0.5$ (long-dashed line), and $B = 0.999$ (continuous line)

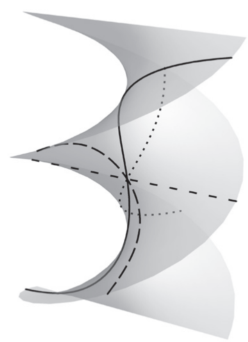

Fig. 3. Geodesics given by (22) on the left-handed helicoid ($\vartheta = 0$) with the following values of parameters: $v_0 = 0$, $B = -0.9$ (dotted line), $B = 0$ (short-dashed line), $B = 0.8$ (long-dashed line), and $B = 0.99$ (continuous line)

For the special case when the integration constant $B = 0$ we have that $F(z,0) = E(z,0) = \arcsin z$, i.e.,

$$t(u) = t_0 + \sinh u, \qquad v(u) = v_0. \tag{26}$$

Then the explicit dependence of the translational degrees of freedom u and v on the time parameter t is given as

$$u(t) = \ln\left(t - t_0 + \sqrt{1 + (t - t_0)^2}\right), \qquad v(t) = v_0. \tag{27}$$

3. Mechanics of infinitesimal gyroscopes

From the general formulation of infinitesimal test bodies moving in Riemannian spaces we know (see Appendix) that the d'Alembert form of the kinetic energy for the infinitesimal gyroscope moving on some minimal surface from the helicoid-catenoid deformation family defined by (3) is given as

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{m}{2}\cosh^2 u\left(\dot{u}^2 + \dot{v}^2\right) + \frac{I}{2}\left(\dot{\psi} - \omega_{\mathrm{dr}}\dot{v}\right)^2 \tag{28}$$

where m and I are respectively the mass and scalar moment of inertia of our gyroscope, ψ describes the relative part of the rotational motion performed with respect to the fixed once and for all reference frames E, whereas ω_{dr} is the drive (or drift) factor describing the part of the rotational motion contained in the frames E themselves. It can be shown that from the general formulas presented in [7] we can obtain that in our case

$$\omega_{\mathrm{dr}} = \frac{1}{2}\frac{g_{vv,u}}{\sqrt{g_{uu}g_{vv}}} = \tanh u. \tag{29}$$

Let us now rewrite the above kinetic energy (28) in the form in which we have explicitly separated the mass factor, i.e.,

$$T(q,\dot{q}) = \frac{m}{2}G_{ij}(q)\frac{\mathrm{d}q^i}{\mathrm{d}t}\frac{\mathrm{d}q^j}{\mathrm{d}t} \tag{30}$$

where $\left(q^i\right) = (u, v, \psi)$ are generalized coordinates (combining the translational and internal degrees of freedom together) and the new metric $G_{ij}(q)$ is given as

$$\begin{bmatrix} \cosh^2 u & 0 & 0 \\ 0 & \cosh^2 u + \dfrac{I}{m}\tanh^2(u) & -\dfrac{I}{m}\tanh(u) \\ 0 & -\dfrac{I}{m}\tanh(u) & \dfrac{I}{m} \end{bmatrix}.$$

The square root of the determinant of the above matrix (i.e., the weight-one volume density) is given by the expression

$$\sqrt{G} = \sqrt{\det\left[G_{ij}\right]} = \sqrt{\frac{I}{m}}\cosh^2 u. \tag{31}$$

The contravariant inverse metric G^{ij} (for which we have the relations $G^{ik}G_{kj} = \delta^i{}_j$) is given as

$$\begin{bmatrix} \operatorname{sech}^2 u & 0 & 0 \\ 0 & \operatorname{sech}^2 u & \tanh u \operatorname{sech}^2 u \\ 0 & \tanh u \operatorname{sech}^2 u & \tanh^2 u \operatorname{sech}^2 u + \dfrac{m}{I} \end{bmatrix}.$$

Therefore, for the potential systems with Lagrangians given as $L(q,\dot{q}) = T(q,\dot{q}) - V(q)$ the Legendre transformation defined by $p_i = \partial L/\partial\dot{q}^i = mG_{ij}(q)\dot{q}^j$ has the form

$$p_u = m\cosh^2 u\,\dot{u}, \qquad p_\psi = I\left(\dot{\psi} - \tanh u\,\dot{v}\right),$$
$$p_v = \left(m\cosh^2 u + I\tanh^2 u\right)\dot{v} - I\tanh u\,\dot{\psi}.$$

Inverting the above expressions we obtain that the generalized velocities $\dot{q}^i = (1/m)\,G^{ij}(q)\,p_j$ are given as

$$\begin{aligned} \dot{u} &= \frac{p_u}{m\cosh^2 u}, \qquad \dot{v} = \frac{p_v + \tanh u\,p_\psi}{m\cosh^2 u}, \\ \dot{\psi} &= \frac{p_\psi}{I} + \tanh u\,\dot{v}, \\ &= \frac{m\cosh^2 u + I\tanh^2 u}{m\cosh^2 u}\,\frac{p_\psi}{I} + \frac{\tanh u}{m\cosh^2 u}\,p_v\,. \end{aligned} \tag{32}$$

Substituting (32) into the expression for the total energy

$$E(q,\dot{q}) = \dot{q}^i\frac{\partial L}{\partial\dot{q}^i} - L \tag{33}$$

we obtain the Hamiltonian $H(q,p) = \mathscr{T}(q,p) + V(q)$, where

$$\mathscr{T}(q,p) = \frac{p_u^2 + \left(p_v + \tanh u\,p_\psi\right)^2}{2m\cosh^2 u} + \frac{p_\psi^2}{2I}\,. \tag{34}$$

4. Geodetic equations of motion

Let us now calculate the Euler–Lagrange equations of motion for the kinetic energy expression given by (28) and some general potential term $V(u,v,\psi)$ depending on all translational and internal degrees of freedom. Therefore, we obtain that

$$\begin{aligned} \dot{p}_u &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot{u}} = m\cosh^2 u\left(\ddot{u} + 2\tanh u\,\dot{u}^2\right), \\ \dot{p}_v &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot{v}} = m\cosh^2 u\left(\ddot{v} + 2\tanh u\,\dot{u}\dot{v}\right) \\ &\qquad - I\tanh u\left(\ddot{\psi} - \tanh u\,\ddot{v}\right) \\ &\qquad - I\operatorname{sech}^2 u\,\dot{u}\left(\dot{\psi} - 2\tanh u\,\dot{v}\right), \\ \dot{p}_\psi &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot{\psi}} = I\left(\ddot{\psi} - \tanh u\,\ddot{v} - \operatorname{sech}^2 u\,\dot{u}\dot{v}\right). \end{aligned} \tag{35}$$

From the other side we have that

$$\begin{aligned}\frac{\partial L}{\partial u} &= m\sinh u \cosh u \left(\dot{u}^2+\dot{v}^2\right) \\ &\quad - I\,\mathrm{sech}^2 u\,\dot{v}\left(\dot{\psi}-\tanh u\,\dot{v}\right) - \frac{\partial V}{\partial u} \\ \frac{\partial L}{\partial v} &= -\frac{\partial V}{\partial v}, \qquad \frac{\partial L}{\partial \psi} = -\frac{\partial V}{\partial \psi}.\end{aligned} \tag{36}$$

Finally, using (35), (36) the Euler–Lagrange equations of motion can be written as

$$\begin{aligned}m\ddot{u} &= \tanh u\left(\left(m+I\mathrm{sech}^4 u\right)\dot{v}^2 - m\dot{u}^2\right) \\ &\quad - I\,\mathrm{sech}^4 u\,\dot{v}\dot{\psi} - \mathrm{sech}^2 u\,\frac{\partial V}{\partial u} \\ \left(m\cosh^2 u + I\tanh^2 u\right)\ddot{v} - I\tanh u\,\ddot{\psi} &= I\mathrm{sech}^2 u\,\dot{u}\dot{\psi} \\ 2m\left(\sinh u\cosh u + \frac{I}{m}\tanh u\,\mathrm{sech}^2 u\right)\dot{u}\dot{v} - \frac{\partial V}{\partial v} \\ \ddot{\psi} - \tanh u\,\ddot{v} &= \mathrm{sech}^2 u\,\dot{u}\dot{v} - \frac{1}{I}\frac{\partial V}{\partial \psi}.\end{aligned} \tag{37}$$

Solving the second and third equations in (37) with respect to $\ddot{v}$ and $\ddot{\psi}$ we obtain that

$$\begin{aligned}m\ddot{v} &= I\mathrm{sech}^4 u\,\dot{u}\dot{\psi} - m\tanh u\left(2+\frac{I}{m}\mathrm{sech}^4 u\right)\dot{u}\dot{v} \\ &\quad - \mathrm{sech}^2 u\left(\frac{\partial V}{\partial v} + \tanh u\,\frac{\partial V}{\partial \psi}\right),\end{aligned} \tag{38}$$

$$\begin{aligned}\ddot{\psi} &= \frac{I}{m}\frac{\tanh u}{\cosh^4 u}\dot{u}\dot{\psi} + \left(\mathrm{sech}^2 u - 2\tanh^2 u - \frac{I}{m}\frac{\tanh^2 u}{\cosh^4 u}\right)\dot{u}\dot{v} \\ &\quad - \frac{1}{I}\frac{\partial V}{\partial \psi} - \frac{\tanh u}{m\cosh^2 u}\left(\frac{\partial V}{\partial v} + \tanh u\,\frac{\partial V}{\partial \psi}\right).\end{aligned} \tag{39}$$

For the special case of the geodetic motion ($V \equiv 0$) the first integrals of the system (37)–(39) can be calculated as

$$\begin{aligned}\dot{u}^2+\dot{v}^2 &= A^2\mathrm{sech}^2 u, \\ \dot{v} &= B\mathrm{sech}^2 u\left(1+\frac{D}{B}\tanh u\right), \\ \dot{\psi} &= C + B\frac{\tanh u}{\cosh^2 u}\left(1+\frac{D}{B}\tanh u\right)\end{aligned} \tag{40}$$

where A, B, C are constants of integration and $D = (I/m)\,C$. Because of the fact that from the second and third equations we can obtain that $\dot{\psi} - \tanh u\,\dot{v} = C$, we can deduce that the kinetic energy (28) is also constant and positively definite, i.e.,

$$\begin{aligned}T &= \frac{m}{2}\cosh^2 u\left(\dot{u}^2+\dot{v}^2\right) + \frac{I}{2}\left(\dot{\psi}-\tanh u\,\dot{v}\right)^2 \\ &= \frac{m}{2}A^2 + \frac{I}{2}C^2 = E.\end{aligned} \tag{41}$$

Finally, performing the integration of the obtained set of three first integrals (40), we obtain the geodetic solutions in the parametric form as

$$\begin{aligned}At(u) &= \int \frac{\cosh^2 u\;\mathrm{d}u}{\sqrt{\cosh^2 u - \left(\frac{B}{A}+\frac{D}{A}\tanh u\right)^2}}, \\ v(u) &= \int \frac{\left(\frac{B}{A}+\frac{D}{A}\tanh u\right)\mathrm{d}u}{\sqrt{\cosh^2 u - \left(\frac{B}{A}+\frac{D}{A}\tanh u\right)^2}}, \\ \psi(u) &= \int \frac{\frac{C}{A}\cosh^2 u + \frac{B}{A}\tanh u + \frac{D}{A}\tanh^2 u}{\sqrt{\cosh^2 u - \left(\frac{B}{A}+\frac{D}{A}\tanh u\right)^2}}\,\mathrm{d}u.\end{aligned} \tag{42}$$

Let us also note that from the first two expressions in (42) we can reobtain geodesics (13), (14) supposing that $C = 0$ (then, also $D = 0$).

We see that in the above expressions (42) the constant A is again simply rescaling the time t and the other integration constants B, C, and D (similarly as in the geodesic case). The same is true with the expression (41) that can be rewritten as

$$\frac{m}{2} + \frac{I}{2}\left(\frac{C}{A}\right)^2 = \frac{E}{A^2}. \tag{43}$$

Therefore, without any loss of generality, later on we can suppose that $A = 1$ with the possibility of reobtaining it in the final expressions with the substitution $t \mapsto At$, $B \mapsto B/A$, $C \mapsto C/A$, $D \mapsto D/A$, and $E \mapsto E/A^2$.

Next, let us consider the situation when $B = 0$, i.e., for geodesics we have that $v(t) = \mathrm{const}$ (i.e., they are simply the meridians) and for geodetics their drift with respect to the corresponding meridian is a manifestation of the influence of the internal degrees of freedom on the translational ones. Moreover, from (43) with $A = 1$ we see that there are two branches

$$C_\pm = \pm|C| = \pm\sqrt{\frac{2E-m}{I}} \tag{44}$$

of the geodetic solutions (42) corresponding to the same value of the total energy E of the infinitesimal rotator, i.e.,

$$\begin{aligned}t(u) &= \int \frac{\cosh^2 u\;\mathrm{d}u}{\sqrt{\cosh^2 u - D^2\tanh^2 u}}, \\ v_\pm(u) &= \pm|D|\int \frac{\tanh u\;\mathrm{d}u}{\sqrt{\cosh^2 u - D^2\tanh^2 u}}, \\ \psi_\pm(u) &= \pm|C|t(u) \\ &\quad \pm|D|\int \frac{\tanh^2 u\;\mathrm{d}u}{\sqrt{\cosh^2 u - D^2\tanh^2 u}}.\end{aligned} \tag{45}$$

The above expressions can be rewritten in the polynomial form with the introduction of the new variable χ, i.e.,

$$
\begin{aligned}
t\left(\chi=\sinh u\right) &= \int \frac{\left(1+\chi^2\right)\,\mathrm{d}\chi}{\sqrt{1+\left(2-D^2\right)\chi^2+\chi^4}}\,, \\
v_{\pm}\left(\chi=\cosh u\right) &= \pm|D|\int \frac{\mathrm{d}\chi}{\sqrt{D^2-D^2\chi^2+\chi^4}}\,, \\
\psi_{\pm}\left(\chi=\sinh u\right) &= \pm|C|t\left(\chi=\sinh u\right) \\
&\quad \pm|D|\int \frac{\chi^2\,\mathrm{d}\chi}{\left(1+\chi^2\right)\sqrt{1+\left(2-D^2\right)\chi^2+\chi^4}}\,.
\end{aligned}
\tag{46}
$$

Let us transform the expressions under the square root signs as

$$
\sqrt{1+\left(2-D^2\right)\chi^2+\chi^4}=\sqrt{\left(1-\frac{\chi^2}{a_+^2}\right)\left(1-\frac{\chi^2}{a_-^2}\right)}, \tag{47}
$$

$$
\frac{1}{|D|}\sqrt{D^2-D^2\chi^2+\chi^4}=\sqrt{\left(1-\frac{\chi^2}{b_+^2}\right)\left(1-\frac{\chi^2}{b_-^2}\right)} \tag{48}
$$

where $a_+^2 \geq a_-^2$, $a_+^2 a_-^2 = 1$, $a_+^2 + a_-^2 = D^2 - 2$ and $b_+^2 \geq b_-^2$, $b_+^2 b_-^2 = b_+^2 + b_-^2 = D^2$.

The explicit form of those coefficients is given as

$$
a_{\pm}=\frac{1}{2}\left(|D|\pm\sqrt{D^2-4}\right), \quad b_{\pm}=\sqrt{|D|a_{\pm}}. \tag{49}
$$

In order to obtain real values of the coefficients $a_{\pm}$ and $b_{\pm}$ in the above expressions we need to suppose that $D^2 \geq 4$, i.e., $|D| \geq 2$. In other words, this is the requirement that is needed to obtain the real solutions a_i, b_i, $i=\overline{1,4}$ of the fourth-order algebraic equations $1+\left(2-D^2\right)x^2+x^4=0$ and $D^2-D^2x^2+x^4=0$ that can be expressed through $a_{\pm}$ and $b_{\pm}$ as

$$
a_{1,2}=\pm a_+, \quad a_{3,4}=\pm a_-, \quad b_{1,2}=\pm b_+, \quad b_{3,4}=\pm b_-. \tag{50}
$$

So, supposing that $D^2 \geq 4$, we obtain that $a_+ \geq 1 \geq a_-$, therefore, $a_-/a_+ = a_-^2 \leq 1$, whereas $b_-/b_+ = \sqrt{a_-/a_+} = a_- \leq 1$.

Then, integrating (46) we obtain that

$$
\begin{aligned}
t(u) &= t_0+\frac{1+a_-^2}{a_-}F\left(\frac{\sinh u}{a_-},a_-^2\right)-\frac{1}{a_-}E\left(\frac{\sinh u}{a_-},a_-^2\right) \\
v_{\pm}(u) &= v_0^{\pm}\pm\sqrt{|D|a_-}F\left(\frac{\cosh u}{\sqrt{|D|a_-}},a_-\right) \\
\psi_{\pm}(u) &= \psi_0^{\pm}\pm|C|t(u) \\
&\quad \pm\ |D|a_-\left(F\left(\frac{\sinh u}{a_-},a_-^2\right)-\Pi\left(a_-^2,\frac{\sinh u}{a_-},a_-^2\right)\right)
\end{aligned}
\tag{51}
$$

where $\Pi(n,z,k)$ is the incomplete elliptic integral of the third kind of the characteristic n and the elliptic modulus k that is given in the Legendre normal form as

$$
\Pi(n,z,k)=\int\limits_0^z \frac{\mathrm{d}x}{\left(1+nx^2\right)\sqrt{1-x^2}\sqrt{1-k^2x^2}}\,. \tag{52}
$$

Again in order to reobtain from (51) the geodesics (26) we need to formally suppose that $C=0$ (then, also $D=0$), therefore, $a_{\pm}=\pm\mathrm{i}$, i.e., $1+a_-^2=0$, $k^2=a_-^4=1$, and $E(z,1)=z$.

Similarly to the geodesic case, the geodetics on the helicoid-catenoid family of surfaces can be also drawn (see Figs. 4, 5) using only the second expression in (51) that can be rewritten with the use of Jacobi's elliptic sine function, i.e., in this way we obtain the connection between u and $v_{\pm}$ variables given as

$$
\frac{1}{\sqrt{|D|a_-}} \leq \frac{\cosh u}{\sqrt{|D|a_-}} = \mathrm{sn}\left(\frac{v_{\pm}-v_0^{\pm}}{\sqrt{|D|a_-}},a_-\right) \leq 1. \tag{53}
$$

The first inequality in (53) is due to the fact that $\cosh u \geq 1$ and the second one is because the range of Jacobi's elliptic sine function (similarly to the one of the trigonometrical sine function) is restricted to the interval from -1 to 1. Hence, the range of the variable u is also restricted to $[-u_{\max}, u_{\max}]$, where

$$
u_{\max}=\operatorname{arccosh}\left(\sqrt{|D|a_-}\right). \tag{54}
$$

Moreover, the full period of the Jacobi's elliptic sine function $\mathrm{sn}(z,k)$ is equal to $4K(k)$, where $K(k) := F(1,k)$ is the complete elliptic integrals of the first kind of the elliptic modulus k. Hence, we can distinguish two halves of the full period:

1) when u is changing forward from $-u_{\max}$ to $u_{\max}$ and
2) when u is changing backward from $u_{\max}$ to $-u_{\max}$.

Then one of the branches of the solution (51) can be assigned to the first half of the full period (e.g., v_+) and the other branch of the solution can be assigned to the second half of the full period (e.g., v_-). In order to obtain a smooth trajectory in the variables u and v we need to introduce the compatibility conditions that glue together both branches of the solution, i.e.,

$$
v_+\left(\pm u_{\max}\right)=v_-\left(\pm u_{\max}\right). \tag{55}
$$

This gives us the relation between the integration constants $v_0^{\pm}$ assigned to both branches of the geodetic solution (51) leaving only one of them to be independent, i.e.,

$$
v_0^+ + 2\sqrt{|D|a_-}K\left(a_-\right)=v_0^-. \tag{56}
$$

We can also relate every geodetic solution to the corresponding geodesic one with $B=0$, i.e., the meridian $v=v_0$ which is drawn as the vertical line on the catenoid (see Fig. 4) and the horizontal line on the helicoid (see Fig. 5). Then the geodetic solution is starting on the corresponding geodesic (meridian) on the level of the parallel $-u_{\max}$, then deviating from it in the positive or negative direction in the variable v, and finally returning symmetrically to the same initial geodesic on the level of the parallel $u_{\max}$ (see Fig. 4). This means that we can express both integration constants $v_0^{\pm}$ through the integration constant v_0 for the corresponding geodesic as

$$
v_0^{\pm}=v_0\mp\sqrt{|D|a_-}K\left(a_-\right). \tag{57}
$$

The similar identification of the branches can be done also for the third solution in (51) describing the behaviour of the

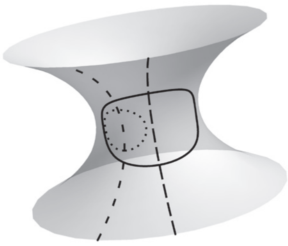

Fig. 4. Geodetics given by (53) on the catenoid ($\vartheta = \pi/2$) with the following values of parameters: $v_0^+ = 0$, $D = 2.1$ (continuous line), $D = 2.9$ (dotted line). Long- and short-dashed lines are describing the geodesics v_0 with respect to which the above two geodetics are constructed

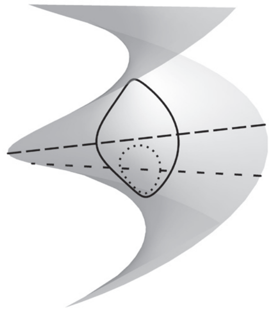

Fig. 5. Geodetics given by (53) on the left-handed helicoid ($\vartheta = 0$) with the following values of parameters: $v_0^+ = 0$, $D = 2.1$ (continuous line), $D = 2.9$ (dotted line). Long- and short-dashed lines are describing the geodesics v_0 with respect to which the above two geodetics are constructed

internal variable $\psi_\pm$ with the compatibility condition

$$\psi_+(u_{\max}) = \psi_-(u_{\max}) \tag{58}$$

producing the relation between the integration constants $\psi_0^\pm$ assigned to both branches of the geodetic solution (51) leaving only one of them to be an independent integration constant.

5. Action-angle analysis

For a Hamiltonian $H(q,p)$ with the kinetic energy term given as (34) the stationary Hamilton–Jacobi equation has the form

$$H\left(q^i, \frac{\partial S_0}{\partial q^i}\right) = E. \tag{59}$$

Let us note that in the expression (34) the variables v and ψ are cyclic, therefore, in the following action-angle analysis we can focus our attention on the models of the potential energy $V(q)$ that does not depend on them, i.e., the corresponding conjugate momenta p_v and p_ψ are constants of motion.

In this way the reduced function S_0 can be decomposed as

$$\begin{aligned} S_0(u,v,\psi;E,l,s) &= S_u(u;E) + S_v(v;l) + S_\psi(\psi;s) \\ &= S_u(u;E) + lv + s\psi \end{aligned}$$

where E, l, and s are three integration constants for our system with three degrees of freedom.

Then, due to the assumed symmetry, the considered partial differential equation (59) is reduced to the ordinary differential equation only for the function $S_u(u;E)$ [8,9], i.e.,

$$\left(\frac{\mathrm{d}S_u}{\mathrm{d}u}\right)^2 = 2m\cosh^2 u\left(E - V(u) - \frac{s^2}{2I}\right) - (l + s\tanh u)^2. \tag{60}$$

Hence, the conjugate momenta are given as $p_v = l$, $p_\psi = s$, and

$$p_u = \pm\sqrt{2m\cosh^2 u\left(E - V(u) - \frac{s^2}{2I}\right) - (l + s\tanh u)^2}$$

whereas the corresponding action variables are expressed as

$$J_v = \int_0^{2\pi} l \,\mathrm{d}v = 2\pi l, \qquad J_\psi = \int_0^{2\pi} s\,\mathrm{d}\psi = 2\pi s,$$

$$J_u = \oint \sqrt{2m\cosh^2 u\left(E - V(u) - \frac{s^2}{2I}\right) - (l + s\tanh u)^2}\,\mathrm{d}u.$$

Substituting $l = J_v/2\pi$ and $s = J_\psi/2\pi$ into the last formula we obtain that the expression under the integral sign is given as

$$\sqrt{2m\cosh^2 u\left(E - V(u) - \frac{J_\psi^2}{8\pi^2 I}\right) - \frac{(J_v + J_\psi \tanh u)^2}{4\pi^2}}.$$

Next, let us choose such kinds of external potentials $V(u)$ that are somehow related to the geometry of the considered manifold, e.g., inversely proportional to the determinant of the internal metric defined by the first fundamental form (5), i.e.,

$$V(u) = \frac{\varphi(u)}{\det[g_{ij}]} = \frac{\varphi(u)}{\cosh^4 u} \tag{61}$$

where $\varphi(u)$ is some function of the variable u, e.g., the one that is mimicking the structure of the kinetic energy term (34), i.e.,

$$\varphi(u) = \frac{\alpha\cosh^2 u + \beta\sinh u\cosh u + \gamma\sinh^2 u}{2m} \tag{62}$$

where α, β, γ are some parameters. Let us note that the factor $1/2m$ in (62) is chosen only for the matter of convenience because now the Hamiltonian $H(q,p)$ can be rewritten as

$$\frac{p_u^2 + p_v^2 + \alpha + (2p_v p_\psi + \beta)\tanh u + (p_\psi^2 + \gamma)\tanh^2 u}{2m\cosh^2 u} + \frac{p_\psi^2}{2I}.$$

Therefore, using (61) with φ given by (62) we can also rewrite

$$J_u = \oint \sqrt{A\cosh^2 u - B - C\tanh u - D\tanh^2 u}\,\mathrm{d}u \tag{63}$$

where the newly introduced constants are defined as

$$A = 2mE - \frac{m}{I}s^2,\ B = l^2 + \alpha,\ C = 2ls + \beta,\ D = s^2 + \gamma. \quad (64)$$

Next, let us use the well-known identities

$$\cosh u = \frac{1 + \tanh^2(u/2)}{1 - \tanh^2(u/2)}, \quad \tanh u = \frac{2\tanh(u/2)}{1 + \tanh^2(u/2)} \quad (65)$$

and transform the independent variable u in (63) as

$$\zeta = \tanh\left(\frac{u}{2}\right), \quad -1 \le \zeta \le 1, \quad \mathrm{d}u = \frac{2\,\mathrm{d}\zeta}{1 - \zeta^2}. \quad (66)$$

In this way we obtain that $J_u = \oint f(\zeta)\ \mathrm{d}\zeta$ where the complex-valued function $f(\zeta)$ is given as

$$f(\zeta) = \frac{2\sqrt{P_8(\zeta)}}{(1-\zeta)^2(1+\zeta)^2(\zeta-\mathrm{i})(\zeta+\mathrm{i})} \quad (67)$$

with $P_8(\zeta)$ being the polynomial

$$a\zeta^8 - b\zeta^7 + c\zeta^6 + b\zeta^5 + d\zeta^4 + b\zeta^3 + c\zeta^2 - b\zeta + a \quad (68)$$

which coefficients are specified by the formulas

$$a = A - B,\ b = 2C,\ c = 4\,(A - D),\ d = 2\,(3A + B + 4D). \quad (69)$$

Next, let us consider the five possibilities depending on the number of conditions between the three action variables J_u, $J_v = 2\pi l$, $J_\psi = 2\pi s$ and the total energy E:

1) The non-degenerate case when the polynomial $P_8(\zeta)$ has no roots equal to ± 1 or $\pm\mathrm{i}$, then $f(\zeta)$ given by (67) has five poles at $a_k = \{1^2, -1^2, \mathrm{i}, -\mathrm{i}, \infty\}$ (the first two are double poles). Using the Cauchy's residue theorem we obtain that

$$\oint_\gamma f(\zeta)\,\mathrm{d}\zeta = -2\pi\mathrm{i}\sum_{k=1}^{n} \mathrm{Res}\,(f, a_k) \quad (70)$$

where γ is some positively-oriented simple closed curve that infinitesimally encircles the branch cut (or cuts, depending on the roots of the polynomial $P_8(\zeta)$) of the complex-valued function $f(\zeta)$. The residues in (70) can be calculated as

$$\mathrm{Res}(f, \pm\mathrm{i}) = \frac{\mathrm{i}}{4}\sqrt{|2\,(a-c)+d|} = \mathrm{i}\sqrt{|s^2+\gamma|}$$

whereas $\mathrm{Res}(f, \pm 1) = 0$ and, because of the fact that $\lim_{|\zeta|\to+\infty} f(\zeta) = 0$, the residue at infinity is given by

$$\mathrm{Res}(f, \infty) = -\lim_{|\zeta|\to+\infty} \zeta f(\zeta) = 0.$$

Substituting the obtained results into (70) we obtain that for the non-degenerate case there is only one condition given as

$$J_u^2 = \left|J_\psi^2 + 4\pi^2\gamma\right| \quad (71)$$

which in the case when $\gamma = 0$ simplifies to $J_u = |J_\psi|$.

2) Three degenerate cases when $d = -2(a+c)$, i.e., $A = 0$, and the polynomial $P_8(\zeta)$ is given by the following expressions:

a) $(1-\zeta)^2(a\zeta^6 + e\zeta^5 + f\zeta^4 + g\zeta^3 + f\zeta^2 + e\zeta + a)$, where $e = 2a - b$, $f = 3a - 2b + c$, and $g = 4a - 2b + 2c$, therefore the function $f(\zeta)$ given by (67) has five poles at $a_k = \{1, -1^2, \mathrm{i}, -\mathrm{i}, \infty\}$

b) $(1+\zeta)^2(a\zeta^6 - e\zeta^5 + f\zeta^4 - g\zeta^3 + f\zeta^2 - e\zeta + a)$, in which the coefficients of the second polynomial fulfil the linear conditions $e = 2a + b$, $f = 3a + 2b + c$, and $g = 4a + 2b + 2c$, meaning that the function $f(\zeta)$ has five poles at $a_k = \{1^2, -1, \mathrm{i}, -\mathrm{i}, \infty\}$

c) $(1-\zeta^2)^2(a\zeta^4 - b\zeta^3 + (2a+c)\zeta^2 - b\zeta + a)$, then the function $f(\zeta)$ has five poles at $a_k = \{1, -1, \mathrm{i}, -\mathrm{i}, \infty\}$.

Again using the Cauchy's residue theorem (70) we obtain that in this case there are two conditions, i.e.,

$$E = \frac{J_\psi^2}{8\pi^2 I} \quad (72)$$

and

$$a)\quad 2J_u = \sqrt{2\left|J_\psi^2 + 4\pi^2\gamma\right|} - \sqrt{\left|\left(J_v + J_\psi\right)^2 + 4\pi^2\,(\alpha+\beta+\gamma)\right|} \quad (73)$$

$$b)\quad 2J_u = \sqrt{2\left|J_\psi^2 + 4\pi^2\gamma\right|} - \sqrt{\left|\left(J_v - J_\psi\right)^2 + 4\pi^2\,(\alpha-\beta+\gamma)\right|} \quad (74)$$

$$c)\quad J_u = \sqrt{\left|J_\psi^2 + 4\pi^2\gamma\right|} - \frac{1}{2}\sqrt{\left|\left(J_v + J_\psi\right)^2 + 4\pi^2\,(\alpha+\beta+\gamma)\right|} - \frac{1}{2}\sqrt{\left|\left(J_v - J_\psi\right)^2 + 4\pi^2\,(\alpha-\beta+\gamma)\right|}. \quad (75)$$

3) There are five degenerate cases when the polynomial $P_8(\zeta)$ reduces respectively to:

a) $(1+\zeta^2)^2(a\zeta^4 + (c-2a)\zeta^2 + a)$, therefore the function $f(\zeta)$ has three poles at $a_k = \{1^2, -1^2, \infty\}$

b) $(1-\zeta)^4(a\zeta^4 + e\zeta^3 + f\zeta^2 + e\zeta + a)$, where $e = 4a - b$ and $f = 6a - 2b$, therefore the function $f(\zeta)$ has four poles at $a_k = \{-1^2, \mathrm{i}, -\mathrm{i}, \infty\}$

c) $(1+\zeta)^4(a\zeta^4 - e\zeta^3 + f\zeta^2 - e\zeta + a)$, where $e = 4a + b$ and $f = 6a + 2b$, therefore the function $f(\zeta)$ has four poles at $a_k = \{1^2, \mathrm{i}, -\mathrm{i}, \infty\}$

d) $(1-\zeta)^4(1+\zeta)^2(a\zeta^2 + (2a-b)\zeta + a)$, therefore the function $f(\zeta)$ has four poles at $a_k = \{-1, \mathrm{i}, -\mathrm{i}, \infty\}$

e) $(1+\zeta)^4(1-\zeta)^2(a\zeta^2 - (2a+b)\zeta + a)$, meaning that the function $f(\zeta)$ has four poles at $a_k = \{1, \mathrm{i}, -\mathrm{i}, \infty\}$.

In this way we obtain the following sets of three conditions

$$a)\quad J_u = 0, \quad J_\psi^2 = -4\pi^2\gamma, \quad J_v J_\psi = -2\pi^2\beta, \tag{76}$$

$$b)\quad J_u = 0, \quad E = \frac{J_\psi^2}{8\pi^2 I}$$
$$(J_v + J_\psi)^2 = -4\pi^2(\alpha+\beta+\gamma) \tag{77}$$

$$c)\quad J_u = 0, \quad E = \frac{J_\psi^2}{8\pi^2 I}$$
$$(J_v - J_\psi)^2 = -4\pi^2(\alpha-\beta+\gamma) \tag{78}$$

$$d)\quad \sqrt{2}J_u = \sqrt{\left|J_\psi^2 + 4\pi^2\gamma\right|} - \sqrt{2\left|J_v J_\psi + 2\pi^2\beta\right|}$$
$$E = \frac{J_\psi^2}{8\pi^2 I}, \quad (J_v + J_\psi)^2 = -4\pi^2(\alpha+\beta+\gamma) \tag{79}$$

$$e)\quad \sqrt{2}J_u = \sqrt{\left|J_\psi^2 + 4\pi^2\gamma\right|} - \sqrt{2\left|J_v J_\psi + 2\pi^2\beta\right|}$$
$$E = \frac{J_\psi^2}{8\pi^2 I}, \quad (J_v - J_\psi)^2 = -4\pi^2(\alpha-\beta+\gamma). \tag{80}$$

4) Two degenerate cases when $P_8(\zeta)$ reduces to:
   a) $a(1-\zeta^4)^2$ and $f(\zeta)$ has three poles at $a_k = \{1, -1, \infty\}$
   b) $a(1-\zeta^2)^4$ and $f(\zeta)$ has three poles at $a_k = \{\mathrm{i}, -\mathrm{i}, \infty\}$.

   In this case we obtain that there are four conditions, i.e.,

$$E = \frac{J_\psi^2}{8\pi^2 I}, \qquad J_v J_\psi = -2\pi^2\beta \tag{81}$$

and

$$a)\ J_u = -\sqrt{\left|J_v^2 + 4\pi^2\alpha\right|}, \quad J_\psi^2 = -4\pi^2\gamma \tag{82}$$

$$b)\ J_u = \sqrt{\left|J_\psi^2 + 4\pi^2\gamma\right|}, \quad J_v^2 + J_\psi^2 = -4\pi^2(\alpha+\gamma). \tag{83}$$

5) The trivial degenerate case when $a = b = c = d = 0$, i.e., the polynomial $P_8(\zeta) = 0$, therefore also the function $f(\zeta) = 0$. In this situation we have five different conditions

$$E = \frac{J_\psi^2}{8\pi^2 I}, \qquad J_u = 0, \tag{84}$$

$$J_v^2 = -4\pi^2\alpha, \quad J_v J_\psi = -2\pi^2\beta, \quad J_\psi^2 = -4\pi^2\gamma. \tag{85}$$

6. Conclusions

In this article we have presented the general description of mechanics of infinitesimal gyroscopes moving on minimal surfaces from the helicoid-catenoid deformation family. The obtained results concerning the classical geodesic and geodetic solutions, as well as the action-angle analysis of the corresponding Hamilton–Jacobi equation (i.e., the description of the problem at the level of the old quantum theory according to the Bohr–Sommerfeld postulates) can be useful for the general theory of shells and membranes, e.g., for description of infinitesimal objects with internal structure moving on different material or biological membranes that can (at least locally) be represented as 2D minimal surfaces embedded into the 3D Euclidean space.

Apart from this a very promising area of applications of our approach is connected to the description of electro-optic properties of cholesteric liquid crystals (confined chiral nematics) placed in some externally applied magnetic or electric fields that was recently developed in [10, 11].

Under confinement and geometric frustration such cholesterics are subjected to an anisotropic environment that leads to the appearance of internal frustrated configurations with topological defects, e.g., some elongated stringlike objects called cholesteric fingers (threads) or helicoids. In this way some low-cost photonic filters/switches or other micro-optical integrated devices can be built that modify the propagation of waves in such waveguides.

Last, but not least – the present study of helicoid-catenoid deformation family of minimal surfaces should be of immediate interest in the studies of beta-barrels which include up to now as models cylinders, one-sheeted hyperboloids, twisted hyperboloids, and catenoids [12]. The helicoid-catenoid family is just next in the row!

Appendix

Relying on the general approach presented in [7, 13] we can introduce the d'Alembert expression for the kinetic energy term of the infinitesimal test body in a differential manifold M as

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{m}{2} g_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} + \frac{1}{2} g_{ij} \frac{De^i{}_A}{Dt}\frac{De^j{}_B}{Dt} J^{AB} \tag{86}$$

where g_{ij} are the components of the metric tensor defined in the manifold M, x^i are the space coordinates of the test body as a whole (the remnant of the centre-of-mass position in the flat-space theory), $e^i{}_A$ are the internal coordinates of the test body being injected into the tangent space T_xM (microphysical space) where it can be identified with linear frames $e_A \in T_xM$, whereas m and J^{AB} are describing the mass and symmetric and positively definite micromaterial inertial tensor respectively. In the situation when it is isotropic, i.e., $J^{AB} = (I/n)\mathrm{Id}_n{}^{AB}$, where Id_n is the identity tensor in $\mathbb{R}^n$ (micromaterial space of the dimension n), then $\mathrm{Tr}(J) = (I/n)\mathrm{Tr}(\mathrm{Id}_n) = I$, therefore I can be called the scalar moment of inertia of our test body.

As it has been shown in [7,13] when we introduce in the manifold M the fixed once and for all fields of linear g-orthonormal ($g_{ij}E^i{}_A E^j{}_B = \mathrm{Id}_{nAB}$) aholonomic reference frames E_A, then at any time instant t and at any geometric point $\mathbf{x}(t)$ the internal configurations e_A can be decomposed with respect to those reference frames E_A as $e(t)_A = E[\mathbf{x}(t)]_B U(t)^B{}_A$, where the matrix U represents the internal variables of our test body. In the special case of gyroscopes taken as test bodies we obtain that U should be given as an orthogonal matrix, i.e., $U^T U = U U^T = \mathrm{Id}_n$, where T denotes the matrix transposition.

Now the covariant derivatives in (86) are given as [7, 13]

$$\frac{De^i{}_A}{Dt} = E^i{}_B \frac{\mathrm{d}U^B{}_A}{\mathrm{d}t} + \frac{DE^i{}_B}{Dt} U^B{}_A = e^i{}_B \widehat{\omega}^B{}_A \tag{87}$$

where the co-moving angular velocity $\widehat{\omega} = \widehat{\omega}_{\mathrm{rl}} + \widehat{\omega}_{\mathrm{dr}}$ can be divided into two parts. Namely, the relative term which refers to the part of internal rotation that is done with respect to the just passed reference frames E

$$\widehat{\omega}_{\mathrm{rl}}{}^B{}_A = U^{-1B}{}_C \frac{\mathrm{d}U^C{}_A}{\mathrm{d}t} \tag{88}$$

and the drive (or drift) term which describes the time rate of the part of internal rotation that is contained in the fields E

$$\widehat{\omega}_{\mathrm{dr}}{}^B{}_A = U^{-1B}{}_C \Gamma^C{}_{DF} U^D{}_A E^F{}_j \frac{\mathrm{d}x^j}{\mathrm{d}t}. \tag{89}$$

Here $\Gamma^C{}_{DF}$ denotes the affine connection with respect to the fields E that is connected to $\Gamma^k{}_{lm}$, i.e., the Levi-Civita affine connection with respect to coordinates x^i in M given by (6), as

$$\Gamma^C{}_{DF} = E^C{}_k \left(\Gamma^k{}_{lm} - E^k{}_A E^A{}_{l,m} \right) E^l{}_D E^m{}_F. \tag{90}$$

Finally, taking any minimal surface of the helicoid-catenoid family defined by (3) as a differential manifold M with the metric tensor g and affine connection Γ specified by (5) and (6), we can use as reference frames E the normalized fields

$$E_u = \frac{1}{\sqrt{g_{uu}}} \frac{\partial}{\partial u}, \qquad E_v = \frac{1}{\sqrt{g_{vv}}} \frac{\partial}{\partial v}. \tag{91}$$

In this way we obtain that for the infinitesimal gyroscopes considered as test bodies moving on the surface M the co-moving angular velocity can be calculated as

$$\widehat{\omega} = \widehat{\omega}_{\mathrm{rl}} + \widehat{\omega}_{\mathrm{dr}} = \left(\frac{\mathrm{d}\psi}{\mathrm{d}t} - \omega_{\mathrm{dr}} \frac{\mathrm{d}v}{\mathrm{d}t} \right) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{92}$$

where the coefficient ω_{dr} is given by (29) and ψ is the internal rotation angle, i.e., the internal configuration has the form

$$U = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}. \tag{93}$$

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