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The use of synthesis methods in position optimisation and selection of tuned mass damper (TMD) parameters for systems with many degrees of freedom

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The paper formulates and formalises a method for selecting parameters of the tuned mass damper (TMD) for primary systems with many degrees of freedom. The method presented uses the properties of positive rational functions, in particular their decomposition, into simple fractions and continued fractions, which is used in the mixed method of synthesis of vibrating mechanical systems. In order to formulate a method of tuning a TMD, the paper discusses the basic properties of positive rational functions. The main assumptions of the mixed synthesis method is presented, based on which the general method of determining TMD parameters in the case of systems with many degrees of freedom was formulated. It has been shown that a tuned mass damper suppresses the desired resonance zone regardless of where the excitation force is applied. The advantages of the formulated method include the fact of reducing several forms of the object's free vibration by attaching an additional system with the number of degrees of freedom corresponding to the number of resonant frequencies reduced. In addition, the tuned mass damper determined in the case of excitation force applied at a single point can be attached to any element of the inertial primary system without affecting the reduction conditions in this way. It results directly from the methodology formalised in the paper. As part of the paper, numerical calculations were performed regarding the tuning of the TMD to the first form of free vibration of a system with 3 degrees of freedom. The parameters determined were subjected to analysis and verification of the correctness of the calculations carried out. For the considered case of a system with 3 degrees of freedom together with a TMD, time responses of displacement, from each floor, were generated to excitation induced by a harmonic force equal to the first form of vibration of the basic system. In addition, in the case of the parameters obtained, the response of the inertial element system to which the TMD was attached to random white noise excitation was determined.

Key words: structural control system, vibration, simple fractions, synthesis, rational functions, dynamic characteristics

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1. Introduction

Nowadays, interest in reducing vibration in various types of building structures, in particular high rise buildings, is growing due to not only horizontal but also vertical construction trends. Buildings are becoming taller, slimmer and thus more likely to be excited by natural phenomena such as earthquakes and winds. The reduction of vibration in such structures subjected to these phenomena takes place through the use of passive, active or semi-active control systems [1–11]. Among these systems, one of the most reliable and simplest is a tuned mass damper (TMD), i.e. a passive control system. Optimal selection of a tuned mass damper, which would be best in reducing vibration, is one of the biggest problems encountered by engineers in the development of TMDs, which is heading in various direction. In [12] Ghaedi et al. have carried out a detailed review of previous and latest research on the reduction of vibrations in building structures. One of the proposed approaches, they offer certain formulas based on simplified models of the building reduced to a system with one degree of freedom and load [1,3,13–16]. On the other hand, the current requirements enforce procedures for including the total structure of the building in the model in order to obtain a TMD's parameters [2,17–23]. The main idea is to tune the frequency of the TMD near the first frequency of the natural structure. The selection of parameters must be such that the damping applies to each point of the structure, irrespective of the place of loading and the attachment of the tuned mass damper. Hahnkamm [1] was one of the first to derive the optimal natural frequency coefficient according to the optimisation criterion and had described this method. Nishihara and Asami [24], Asami et al. [25] derived exact series solutions for the stiffness and damping coefficient. Nigdeli et al. [26] proposed an optimisation methodology to find the optimal period and damping of base isolation systems. In Sinan Melih Nigdeli, Gebrail Bekdas [27] proposed methods for selecting the optimal TMD frequency ratio and a superstructure with an optimal damping factor. In Shen et al. [2,17] and Wang et al. [18], negative stiffness was introduced to optimise parameters. Recently, combining a TMD with an inverter has been very popular. This issue was dealt with in [20], Hu et al. [21], Shen et al. [22], Brzeski et al. [23]. In addition to papers dealing with methods of optimal determination of TMD parameter, the impact of the place of TMD attachment on primary system's response was also dealt with [11,27]. The impact of specially developed two-way BTMD dampers on skyscraper vibration was also investigated [19]. This paper formulates a method for selecting the parameters of a tuned mass damper by tuning to the natural frequency of the structure with many degrees of freedom. In order to formalise the task of tuning the parameters of a tuned mass damper, a mixed method of mechanical systems synthesis was used, in particular the decomposition of dynamic characteristics of the examined building [5]. This approach makes it possible to employ the properties of positive rational functions [28] and in particular the methods of their decomposition to determine the parameters of a tuned mass damper. The mixed method of synthesis [5–8] allows for such a decomposition of dynamic characteristics that allows presenting a convenient form of a characteristic function corresponding to the primary structure together with a tuned mass damper presentation of the characteristic function in the form of dynamic characteristics. On the basis of the proposed decomposition of the dynamic characteristics of the structure together with the characteristics of the sought tuned mass damper, its parameters are determined. The properties of positive rational functions used in the paper and the mixed synthesis method allow to tune the parameters of a tuned mass damper so that the reduction of vibration is not affected by the place of application of the excitation force. In addition, the presented method makes it possible to determine the place of attachment of the TMD and select its parameters due to the optimal location of resonant frequencies of the system with a passive vibration damper. The goal of the paper is to formulate a method for determining the parameters of a tuned mass damper for systems with many degrees of freedom. The paper does not deal with the analysis of natural phenomena that cause vibration, instead it only focuses on formalising a method for tuning a TMD's parameters. In order to check the correctness of the obtained results, numerical calculations were carried out for a system with three degrees of freedom and dynamic excitation with excitation using the first harmonic of the examined system was adopted.

2. The method of selecting the parameters of a tuned mass damper for systems with $n \geqslant 2$ degrees of freedom

This chapter formulates a method of solving the problem of selection and tuning of a tuned mass damper's parameters in the case of systems with $n \geqslant 2$ degrees of freedom. The paper uses the properties of characteristic functions describing vibrating systems, in particular their decomposition into a continued fraction and a simple fraction. Both methods of decomposition of dynamic characteristics in the form of positive rational functions are included in the methods of synthesis of discrete vibrating systems. The combination of the decomposition of dynamic characteristics into simple fractions and a continued fraction enabled the authors to introduce a mixed synthesis method. The mixed method has become the starting point for formulating the task of selecting a tuned mass damper's parameters in systems with many degrees of freedom.

2.1. Mixed synthesis method

The dynamic characteristics of the considered class of vibrating systems are positive rational functions of a complex variable in the form of the function of stiffness, compliance, mobility and slowness in accordance with the theorems quoted after [28].

Theorem 1 *Function $U(s)$ is a dynamic characteristic if and only if it is a measurable real positive function of variable s (s – Laplace operator) satisfying the conditions:*

$$\begin{cases} \operatorname{Im} U(s) = 0, & \textit{if } \operatorname{Im} s = 0, \\ \operatorname{Re} U(s) \geqslant 0, & \textit{if } \operatorname{Re} s \geqslant 0. \end{cases}$$

Theorem 2 *Each positive-real rational function can be implemented as dynamic characteristics of a model of a mechanical system with concentrated parameters, built of inertial, elastic or damping mechanical double-connectors.*

Due to the application of positive-real rational functions in the case of tuning the parameters of a tuned mass damper, it will be appropriate to provide these of their characteristics that result directly from the quoted theorems, which will be used in the paper, i.e.:

- *The sum of the finite number of positive-real rational functions is the positive-real rational function; this property was used in the decomposition of dynamic characteristics into simple fractions.*

- *The product of the positive-real rational function and a positive constant is the positive-real rational function; this property is used to present dynamic characteristics in normal form and to scale them.*

- *The inverse of the positive-real rational function is the positive-real rational function; i.e.*

$$U(s) = \frac{1}{V(s)}\,. \tag{1}$$

The aforementioned mixed synthesis method is a combination of known methods for the synthesis of mechanical (electrical) systems such as the decomposition of characteristic functions into a continued fraction and simple fractions. The starting point in the case of using any decomposition of a positive rational function in the synthesis of mechanical or electrical systems is the dynamic characteristics of the tested systems. On the basis of these functions, it is possible to identify parameters by reducing the examined system to a structure obtained as a result of the decomposition of dynamic characteristics. The structure obtained results directly from the applied method of decomposition of the function describing the system. And so, as a result of the applied decomposition of the positive rational function of a complex variable, systems with a cascade structure are obtained in the case of a decomposition into a continued fraction and a branched structure in the case of a decomposition into simple fractions. The combination of methods of decomposition of rational functions into a simple fraction and a continued fraction enables obtaining a branched cascade structure of the system. In this case, the structure obtained may correspond to the model of the tested system together with the tuned mass damper determined.

In order to discuss the mixed method, it was assumed that the dynamic characteristics considered describe the properties of a fixed vibrating system without damping with n degrees of freedom in the form of the set slowness function $U(s)$. Characteristics $U(s)$ presented in general form must meet the property of the positive-real rational function. Such conditions will be met in the case of dynamic characteristics of the analysed system determined at the place of excitation, which takes the form of:

$$U(s) = \frac{c_k s^k + c_{k-2} s^{k-2} + \cdots + c_0}{d_l s^l + d_{l-2} s^{l-2} + \cdots + d_1 s}, \quad (l = k - 1), \tag{2}$$

where: k, l – natural numbers, s – Laplace operator, d_l, c_k – real numbers.

Function $U(s)$ can also be represented as the quotient of the product of resonance frequencies to the product of anti-resonance frequencies in the form:

$$U(s) = \frac{1}{V(s)} = \frac{1}{sY(s)} = \frac{H \prod_{i=1}^{n} \left(s^2 + \omega_{bpi}^2\right)}{s \prod_{i=1}^{n-1} \left(s^2 + \omega_{zi}^2\right)}, \tag{3}$$

where: ω_{bpi} – frequency values of the basic system, ω_{zi} – anti-resonance frequency values of the basic system, $n = \frac{k}{2}$ – number of resonance frequencies of the system, H – proportionality factor, $V(s)$ – mobility of the analysed system, $Y(s)$ – compliance function of the analysed system.

The characteristic function obtained (3) is decomposed into simple fractions. As a result of its decomposition, slowness $U(s)$ takes the form of the sum of simple fractions:

$$\frac{U(s)}{H} = k_\infty s + \frac{k_0}{s} + \frac{A_1}{s - i\omega_{z1}} + \frac{A_2}{s + i\omega_{z1}} + \frac{A_3}{s - i\omega_{z2}} + \frac{A_4}{s + i\omega_{z2}} + \cdots + \frac{A_{2n-3}}{s - i\omega_{zn-2}} + \frac{A_{2n-4}}{s + i\omega_{zn-2}}. \tag{4}$$

Values of residues k_∞, k_0, A_1, A_2, A_3, A_4, $\ldots$, A_{2n-3}, A_{2n-4} are determined using:

$$
\begin{gathered}
k_\infty = \lim_{s\to\infty} \frac{U(s)}{s},\\
k_0 = \lim_{s\to 0} sU(s),\\
A_1 = \lim_{s\to i\omega_{z1}} (s - i\omega_{z1})\, U(s), \quad A_2 = \lim_{s\to -i\omega_{z1}} (s + i\omega_{z1})\, U(s),\\
A_3 = \lim_{s\to i\omega_{z2}} (s - i\omega_{z2})\, U(s), \quad A_4 = \lim_{s\to -i\omega_{z2}} (s + i\omega_{z2})\, U(s),\\
\vdots\\
A_{2n-3} = \lim_{s\to i\omega_{zn-2}} (s - i\omega_{zn-2})\, U(s), \quad A_{2n-4} = \lim_{s\to -i\omega_{zn-2}} (s + i\omega_{zn-2})\, U(s).
\end{gathered}
\tag{5}
$$

Knowing that the considered characteristics (2) are a measurable positive-real function, i.e. all residues on the imaginary axis are positive-real, and A_1, A_2, $\ldots$, A_{2n-3}, A_{2n-4} are the values of conjugate numbers, we can write:

$$
A_1 = A_2 = \frac{k_1}{2}, \quad A_3 = A_4 = \frac{k_2}{2}, \quad \cdots, \quad A_{2n-3} = A_{2n-4} = \frac{k_n}{2}
$$

from where we get:

$$
\begin{gathered}
\frac{A_1}{s - i\omega_{z1}} + \frac{A_2}{s + i\omega_{z1}} = \frac{k_1 s}{s^2 + \omega_{z1}^2},\\
\frac{A_3}{s - i\omega_{z2}} + \frac{A_4}{s + i\omega_{z2}} = \frac{k_2 s}{s^2 + \omega_{z2}^2},\\
\vdots\\
\frac{A_{2n-3}}{s - i\omega_{zn}} + \frac{A_{2n-4}}{s + i\omega_{zn}} = \frac{k_n s}{s^2 + \omega_{zn}^2}.
\end{gathered}
\tag{6}
$$

Finally, the sum of simple fractions of the slowness function (2) after taking into account (5) in (3) takes the form:

$$
\frac{U(s)}{H} = k_\infty s + \frac{k_0}{s} + \frac{k_1 s}{s^2 + \omega_{z1}^2} + \frac{k_2 s}{s^2 + \omega_{z2}^2} + \cdots \frac{k_n s}{s^2 + \omega_{zn}^2}.
\tag{7}
$$

The physical interpretation of the decomposition of the characteristic function (7) into the sum of simple fractions is a vibrating system, in this case a fixed one, with a branched structure corresponding to $n-1$ harmonic oscillators attached to an inertial element with the value of k_∞ and suspended on an elastic element of with the value of k_0.

The mixed method uses the decomposition of dynamic characteristics into simple fractions, which can be presented as the sum of any rational functions. This approach makes it possible to obtain a vibrating system with any branched cascade structure, whose characteristics correspond to the function subjected to decomposition into simple fractions. By adding any components of the sum (7), rational functions (8)–(11) are determined, which are interpreted physically as subsystems with cascade structure obtained as a result of the decomposition of functions into a continued fraction (Fig. 1).

$$\frac{U(s)}{H} = k_\infty s + \frac{k_0}{s} + \frac{k_1 s}{s^2+\omega_{z1}^2} + \frac{B_1 s^{2n-4} + B_2 s^{2n-6} + \cdots + B_0 s}{\prod_{i=2}^{n-2}\left(s^2+\omega_{zi}^2\right)}, \tag{8}$$

$$\frac{U(s)}{H} = k_\infty s + \frac{k_0}{s} + \frac{k_1 s}{s^2+\omega_{z1}^2} + \frac{k_2 s}{s^2+\omega_{z2}^2} + \frac{C_1 s^{2n-6} + C_2 s^{2n-8} + \cdots + C_0 s}{\prod_{i=2}^{n-2}\left(s^2+\omega_{zi}^2\right)}, \tag{9}$$

$$\frac{U(s)}{H} = k_\infty s + \frac{k_0}{s} + \frac{D_1 s^3 + D_2 s}{(s^2+\omega_{z1}^2)(s^2+\omega_{z2}^2)} + \frac{C_1 s^{2n-6} + C_2 s^{2n-8} + \cdots + C_0 s}{\prod_{i=2}^{n-2}\left(s^2+\omega_{zi}^2\right)}, \tag{10}$$

$$\frac{U(s)}{H} = k_\infty s + \frac{E_1 s^3 + E_2 s + E_0}{(s^2+\omega_{z1}^2)(s^2+\omega_{z2}^2)} + \frac{C_1 s^{2n-6} + C_2 s^{2n-8} + \cdots + C_0 s}{\prod_{i=2}^{n-2}\left(s^2+\omega_{zi}^2\right)}, \tag{11}$$

where: $m_\infty, m_{11}, m_{1(n-1)}, m_{1n}, \ldots, m_{(n-1)(n-1)}$ – value of the inertial components of the sought system, c_{10}, c_{20}, c_{n0}, $c_{(n+1)0}$ – values of the springs restrained components, c_{11}, $c_{1(n-1)}$, c_{1n}, $c_{(n-1)(n-1)}$ – values of the springs components of the sought system.

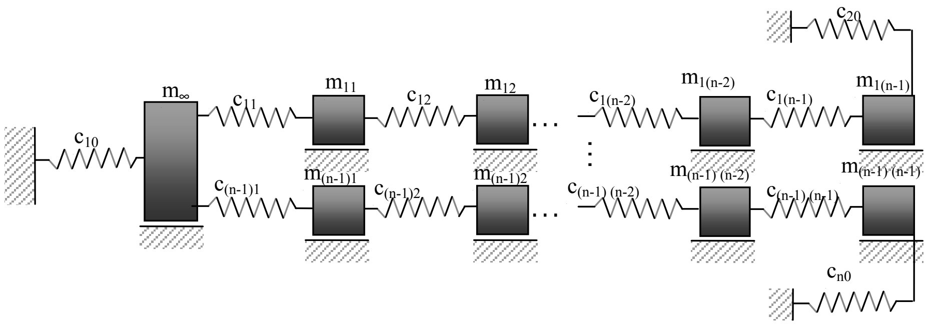

Figure 1: Structure of the system's model obtained as a result of using the mixed method

The presented functions (8)–(11) are not the only ones that can be obtained from dynamic characteristics (7). The number of rational functions created as components of the sum of the considered characteristics (2) depends on the real system or model of such a system in which identification or synthesis is carried out, but also on the experience of the design engineer. One form of the characteristic functions obtained as a result of the decomposition using the mixed method can take the form (12). The sum of the rational function $U(s)$ and a simple fraction whose denominator assumes a zero value in the characteristic pole $U(s)$ ($\omega_{zi} = \omega_{bi}$), presented in this way is synonymous with the characteristics of the reduced system using a tuned mass damper.

$$U1(s) = U(s) + \frac{ks}{s^2 + \omega_{z1}^2}, \tag{12}$$

$$U1(s) = m_1 s + \cfrac{1}{\cfrac{s}{c_1} + \cfrac{1}{m_2 s + \cfrac{1}{\cfrac{s}{c_2} + \cfrac{1}{m_3 s + \cfrac{1}{\cfrac{s}{c_3} + \cfrac{1}{m_4 s + \cdots \cfrac{1}{m_n s + \cfrac{c_n}{s}}}}}}}} + \frac{ks}{s^2 + \omega_{z1}^2}, \tag{13}$$

where: $m_1, m_2, m_3, m_4, \ldots, m_n$ – value of inertial elements of the system, $c_1, c_2, c_3, c_4, \ldots, c_n$ – value of elastic elements of the system, $\frac{ks}{s^2 + \omega_{z1}^2}$ – characteristics of the tuned mass damper, ω_{z1} – eliminated vibration frequency of the tested system = vibration frequency of the damper.

2.2. Selection of tuned mass damper parameters

Let's assume that dynamic characteristics of the system subjected to vibration reduction are specified $U(s)$. In this case, the system characteristics including a tuned mass damper take the form:

$$U_R(s) = U(s) + \frac{ks}{s^2 + \omega_{z01}^2}, \tag{14}$$

The decomposition of characteristics $U_R(s)$ shown is synonymous with the characteristics of a system with a tuned mass damper. Frequency ω_{z01} occurring in the simple fraction's denominator is the value of the dangerous resonance frequency of the system subjected to vibration reduction with characteristics $U(s)$. In order to determine the parameters of the damper used, the natural frequencies (resonance and anti-resonance frequencies) of the tested structure should be calculated. In this case, the rigidity matrix of the examined system subjected to vibration reduction is determined, in the form:

$$\mathbf{Z}(s) = \begin{bmatrix} m_1 s^2 + (c_{11}) & -(c_{12}) & \cdots & -(c_{1n}) \\ -(c_{21}) & m_2 s^2 + (c_{22}) & \cdots & -(c_{2n}) \\ \vdots & \vdots & \cdots & \vdots \\ -(c_{n1}) & -(c_{n2}) & \cdots & m_n s^2 + (c_{nn}) \end{bmatrix}. \tag{15}$$

Algebraic complement $D_{nn}(Z)$ is determined based on the matrix relative to the inertial element m_n to which a tuned mass damper will be attached:

$$D_{nn}(\mathbf{Z}) = (-1)^{n+n} |\mathbf{Z}_{nn}|, \tag{16}$$

$$Y(s) = \frac{X_n(s)}{F_n(s)} = \frac{D_{nn}(\mathbf{Z})}{\det Z(s)}, \tag{17}$$

where: $F_n(s)$ – excitation force acting on n inertial element, $X_n(s)$ – response of the n-th inertial element system.

Using the relationships (16) and (17), resonance ω_{bpi} and anti-resonance ω_{zi} frequencies are determined, which will be used to determine the characteristic function of the examined system in the form of slowness $U(s)$ (18):

$$U(s) = \frac{1}{V(s)} = \frac{1}{sY(s)} = \frac{M(s)}{L(s)} = \frac{H \prod\limits_{i=1}^{n} \left(s^2 + \omega_{bpi}^2\right)}{s \prod\limits_{i=1}^{n-2} \left(s^2 + \omega_{zi}^2\right)}, \tag{18}$$

where: $M(s)$ – characteristic equation (polynomial) of the system subjected to reduction, $L(s)$ – denominator (polynomial) of a characteristic function $U(s)$.

Among the resonance frequencies of the building, the dangerous frequency value is determined for which the tuned mass damper will be selected. The frequency determined, which in the new system adopts the value of anti-resonance frequency ω_{z01}, is added in the form of a simple fraction to the characteristics of the analysed building $U(s)$ in the form of:

$$U(s)+\frac{ks}{s^2+\omega_{z01}^2}=\frac{H\prod_{i=1}^{n}\left(s^2+\omega_{bpi}^2\right)}{s\prod_{i=1}^{n-2}\left(s^2+\omega_{zi}^2\right)}+\frac{ks}{s^2+\omega_{z01}^2}$$

$$=\frac{H\prod_{i=1}^{n}\left(s^2+\omega_{bpi}^2\right)}{s\prod_{i=1}^{n-2}\left(s^2+\omega_{zi}^2\right)}+\frac{1}{\frac{s}{k}+\frac{1}{\frac{k}{\omega_{z01}^2}s}}$$

$$=\frac{H\prod_{i=1}^{n}\left(s^2+\omega_{bpi}^2\right)}{s\prod_{i=1}^{n-2}\left(s^2+\omega_{zi}^2\right)}+\frac{1}{\frac{s}{c_{TMD}}+\frac{1}{m_{TMD}s}}=U_R(s). \tag{19}$$

Dynamic characteristics (19), taking into account several dampers to reduce vibrations of a structure loaded with many harmonics, take the form:

$$U(s)+\frac{k_1 s}{s^2+\omega_{z01}^2}+\frac{k_2 s}{s^2+\omega_{z02}^2}+\cdots+\frac{k_n s}{s^2+\omega_{z0n}^2}$$
$$=\frac{H\prod_{i=1}^{n}\left(s^2+\omega_{bpi}^2\right)}{s\prod_{i=1}^{n-2}\left(s^2+\omega_{zi}^2\right)}+\frac{1}{\frac{s}{c_{TMD1}}+\frac{1}{m_{TMD1}s}}+\frac{1}{\frac{s}{c_{TMD2}}+\frac{1}{m_{TMD2}s}}$$
$$+\cdots+\frac{1}{\frac{s}{c_{TMDn}}+\frac{1}{m_{TMDn}s}}, \tag{20}$$

where: k_n – the coefficient sought for determining the parameters of the tuned mass damper, ω_{z0n} – value of the reduced frequency of the analysed system, c_{TMD} – value of the sought elastic element of the tuned mass damper, m_{TMD} – value of the sought inertial element of the tuned mass damper.

The determined dynamic function (19) of the system has zeros (anti-resonance frequencies) in the place of zeros of the examined system ω_{zi} and zero at the place of the reduced natural frequency of the basic system ω_{z0n}. The poles of the new n+1 characteristic are unknown in value compared to the output system. One of the frequencies is assumed as a known value (ω_{bp}), which can be treated as an advantage, but it should be remembered that all frequencies in terms of values alternate with the values of known zeros of the characteristic. This is the case when excitation and response are measured relative to the same inertial element. The adopted assumptions in accordance with the conditions presented above take the form of the following relationship:

$$\frac{M(s)}{L(s)} + \frac{ks}{s^2 + \omega_{z01}^2} = \frac{H(s^2 + \omega_{bp}^2)\prod_{i=1}^{n}(s^2 + \omega_{bri}^2)}{s(s^2 + \omega_{z01}^2)\prod_{i=1}^{n-1}(s^2 + \omega_{zi}^2)}, \tag{21}$$

where: ω_{z01} – value of the reduced frequency of the analysed system, ω_{bp} – value of the adopted frequency or value of the frequency of the basic system, ω_{bri} – values of the system's frequency after reduction, ω_{zi} – values of the anti-resonance frequency of the basic system.

First of the components of the sum of the expression (21) should be multiplied by the quotient of the value of inertial elements of the analysed system starting from the second element $\prod_{i=2}^{n} m_i \Big/ \prod_{i=2}^{n} m_i$ and the other by coefficient H. The mass of the system relative to which the dynamic characteristic is determined $U(s)$ is marked as the first inertial element of the considered system. In the case of a system with one degree of freedom, the product $\prod_{i=2}^{n} m_i$ equals unity. As a result of applying the above, equation (21) takes the form (22):

$$\frac{M(s)\dfrac{1}{\prod_{i=2}^{n} m_i}}{L(s)\dfrac{1}{\prod_{i=2}^{n} m_i}} + \frac{Hks}{s^2 + \omega_{z01}^2} = \frac{H\left(s^2 + \omega_{bp}^2\right)\prod_{i=1}^{n}\left(s^2 + \omega_{bri}^2\right)}{s\left(s^2 + \omega_{z01}^2\right)\prod_{i=1}^{n-1}\left(s^2 + \omega_{zi}^2\right)}, \tag{22}$$

where:

$$M(s)\frac{1}{\prod\limits_{i=2}^{n} m_i} = m_1 \prod_{i=1}^{n}\left(s^2+\omega_{bpi}^2\right) = M1(s), \tag{23}$$

$$L(s)\frac{1}{\prod\limits_{i=2}^{n} m_i} = \prod_{i=1}^{n-1}\left(s^2+\omega_{zi}^2\right) = L1(s), \tag{24}$$

$$\frac{M1(s)(s^2+\omega_{z01}^2)+L1(s)Hks}{L1(s)(s^2+\omega_{z01}^2)} = \frac{H\left(s^2+\omega_{bp}^2\right)\prod\limits_{i=1}^{n}\left(s^2+\omega_{bri}^2\right)}{s\left(s^2+\omega_{z01}^2\right)\prod\limits_{i=1}^{n-1}\left(s^2+\omega_{zi}^2\right)}, \tag{25}$$

$$\frac{M1(s)(s^2+\omega_{z01}^2)+L1H(s)ks}{s(s^2+\omega_{z01}^2)\prod\limits_{i=1}^{n-1}(s^2+\omega_{zi}^2)} = \frac{H(s^2+\omega_{bp}^2)\prod\limits_{i=1}^{n}(s^2+\omega_{bri}^2)}{s(s^2+\omega_{z01}^2)\prod\limits_{i=1}^{n-1}(s^2+\omega_{zi}^2)}, \tag{26}$$

m_1 – value of the inertial element to which the tuned mass damper is attached,
m_i – value of the i-th inertial element.

The denominators of both rational functions of the expression (26) are the same, therefore the determination of the tuned mass damper's parameters and the resonance frequencies of the obtained system boils down to comparing the numerators of both characteristics (27).

Taking into account (28), in (29) we receive:

$$M1(s)(s^2+\omega_{z01}^2)+s\prod_{i=1}^{n-1}\left(s^2+\omega_{zi}^2\right)Hks = H\left(s^2+\omega_{bp}^2\right)\prod_{i=1}^{n}\left(s^2+\omega_{bri}^2\right), \tag{27}$$

$$\begin{aligned} m_1(s^2+\omega_{z01}^2)\prod_{i=1}^{n}\left(s^2+\omega_{bpi}^2\right) = H\Bigg(&\left(s^2+\omega_{bp}^2\right)\prod_{i=1}^{n}\left(s^2+\omega_{bri}^2\right) \\ &- s\prod_{i=1}^{n-1}(s^2+\omega_{zi}^2)ks\Bigg), \end{aligned} \tag{28}$$

from where:

$$m_1 = H. \tag{29}$$

After taking into account (29) in (28) the form obtained:

$$\begin{aligned}\left(s^2+\omega_{z01}^2\right)\prod_{i=1}^{n}\left(s^2+\omega_{bpi}^2\right)+s\prod_{i=1}^{n-1}(s^2+\omega_{zi}^2)ks\\ =\left(s^2+\omega_{bp}^2\right)\prod_{i=1}^{n}\left(s^2+\omega_{bri}^2\right),\end{aligned} \tag{30}$$

and assuming that

$$\begin{aligned}&\omega_{z01}=\omega_{b1}, \quad \omega_{bp1}=\omega_{b2}, \quad \omega_{bp2}=\omega_{b3}, \quad \cdots, \quad \omega_{bpn}=\omega_{bn+1},\\ &\omega_{bp}=\omega_{br1}, \quad \omega_{br1}=\omega_{br2}, \quad \omega_{br2}=\omega_{br3}, \quad \cdots, \quad \omega_{brn}=\omega_{brn+1},\end{aligned}$$

the equation (30) is simplified to the expression (31)

$$\prod_{i=1}^{n+1}\left(s^2+\omega_{bi}^2\right)+s\prod_{i=1}^{n-1}(s^2+\omega_{zi}^2)k=\prod_{i=1}^{n+1}\left(s^2+\omega_{bri}^2\right). \tag{31}$$

Based on the relationship (31) and (29), a system of n+2 equations is determined in the case of vibration reduction in a structure with n degrees of freedom using a tuned mass damper:

$$\begin{gathered}m_{1i}=H,\\ \sum_{i=1}^{n+1}\omega_{bri}^2=\sum_{i=1}^{n+1}\omega_{bi}^2+k,\\ \sum_{j=1}^{n}\omega_{brj}^2\sum_{i=j+1}^{n+1}\omega_{bri}^2=\sum_{j=1}^{n}\omega_{bj}^2\sum_{i=j+1}^{n+1}\omega_{bi}^2+k\sum_{i=1}^{n-1}\omega_{zi}^2,\\ \sum_{l=1}^{n-1}\omega_{brl}^2\sum_{j=l+1}^{n}\omega_{brj}^2\sum_{i=j+1}^{n+1}\omega_{bri}^2=\sum_{l=1}^{n-1}\omega_{bl}^2\sum_{j=l+1}^{n}\omega_{bj}^2\sum_{i=j+1}^{n+1}\omega_{bi}^2+k\sum_{j=1}^{n-2}\omega_{zj}^2\sum_{i=j+1}^{n-1}\omega_{zi}^2,\end{gathered} \tag{32}$$

(Eq. (32) cont.)

$$\sum_{s=1}^{n-2}\omega_{brs}^2\sum_{l=s+1}^{n-1}\omega_{brl}^2\sum_{j=l+1}^{n}\omega_{brj}^2\sum_{i=j+1}^{n+1}\omega_{bri}^2=\sum_{s=1}^{n-2}\omega_{bs}^2\sum_{l=s+1}^{n-1}\omega_{bl}^2\sum_{j=l+1}^{n}\omega_{bj}^2\sum_{i=j+1}^{n+1}\omega_{bi}^2 + k\sum_{l=1}^{n-3}\omega_{zl}^2\sum_{j=l+1}^{n-2}\omega_{zj}^2\sum_{i=j+1}^{n-1}\omega_{zi}^2,$$

$$\vdots$$

$$\sum_{z=1}^{2}\omega_{brz}^2\sum_{k=z+1}^{3}\omega_{brk}^2\cdots\sum_{j=l+1}^{n}\omega_{brj}^2\sum_{i=j+1}^{n+1}\omega_{bri}^2=\sum_{z=1}^{2}\omega_{bz}^2\sum_{k=z+1}^{3}\omega_{bk}^2\cdots\sum_{j=l+1}^{n}\omega_{bj}^2\sum_{i=j+1}^{n+1}\omega_{bi}^2 + k\sum_{k=1}^{1}\omega_{zk}^2\sum_{s=k+1}^{2}\omega_{zs}^2\cdots\sum_{j=l+1}^{n-2}\omega_{zj}^2\sum_{i=j+1}^{n-1}\omega_{zi}^2,$$

$$\prod_{i=1}^{n+1}\omega_{bri}^2=\prod_{i=1}^{n+1}\omega_{bi}^2 .$$

This manner of presenting individual components of the system of equations is a convenient tool for determining sought values of the parameter k and resonance frequencies of the system with a tuned mass damper attached. The task formulated in this way boils down to determining the value of the coefficient k according to the adopted system of equations (32). The calculations can also be carried out by determining only the resonance frequencies of the system after reduction, while the coefficient k can be determined from the dependence on the residual of the function $U_R(s)$ in the value of reduced frequency $i\omega_{z01}$ (33), (34):

$$\frac{k}{2}=\lim_{z\to i\omega_{z01}}(s-i\omega_{z01})\,\frac{H\left(s^2+\omega_{bp}^2\right)\prod\limits_{i=1}^{n}\left(s^2+\omega_{bi}^2\right)}{s\left(s^2+\omega_{z01}^2\right)\prod\limits_{i=1}^{n-2}\left(s^2+\omega_{zi}^2\right)}, \tag{33}$$

$$\frac{k}{2}=\lim_{z\to -i\omega_{z01}}(s+i\omega_{z01})\,\frac{H\left(s^2+\omega_{bp}^2\right)\prod\limits_{i=1}^{n}\left(s^2+\omega_{bi}^2\right)}{s\left(s^2+\omega_{z01}^2\right)\prod\limits_{i=1}^{n-2}\left(s^2+\omega_{zi}^2\right)}. \tag{34}$$

The above dependencies result directly from the properties of positive-real rational functions – all residues on the imaginary axis are real positive and are the values of the residues of conjugated numbers.

3. Numeric description

In order to check the correct operation of the formulated method of determining the parameters of the tuned mass damper, numerical calculations were carried out for the vibration system with $n \geqslant 2$ degrees of freedom. The analysed system has three degrees of freedom and its parameters have the following values $m_1 = m_2 = m_3 = 1$ kg, $c_1 = 100$ N/m, $c_2 = 150$ N/m, $c_3 = 200$ N/m, (Fig. 2). Based on the parameters adopted in this way, the rigidity matrix of the system adopted was determined in the form:

$$\mathbf{Z}(s) = \begin{bmatrix} m_1 s^2 + c_1 & -c_1 & 0 \\ -c_1 & m_2 s^2 + c_1 + c_2 & -c_2 \\ 0 & -c_2 & m_2 s^2 + c_2 + c_3 \end{bmatrix}. \tag{35}$$

Using the matrix (35), the dynamic characteristic relative to the inertial element is determined m_1 (according to Eqs. (15)–(18)) to which the sought tuned mass damper will be attached to reduce loads caused by harmonic force equal to the first form of system vibration $F_1 1 \sin(5.66t)$ (Fig. 1). Force F_1 acts on the inertial element m_1 causing resonance in the system (Fig. 3).

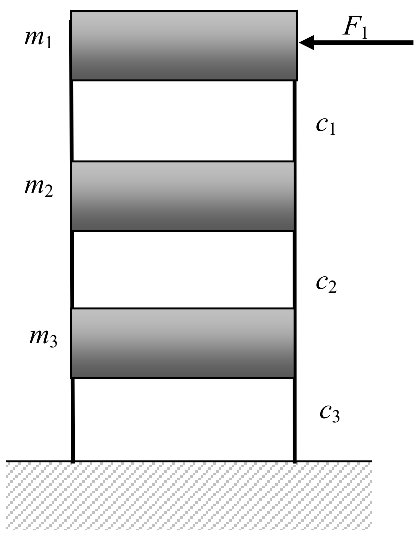

Figure 2: Model of the analysed building

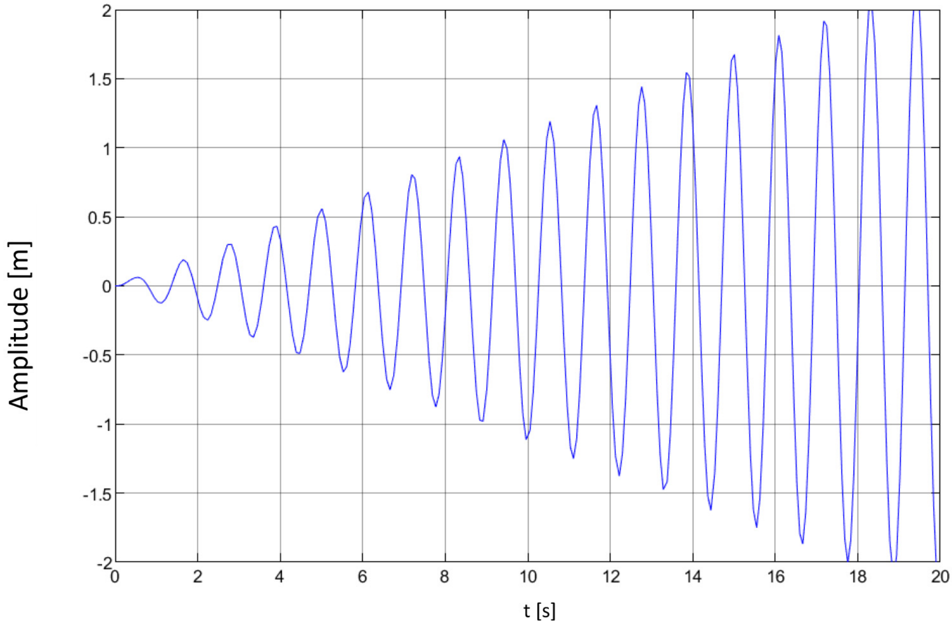

Figure 3: Response of the analysed system to excitation using harmonic force F_1

The system characteristics determined take the form of the slowness function in the following form:

$$U(s) = \frac{1}{sY_{11}(s)} = \frac{F_1(s)}{X_1(s)} = \frac{\left(s^2+\omega_{bp1}^2\right)\left(s^2+\omega_{bp2}^2\right)\left(s^2+\omega_{bp3}^2\right)}{s\left(s^2+\omega_{z1}^2\right)\left(s^2+\omega_{z2}^2\right)}$$
$$= \frac{\left(s^2+5.66^2\right)\left(s^2+14.14^2\right)\left(s^2+21.63^2\right)}{s\left(s^2+11.91^2\right)\left(s^2+21.40^2\right)}. \quad (36)$$

Function (36) is used to determine the slowness $U_R(s)$ of the newly created system with a tuned mass damper in the form of:

$$U_R(s) = U(s) + \frac{ks}{s^2+\omega_{z01=b1}^2} = \frac{\left(s^2+5.66^2\right)\left(s^2+14.14^2\right)\left(s^2+21.63^2\right)}{s\left(s^2+11.91^2\right)\left(s^2+21.40^2\right)}$$
$$+ \frac{ks}{s^2+5.66^2}. \quad (37)$$

The set of equations (32) in the case of the analysed system is as follows:

$$\begin{gathered}
H = 1, \\
\sum_{i=1}^{4}\omega_{bri}^2 = \sum_{i=1}^{4}\omega_{bi}^2 + k, \\
\sum_{j=1}^{3}\omega_{brj}^2 \sum_{i=j+1}^{4}\omega_{bri}^2 = \sum_{j=1}^{3}\omega_{bj}^2 \sum_{i=j+1}^{4}\omega_{bi}^2 + k\sum_{i=1}^{2}\omega_{zi}^2, \\
\sum_{l=1}^{2}\omega_{brl}^2 \sum_{j=l+1}^{3}\omega_{brj}^2 \sum_{i=j+1}^{4}\omega_{bri}^2 = \sum_{l=1}^{2}\omega_{bl}^2 \sum_{j=l+1}^{3}\omega_{bj}^2 \sum_{i=j+1}^{4}\omega_{bi}^2 + k\sum_{j=1}^{1}\omega_{zj}^2 \sum_{i=j+1}^{2}\omega_{zi}^2, \\
\prod_{i=1}^{4}\omega_{bri}^2 = \prod_{i=1}^{4}\omega_{bi}^2,
\end{gathered} \tag{38}$$

where: $\omega_{b1} = 5.66$ rad/s, $\omega_{b2} = 5.66$ rad/s, $\omega_{b3} = 14.14$ rad/s, $\omega_{b4} = 21.63$ rad/s, $\omega_{br1} = 5.2$ rad/s, $\omega_{z1} = 11.91$ rad/s, $\omega_{z1} = 21.40$ rad/s and ω_{br2}, ω_{br3}, ω_{br4} – values of the sought frequencies of natural vibrations of the newly created system.

In order to solve the system of equations, one of the frequencies of the system sought with a tuned mass damper was adopted (Fig. 4). It has been assumed that the adopted frequency is equal to $\omega_{br1} = 5.2$ rad/s. In the case of assumptions formulated in this manner, the following values of the obtained system were determined in the form:

resonant frequencies

$$\begin{aligned}
\omega_{br1} &= 5.2\,\frac{\text{rad}}{\text{s}}, & \omega_{br2} &= 6.17\,\frac{\text{rad}}{\text{s}}, \\
\omega_{br3} &= 14.16\,\frac{\text{rad}}{\text{s}}, & \omega_{br4} &= 21.63\,\frac{\text{rad}}{\text{s}},
\end{aligned} \tag{39}$$

and coefficient

$$k = 1.218. \tag{40}$$

Based on the determined coefficient k the parameters of the tuned mass damper are selected in accordance with the decomposition of its characteristics into an elastic and inertial element (41).

$$\begin{aligned}
\frac{ks}{s^2+\omega_{b1}^2} &= \frac{1.218s}{s^2+5.66^2} = \frac{1.218s}{s^2+32.036} \\
&= \frac{1}{\dfrac{s}{1.218}+\dfrac{1}{0.038s}} = \frac{1}{\dfrac{s}{c_{TMD}}+\dfrac{1}{m_{TMD}s}},
\end{aligned} \tag{41}$$

where: $c_{TMD} = 1.218$ N/m – value of the elastic element of the tuned mass damper, $m_{TMD} = 0.038$ kg – value of the inertial element of the tuned mass damper.

The sought tuned mass damper that reduces the first form of natural vibrations in the system is assumed to be attached to the inertial element m_1 (for which the dynamic characteristics in the form of slowness were determined (36)), which is shown in Fig. 4.

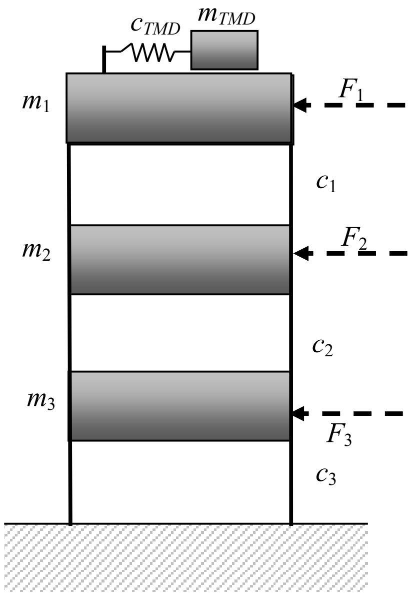

Figure 4: The resulting model of the system and possible attachment points of the vibration excitation force

On the basis of the obtained system (Fig. 4) dynamic characteristics in the form of slowness in the case of the excitation force acting on individual floors of the building were determined. The results are presented in the form of diagrams of the inertial elements' responses in the analysed building. In addition, for each of the considered cases of attaching the force exciting the first harmonic vibration, resonance and anti-resonance frequencies (zeros) of the dynamic characteristics (function of variable s) of the system were determined $Y_{11}(s)$, $Y_{12}(s)$, $Y_{13}(s)$ responses of the $x_1(t)$ inertial element m_1 to excitation force $F_1(t) = F_2(t) = F_3(t) = 1\sin(5.66t)$. As a result of calculations for the excitation force $F_1(t)$ acting on the inertial element m_1, resonance frequency values consistent with the values (39) and the following anti-resonance frequencies were obtained:

$$\omega_{zr1} = 5.66\ \frac{\text{rad}}{\text{s}}, \qquad \omega_{zr2} = 11.91\ \frac{\text{rad}}{\text{s}}, \qquad \omega_{zr3} = 21.40\ \frac{\text{rad}}{\text{s}}. \tag{42}$$

Figure 5 shows the responses of inertial elements of the resulting system caused by dynamic force $F_1(s)$ (Fig. 4) with a frequency equal to the first of natural vibration of the analysed building.

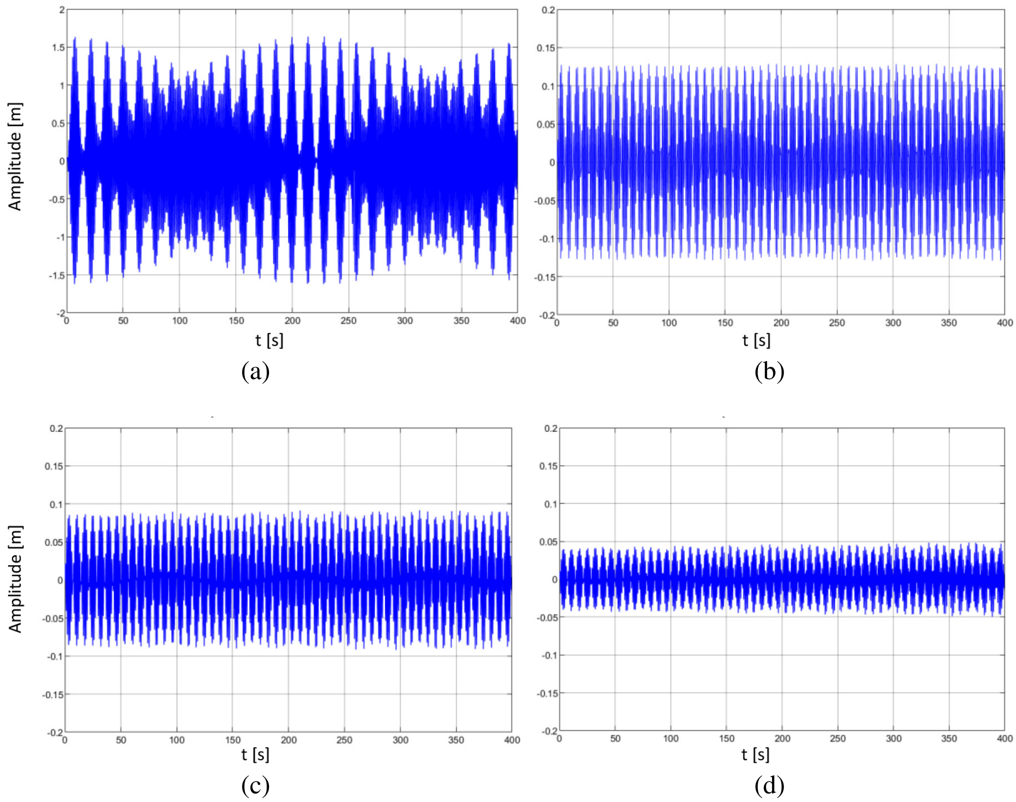

Figure 5: System response to excitation $F_1(s)$: a) damper, b) mass element m_1, c) mass element m_2, d) mass element m_3

In the analysis of the system (Fig. 6) in the case of the excitation force $F_2(s)$ acting on the inertial element m_2 the resonance frequency values consistent with the values (39) and the following anti-resonance frequencies were obtained:

$$\omega_{zr1} = 5.66\ \frac{\text{rad}}{\text{s}}, \qquad \omega_{zr2} = 18.7\ \frac{\text{rad}}{\text{s}}. \tag{43}$$

Below is the response of inertial elements of the system (Fig. 4) caused by dynamic force with a frequency equal to the first form of natural vibrations of the analysed building.

As a result of the analysis carried out in the case of the excitation force $F_3(s)$ acting on the inertial element m_3 the resonance frequency values consistent with

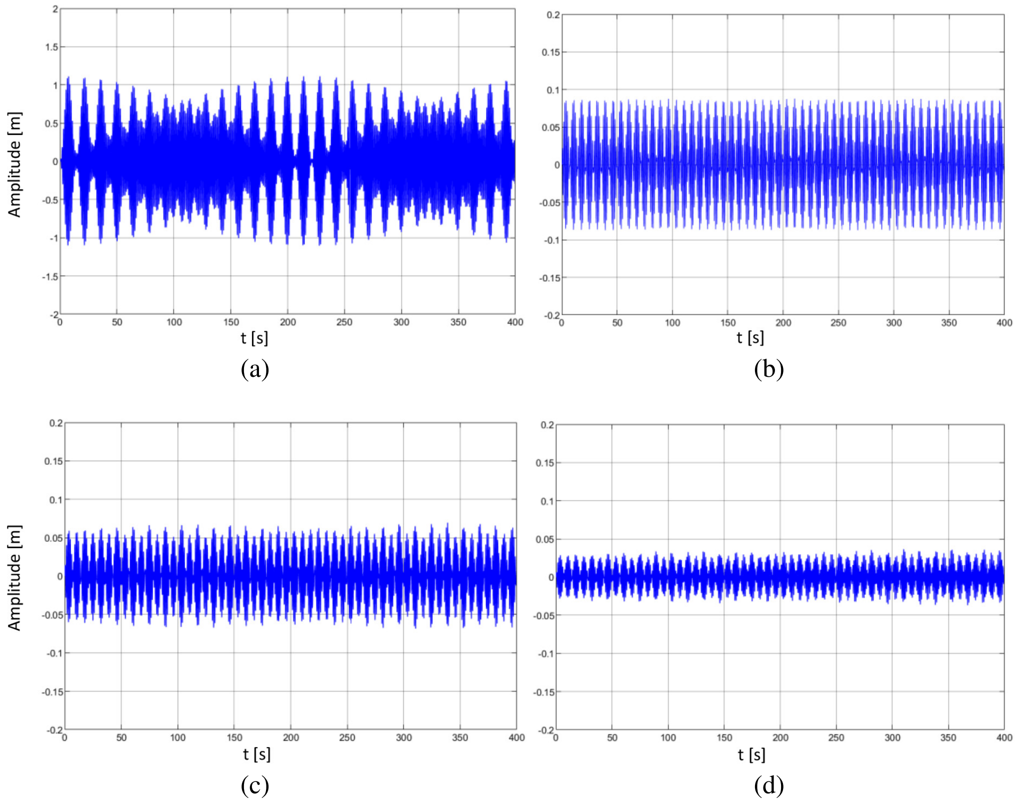

Figure 6: System response to excitation $F_2(s)$: a) damper, b) mass element m_1, c) mass element m_2, d) mass element m_3

the values (39) and the following anti-resonance frequencies were obtained:

$$\omega_{zr1} = 5.66\ \frac{\text{rad}}{\text{s}}\,. \tag{44}$$

Figure 7 presents the response of individual inertial elements of the system (Fig. 4) caused by dynamic force with a frequency equal to the first form of natural vibration of the analysed building acting on the inertial element m_3.

It should be noted that in each of the considered cases of characteristic functions $Y_{11}(s)$, $Y_{12}(s)$, $Y_{13}(s)$ the zero (anti-resonance frequency) occurs in $\omega_{zr1} = 5.66$ rad/s. The value of the repeated resonance frequency indicates the correctness of the determined parameters of the tuned mass damper. Based on the calculations carried out, it can be stated that the formulated method enables the determination of the tuned mass damper's parameters regardless of the location of the excitation force. The figures below show the dynamic characteristics determined for a system with a tuned mass damper (Fig. 8b–Fig. 8d) in the case of

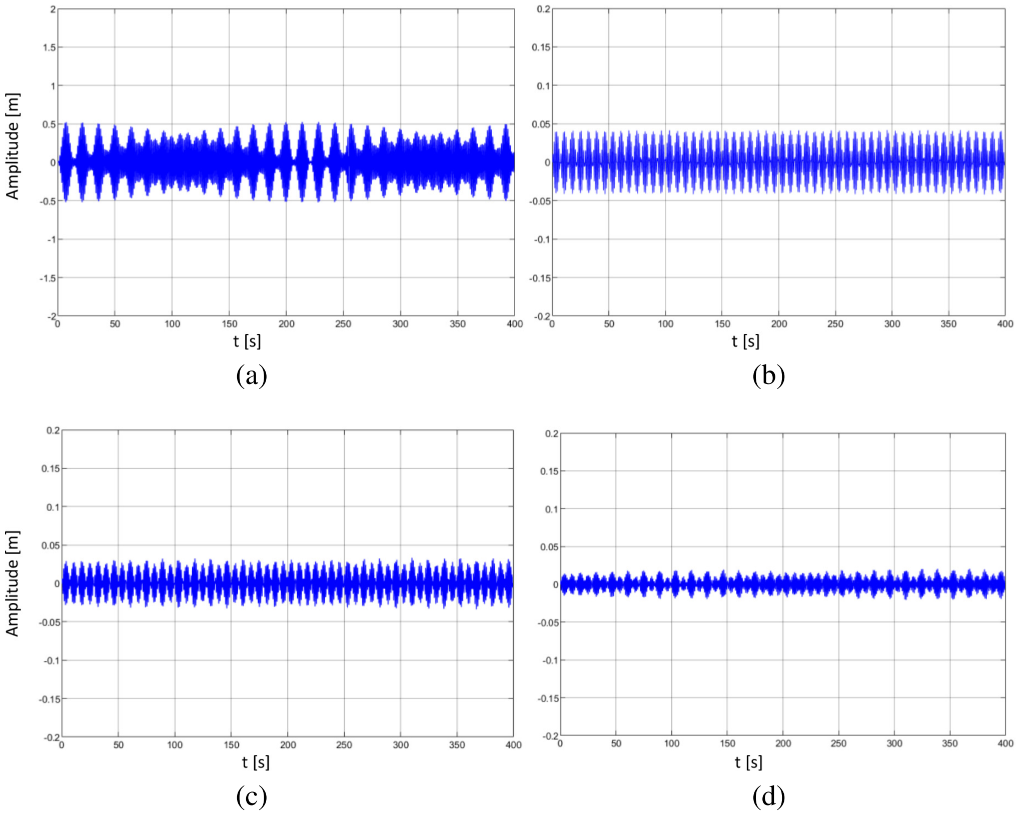

Figure 7: System response to excitation $F_2(s)$: a) damper, b) mass element m_1, c) mass element m_2, d) mass element m_3

a different attachment of dynamic force compared to the characteristics (Fig. 8a) of the output object.

In the case of determined values of parameters of an inertial and elastic tuned mass damper, the value of the potential damping type element was calculated assuming that its value is directly proportional to the obtained value c_{TMD} = 1.218 N/m.

$$b_{TMD} = \alpha c_{TMD} = 0.0609 \, \frac{\mathrm{Ns}}{\mathrm{m}} \, . \tag{45}$$

The primary system with a designated damper, taking into account a damping double-connector with the value of b_{TMD} connected in parallel with the elastic element c_{TMD} (Voigt-Kelvin), was subjected to random white noise loading with value 1.00 for height of PSD and 0.003 for correlation time of noise. As the result of the analysis, a response of the system on the inertial element m1 to which the TMD was suspended was obtained, as shown in Fig. 9b.

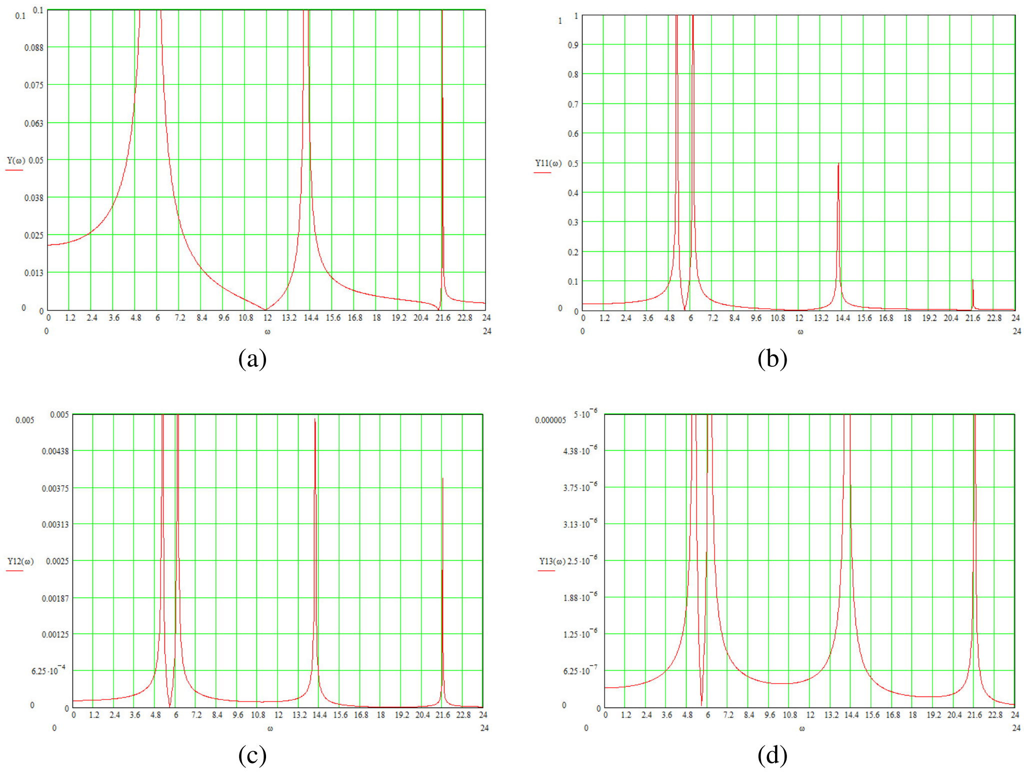

Figure 8: Dynamic characteristics $Y(\omega)$ m/N of the basic system (a) and a system with a damper (b, c, d) as a function of frequency ω rad/s: a) susceptibility of the basic system, b) susceptibility of the system with a damper $Y_{11}(s)$, c) susceptibility of the system with a damper $Y_{12}(s)$, d) susceptibility of the system with a damper $Y_{13}(s)$

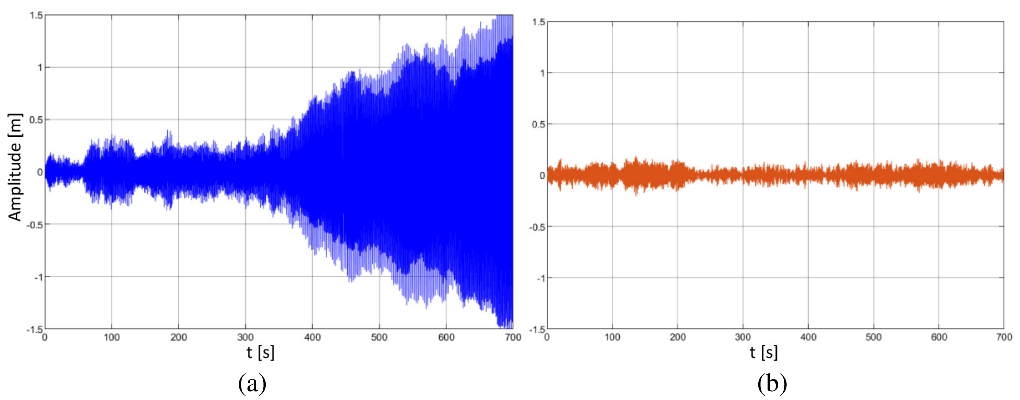

Figure 9: System response to random forcing of the primary system (a) and the system with a TMD (b)

A comparison of the described method (blue line) with the classic method of selecting eliminator parameters (red line) was also carried out (Fig. 10). The functions were determined according to the relationships derived in [14]. In both cases, dynamic characteristics were generated for identical mass and damping factor of the eliminator. The characteristic function of the mobility $Y(\omega)$ m/N, the mechanical impedance $V(\omega)$ m/sN and accelerance inertance $A(\omega)$ m/s^{2}N are shown in Fig. 10.

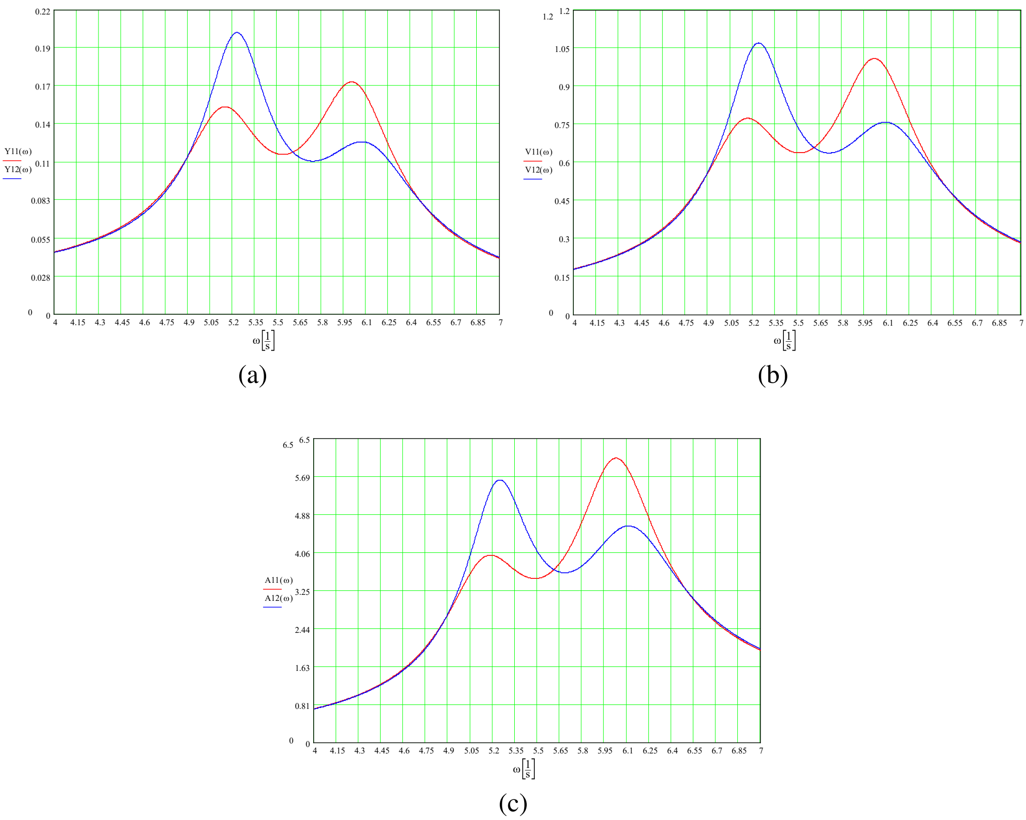

Figure 10: Comparison of TMD tuning methods to the first harmonic $\omega_{zr1} = 5.66$ rad/s: a) the mobility $Y(\omega)$ m/N, b) the mechanical impedance $V(\omega)$ m/sN, c) the accelerance inertance $A(\omega)$ m/s^{2}N

4. Conclusions

The paper aims at formulating and formalising the general method of selecting parameters of a tuned mass damper for vibrating systems with $n \geqslant 2$ degrees of freedom. The parameters of the analysed building were selected so as to present the process of selecting the damper's parameters in a clear way. This does not mean, however, that the considered system cannot have other values of its inertial, elastic or energy dissipating elements. As can be seen from the calculations carried out, the presented method concerns not only the determination of the tuned mass damper reducing the first form of natural vibrations of the analysed building, but also subsequent forms. In the case of discrete systems, this number may be equal to the number of degrees of freedom of the analysed building. Another of the advantages of the formulated method is the reduction of several forms of the building's natural vibrations by attaching to it an additional system with a number of degrees of freedom corresponding to the number of resonant frequencies reduced. In addition, as demonstrated in the paper, the location of the excitation force in the system does not matter in the method used. The damper's parameters are selected so that in each case the tuned mass damper will always reduce the desired frequency. Based on the above, it can be concluded that in the event of an excitation force acting only at the point of attachment of the designated damper, the vibration reduction system can be mounted to any mass, and its values multiplied by any factor whose value does not exceed a value the mass of the entire system. The determined TMD parameters can be successfully applied to any level of the considered system, however, these parameters will not be optimal. In order to adjust the eliminator parameters with regard to its location, consider the input–output characteristics of the system at the TMD connection point and then determine its parameters. The place of attachment of the tuned mass damper, i.e. the inertial element of the basic system, affects the values of the damper's parameters. They are then directly proportional to the mass value and depend on the zeros of the dynamic characteristic. The presented method can be successfully used in conjunction with the currently popular inverters. In the case of reduction of vibrations in tall structures caused by earthquakes or wind gusts (random forcing), the selection of tuned mass damper can be made in the form of an additional system with a number of degrees of freedom equal to the next first significant forms of natural vibrations of the analysed building (structure).

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