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The unique solvability of stationary and non-stationary incompressible melt models in the case of their linearization

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The article presents ε-approximation of hydrodynamics equations' stationary model along with the proof of a theorem about existence of a hydrodynamics equations' strongly generalized solution. It was proved by a theorem on the existence of uniqueness of the hydrodynamics equations' temperature model's solution, taking into account energy dissipation. There was implemented the Galerkin method to study the Navier–Stokes equations, which provides the study of the boundary value problems correctness for an incompressible viscous flow both numerically and analytically. Approximations of stationary and non-stationary models of the hydrodynamics equations were constructed by a system of Cauchy–Kovalevsky equations with a small parameter ε. There was developed an algorithm for numerical modelling of the Navier–Stokes equations by the finite difference method.

Key words: Navier–Stokes equations, hydrodynamic, approximations, mathematical models, incompressible melt

1. Introduction

Numerous hydrodynamic paradoxes point to the long and thorny path that has been covered since its inception. The first long stage was associated with the study and research of ideal incompressible liquid's potential flows. Mathematical methods of their research using the theory of complex variable functions seemed almost perfect. Imperfection of the ideal liquid theory was indicated by the famous Euler-d'Alembert paradox: the total force acting on a body flowing around a potential flow is equal to zero. Then there was created a mathematical model of a viscous incompressible fluid with its basic Navier–Stokes equations. Proposed section outlines various methods for solving and studying the Navier–Stokes

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equations [1–15]. Each considered work offers its own method, but it must be borne in mind that it can be used, perhaps with some necessary modifications.

Asymptotic decomposition and the conditions of their convergence for the class of incompressible viscous liquids under non-standard boundary conditions are researched in the paper [1]. There are introduced generalized elliptic systems of hydrodynamic type's equations, which under certain conditions are transformed into the Navier–Stokes equations. There is proposed a method of the homogenization theory, which allows a numerical study of the eigenvalue problem in nonhomogeneous fields. For the description of nonhomogeneity a large number of holes of εsize is considered. As shown in the paper, at $\varepsilon \to 0$ solution is proved convergent. Tending to zero is the homogenization essence.

There is presented a spectral method of the Navier–Stokes equations numerical integration for an incompressible viscous liquid in the paper [2]. Solution has been decomposed into Chebyshev polynomials for the main flow, and the Fourier transformation is applied for the transverse flow. It is necessary to note the peculiarity of proposed algorithm. It lies in the fact that there is used a special iterative procedure. According to the authors of the paper, proposed algorithm can be used in modeling the internal and external boundary flow's layers. At the same time there is the possibility of viscosity coefficient variation. According to the authors, spectral method is more economical than the well-known finite difference methods.

There are considered some fundamental questions of the incompressible liquid dynamics in the paper [3]. All things considered an interstructural review on the Navier–Stokes equations is made where special attention is paid to computational problems.

Also, there are considered mixed boundary value problems for evolutionary equations in the paper. Various boundary and initial conditions that are used in the calculations are presented.

Viscoelastic theory's distinctive feature, which has received widespread at this time, is a unified liquid and solid states description. In this regard, the article sets and solves the following problems: approximation of stationary and non-stationary models of hydrodynamic equations in order to reduce the nonlinear Navier–Stokes equations to the system of Cauchy–Kovalevsky equations; building finite-difference schemes for Navier–Stokes equations; developing an algorithm for numerical integration of hydrodynamic equations, allowing to predict the technological parameters of metal melt casting.

2. Problem formulation. Nonlinear stationary Navier–Stokes equations

In the article, we establish one of the important aspects of the Navier–Stokes equations' theory: the unique stationary problems' solvability in the case of their linearization. This is most easily done in a Hilbert space with a well-defined extension of the solution concept, which will be described below. The studies conducted in this chapter allow us to state the fact that not only considering problems' unique solvability, but also the possibility of applying approximate methods for finding these solutions, for example, the Galerkin method.

In a limited area $\Omega \subset R^3$ with a smooth border S we consider the following system of nonlinear stationary equations that is a representation of a mathematical model of the incompressible melt's motion:

$$\begin{aligned}(\rho\upsilon\cdot\nabla)\upsilon &= \mu\Delta\upsilon - \nabla p + \lambda(\nabla\rho\cdot\nabla)\upsilon + \lambda(\upsilon\cdot\nabla)\nabla\rho \\ &\quad - \lambda^2 div\left(\left(\frac{1}{\rho}\cdot\nabla\rho\cdot\nabla\right)\rho\right) + \rho f, \qquad (1)\\ (\upsilon\cdot\nabla)\rho &= \lambda\Delta\rho, \qquad (2)\\ div\,\upsilon &= 0, \qquad (3)\end{aligned}$$

with the boundary conditions:

$$\upsilon|_S = 0, \qquad \rho|_S = \rho_S(x), \qquad (4)$$

where $\upsilon(x) = \upsilon(x_1, x_2, x_3)$– velocities' vector function, $\rho(x) = \rho(x_1, x_2, x_3)$ – density field, $p(x) = p(x_1, x_2, x_3)$ – melt pressure field, $f(x) = f(x_1, x_2, x_3)$ – mass force vector, λ, μ – diffusion and viscosity coefficients, and $\lambda > 0$, $\mu > 0$, $S = \partial\Omega$ – sufficiently smooth border area Ω.

Problem's solvability (1)–(4) was researched in the works [4–7]. It is known that the system of equations (1)–(3) is not evolutionary (i.e. it is not a system of Cauchy–Kovalevskaya type), and therefore direct application of numerical methods is difficult.

To solve the difficulty, we will consider another model of an nonhomogeneous melt, which is an approximation of the original model (1)–(4) with a small parameter ε ($\varepsilon > 0$).

So, let's consider the following task:

$$\begin{aligned}(\rho^\varepsilon\upsilon^\varepsilon\cdot\nabla)\upsilon^\varepsilon &= \mu\Delta\upsilon^\varepsilon - \nabla p^\varepsilon + \lambda(\nabla\rho^\varepsilon\cdot\nabla)\upsilon^\varepsilon + \lambda(\upsilon^\varepsilon\cdot\nabla)\nabla\rho^\varepsilon \\ &\quad - \lambda^2 div\left(\left(\frac{1}{\rho^\varepsilon}\cdot\nabla\rho^\varepsilon\cdot\nabla\right)\rho^\varepsilon\right) + \rho^\varepsilon f - \frac{1}{2}\rho^\varepsilon\upsilon^\varepsilon div\,\upsilon^\varepsilon, \qquad (5)\\ (\upsilon^\varepsilon\cdot\nabla)\rho^\varepsilon &= \lambda\Delta\rho^\varepsilon, \qquad (6)\\ \varepsilon p^\varepsilon + div\,\upsilon^\varepsilon &= 0, \qquad (7)\end{aligned}$$

with the boundary conditions:

$$\upsilon^\varepsilon|_S = 0, \qquad \rho^\varepsilon|_S = \rho_S(x). \qquad (8)$$

As it is known, the system of equations (5)–(8) is a system of Cauchy–Kovalevskaya type.

We recall that R^n – Euclidean space; $L_2(\Omega)$ is a Hilbert space; $L_p(\Omega)$, $1 < p \leqslant 6$ is a Banach space; $W_2^1(\Omega)$ is a space consisting of elements $L_2(\Omega)$, having squarely summable over Ω generalized first order derivatives; $W_2^2(\Omega)$ is a space consisting of elements $L_2(\Omega)$, having squarely summable over Ω generalized derivatives of the first and second orders; a space $\overset{0}{W}{}_2^1(\Omega)$ – subspace $W_2^1(\Omega)$ and is the closure of infinitely differentiable finite vector functions' set [6].

Definition 1 *Strongly generalized solution of the problem (5)–(8) is called the set of functions $\{\upsilon^\varepsilon(x), \rho^\varepsilon(x), p^\varepsilon(x)\}$, which satisfies the following conditions:*

1) $\upsilon^\varepsilon(x) \in \overset{0}{W}{}_2^1(\Omega)$, $\rho^\varepsilon(x) \in W_2^{(}\Omega)$, $0 < m \leqslant \rho^\varepsilon(x) \leqslant M < \infty$;

2) $\forall\varphi(x) \in \overset{0}{W}{}_2^1(\Omega)$ *– integral equality is fulfilled:*

$$\int\limits_\Omega \Big\{\rho^\varepsilon(\upsilon^\varepsilon\cdot\nabla)\varphi^\varepsilon\cdot\upsilon^\varepsilon - \mu(\nabla\upsilon^\varepsilon,\nabla\varphi) - \lambda(\nabla\rho^\varepsilon\cdot\nabla)\varphi\cdot\upsilon^\varepsilon$$
$$+\frac{1}{2}\rho^\varepsilon div\,\upsilon^\varepsilon\cdot(\upsilon^\varepsilon\varphi) - \lambda\left(\varphi\cdot\nabla\right)\rho^\varepsilon div\,\upsilon^\varepsilon - \lambda\left(\upsilon^\varepsilon\cdot\nabla\right)\varphi\cdot\nabla\rho^\varepsilon$$
$$+p^\varepsilon div\,\varphi + \lambda^2\left(\left(\frac{1}{\rho^\varepsilon}\cdot\nabla\rho^\varepsilon\cdot\nabla\right)\rho^\varepsilon\right)\nabla\varphi - \rho^\varepsilon f\varphi\Big\}\mathrm{d}x = 0,$$

3) Equations (6), (7) and the boundary conditions (8) are fulfilled almost everywhere in the Ω where possible.

Let's formulate the main result.

Theorem 1 *If $f \in L_{\frac{6}{5}}(\Omega)$, $\rho_S \in W_2^{3/2}(S)$, then with a sufficiently small λ:*

$$\lambda \leqslant \alpha = \min\left\{\frac{M}{16}\frac{m^2}{C_1m^2 + C_2M^2}, \frac{\mu}{M-m}\right\},$$

there exists at least one strongly generalized solution of problems (5)–(8), where C_1, C_2 are constants that depend only on the task data and do not depend on the functions υ^ε, ρ^ε, p^ε.

Proof. The proof of the theorem consists of three stages: obtaining a priori assessments using the Galerkin method and limit transfer.

First, we obtain the necessary priori estimates. We multiply (6) on $\Delta\rho^\varepsilon(x)$ scalar in $L_2(\Omega)$:

$$\lambda\|\Delta\rho^\varepsilon\|^2_{L_2(\Omega)} = \int\limits_\Omega (v^\varepsilon \cdot \Delta)\rho^\varepsilon \cdot \Delta\rho^\varepsilon \cdot \mathrm{d}x. \tag{9}$$

Integration in parts is applicable to the right side:

$$\int\limits_\Omega (v^\varepsilon \cdot \nabla)\,\rho^\varepsilon \cdot \Delta\rho^\varepsilon \mathrm{d}x = -\int\limits_\Omega \nabla\left((v^\varepsilon \cdot \nabla)\rho^\varepsilon\right) \cdot \nabla\rho^\varepsilon \mathrm{d}x + \int\limits_S (v^\varepsilon \cdot \nabla)\,\rho^\varepsilon \frac{\partial\rho^\varepsilon}{\partial n}\mathrm{d}S =$$
$$= -\int\limits_\Omega (\nabla v^\varepsilon \cdot \nabla)\,\rho^\varepsilon \cdot \nabla\rho^\varepsilon \mathrm{d}x - \frac{1}{2}\int\limits_\Omega (v^\varepsilon \cdot \nabla)|\Delta\rho^\varepsilon|^2 \mathrm{d}x \leqslant C\|v^\varepsilon_x\| \cdot \|\nabla\rho^\varepsilon\|^2 \leqslant$$
$$\leqslant C\,\|v^\varepsilon_x\|\left(\|\rho^\varepsilon\| + \max|\rho^\varepsilon| \cdot \|\Delta\rho^\varepsilon\|\right).$$

From Eq. (6) according to the maximum principle we get:

$$\exists m, M:\ 0 < m \leqslant \rho^\varepsilon(x) \leqslant M < \infty,$$

Then we have the following assessment:

$$\int (v^\varepsilon \cdot \nabla)\,\rho^\varepsilon \cdot \Delta\rho^\varepsilon \mathrm{d}x \leqslant \|v^\varepsilon_x\|\left(C_1 + C_2\|\Delta\rho^\varepsilon\|\right) \leqslant \delta\|\Delta\rho^\varepsilon\|^2 + C(\delta)\|v^\varepsilon_x\| + C.$$

We take $\delta = \dfrac{\lambda}{2}$, then from Eq. (9) it follows:

$$\frac{\lambda}{2}\|\Delta\rho^\varepsilon\|^2 \leqslant C(\lambda)\|v^\varepsilon_x\|^2 + C. \tag{10}$$

Now let's multiply Eq. (5) on the function $v^\varepsilon(x)$ scalar in space $C_2(\Omega)$, we get the assessment:

$$\frac{1}{2}\int\limits_\Omega (\rho^\varepsilon v^\varepsilon \cdot \nabla)\,|v^\varepsilon|^2\,\mathrm{d}x + \mu\|v^\varepsilon_x\|^2 = \int\limits_\Omega \rho^\varepsilon f v^\varepsilon \mathrm{d}x + \int\limits_\Omega p^\varepsilon div\, v^\varepsilon \mathrm{d}x +$$
$$+\frac{\lambda}{2}\int\limits_\Omega (\nabla\rho^\varepsilon \cdot \nabla)\,|v^\varepsilon|^2\,\mathrm{d}x + \lambda\int\limits_\Omega (v^\varepsilon \cdot \nabla)\,\nabla\rho^\varepsilon \cdot v^\varepsilon \mathrm{d}x +$$
$$+\lambda^2\int\limits_\Omega \left(\frac{1}{\rho^\varepsilon} \cdot \nabla\rho^\varepsilon \cdot \nabla\right)\rho^\varepsilon \cdot \nabla v^\varepsilon \mathrm{d}x.$$

From above we have:

$$\mu\|\upsilon_x^\varepsilon\|^2_{C_2(\Omega)}+\frac{1}{\varepsilon}\|div\,\upsilon^\varepsilon\|^2_{C_2(\Omega)}=$$
$$=\int\limits_\Omega\left\{-\lambda\left(\upsilon^\varepsilon\cdot\nabla\right)\upsilon^\varepsilon\cdot\nabla\rho^\varepsilon+\lambda\left(\upsilon^2\cdot\nabla\right)\rho^\varepsilon div\,\upsilon^\varepsilon\right\}\mathrm{d}x+$$
$$+\int\limits_\Omega\left\{\lambda^2\left(\left(\frac{1}{\rho^\varepsilon}\cdot\nabla\rho^\varepsilon\cdot\nabla\right)\rho^\varepsilon\right)\cdot\nabla\upsilon^\varepsilon+\rho^\varepsilon f\upsilon^\varepsilon\right\}\mathrm{d}x.$$

Further, by estimating the integral terms in the same way as in [16], we obtain:

$$\mu\|\upsilon_x^\varepsilon\|^2+\frac{1}{\varepsilon}\|div\,\upsilon^\varepsilon\|^2\leqslant\frac{\lambda^2}{m}M\|\Delta\rho^\varepsilon\|\cdot\|\upsilon_x^\varepsilon\|+\frac{\lambda^2}{m}\|\rho^\varepsilon\|\cdot\|\upsilon_x^\varepsilon\|+$$
$$+C_1\|f\|_{L_{\frac{6}{5}}(\Omega)}\cdot\|\upsilon_x^\varepsilon\|+\lambda\frac{M-m}{2}\|\upsilon_x^\varepsilon\|^2.$$

Here we assume that the inequality $\mu-\lambda\frac{M-m}{2}\geqslant\frac{M}{2}$ is true, which implies: $\lambda\leqslant\frac{M-m}{2}$.

By using Jung's inequality repeatedly we obtain:

$$\frac{\mu}{2}\|\upsilon_x^\varepsilon\|^2+\frac{1}{\varepsilon}\|div\,\upsilon^\varepsilon\|^2\leqslant\delta\|\Delta\rho\|^2+\frac{M^2}{m^2}\lambda^4C(\delta)\|\upsilon_x\|^2+\delta_1\|\upsilon_x\|^2+$$
$$+C\left(\lambda,m,M,\delta_I\right)+\frac{\mu}{4}\|\upsilon_x\|^2+C_1(\mu)\|f\|^2_{L_{\frac{6}{5}}}.$$

Let's choose $\delta=\lambda^3$, $\delta_1=\frac{\mu}{8}$ and, taking into account (10), we have:

$$\frac{\mu}{2}\|\upsilon_x^\varepsilon\|^2+\frac{1}{\varepsilon}\|div\,\upsilon^\varepsilon\|\leqslant C\lambda\|\upsilon_x\|^2+\lambda^2\|\upsilon_x\|^2+C_2\frac{M^2}{m^2}+\lambda\|\upsilon_x\|^2+C_1(\mu)\|f\|^2_{L_{\frac{6}{5}}}+C.$$

Further let's assume that the following conditions are held:

$$\gamma=\frac{\mu}{32\lambda^2}\quad\text{and}\quad\gamma\leqslant\frac{\mu}{16}\cdot\frac{m^2}{C_1m^2+C_2M^2}.$$

Then:

$$\frac{\mu}{32}\|\upsilon_x^\varepsilon\|^2+\frac{c}{\varepsilon}\|div\,\upsilon^\varepsilon\|^2\leqslant C_1(\mu)\|f\|^2_{L_{\frac{6}{5}}}+C.$$

Thus, as a result, we obtain the assessment:

$$\|\upsilon_x^\varepsilon\|^2 + \frac{1}{\varepsilon}\|div\,\upsilon^\varepsilon\|^2 \leqslant C < \infty, \tag{11}$$

with enough smallness of γ:

$$\lambda \leqslant \alpha = \min\left\{\frac{\mu}{16}\cdot\frac{m^2}{C_1 m^2 + C_2 M^2}, \frac{\mu}{M-m}\right\}. \tag{12}$$

From the embedding theorems [6] it follows:

$$\upsilon^\varepsilon(x) \in L_p(\Omega), \qquad 1 < p \leqslant 6. \tag{13}$$

And from (10) taking into account (11) it follows:

$$\|\Delta\rho^\varepsilon\|^2 \leqslant C < \infty. \tag{14}$$

By virtue of Eq. (6) we get:

$$\rho^\varepsilon(x) \in L_p(\Omega), \qquad 1 < p \leqslant 6. \tag{15}$$

Further, assessing similarly to [16] the p^ε in the negative form, we have:

$$\|p^\varepsilon\| \leqslant C\,\|\nabla p^\varepsilon\| < \infty. \tag{16}$$

Now let's proceed to the second stage i.e. Galerkin method for constructing approximate solutions.

Let $\{\omega_i\}$ be basis in a space $L_2(\Omega)$ from the problem:

$$\begin{cases} \mu\Delta\omega_i - \nabla p_i = \lambda_i\omega_i\,, \\ \varepsilon p_i + div\,\omega_i = 0, \\ \omega_i|_S = 0. \end{cases} \tag{17}$$

Approximate solution $\upsilon^{N,\varepsilon}$, $\rho^{N,\varepsilon}$, $P^{N,\varepsilon}$ is present in the form:

$$\upsilon^{N,\varepsilon} = \sum_{i=1}^{N} \xi_k^N \omega_k\,, \tag{18}$$

where density and pressure are the classic solution to the problem:

$$\begin{cases} \left(\upsilon^{N,\varepsilon}\cdot\nabla\right)\rho^{N,\varepsilon} = \lambda\Delta\rho^{N,\varepsilon}, \\ \rho^{N,\varepsilon}|_S = \rho_S(x), \end{cases} \tag{19}$$

$$\varepsilon p^{N,\varepsilon} + div\, \upsilon^{N,\varepsilon} = 0. \tag{20}$$

Values of the numbers ξ_k^N are taken from the following system of equations:

$$\Big(\rho^{N,\varepsilon}\left(\upsilon^{N,\varepsilon}\cdot\nabla\right)\upsilon^{N,\varepsilon} - \mu\Delta\upsilon^{N,\varepsilon} - \lambda\left(\nabla\rho^{N,\varepsilon}\cdot\nabla\right)\upsilon^{N,\varepsilon} - \lambda\left(\upsilon^{N,\varepsilon}\cdot\nabla\right)\rho^{N,\varepsilon} +$$
$$+\lambda^2 div\left(\left(\frac{1}{\rho^{N,\varepsilon}}\cdot\nabla\rho^{N,\varepsilon}\cdot\nabla\right)\rho^{N,\varepsilon}\right) - \frac{1}{2}\rho^{N,\varepsilon}\upsilon^{N,\varepsilon} div\,\upsilon^{N,\varepsilon}, \omega_i\Big) = 0, \tag{21}$$
$$i = \overline{1, N}.$$

Using the Brauer Lemma, we prove the existence of a solution to the problems (18)–(21) and show that for approximate solutions $\upsilon^{N,\varepsilon}$, $\rho^{N,\varepsilon}$, $P^{N,\varepsilon}$ a priori estimates (9), (11), (13)–(16) are true. Then from sequences $\{\upsilon^{N,\varepsilon}\}$, $\{\rho^{N,\varepsilon}\}$, $\{p^{N,\varepsilon}\}$ we can identify the subsequences for which the following are true:

$$\begin{array}{llll}
\rho^{N,\varepsilon} \to \rho^{\varepsilon} & *\,\text{weakly} & \text{in the } L_\infty(\Omega), & \\
\dfrac{1}{\rho^{N,\varepsilon}} \to \dfrac{1}{\rho^{\varepsilon}} & *\,\text{weakly} & \text{in the } L_\infty(\Omega), & \\
\rho^{N,\varepsilon} \to \rho^{\varepsilon} & \text{weakly} & \text{in the } W_2^2(\Omega), & \\
\rho^{N,\varepsilon} \to \rho^{\varepsilon} & \text{strongly} & \text{in the } L_p(\Omega), & 1 < p \leqslant 6, \\
\upsilon^{N,\varepsilon} \to \upsilon^{\varepsilon} & \text{weakly} & \text{in the } W_2^1(\Omega), & \\
\upsilon^{N,\varepsilon} \to \upsilon^{\varepsilon} & \text{strongly} & \text{in the } L_p(\Omega), & 1 < p \leqslant 6, \\
p^{N,\varepsilon} \to p^{\varepsilon} & \text{weakly} & \text{in the } L_2(\Omega). &
\end{array}$$

By going to the limit of the selected sequences in the integral identity that is corresponding to the integral identity in Definition 1 and in (19)–(20) we conclude that the limit functions υ^{ε}, ρ^{ε}, p^{ε} are a strongly generalized solution of the problem (5)–(8).

The Theorem 1 is proved. □

Theorem 2 *Let all conditions of Theorem 1 be fulfilled, then the strongly generalized solution of the problem (5)–(8) at $\varepsilon \to 0$ converges to a strongly generalized solution of the problem (1)–(4).*

Proof. By virtue of the obtained necessary prior assessments, we have:

$$\begin{array}{llll}
\dfrac{1}{\rho^{\varepsilon}} \to \dfrac{1}{\rho} & *\,\text{weakly} & \text{in the } L_\infty(\Omega), & \\
\rho^{\varepsilon} \to \rho & *\,\text{weakly} & \text{in the } L_\infty(\Omega), & \\
\rho^{\varepsilon} \to \rho & \text{weakly} & \text{in the } eW_2^2(\Omega), & \\
\rho^{\varepsilon} \to \rho & \text{strongly} & \text{in the } L_p(\Omega), & 1 < p \leqslant 6, \\
\upsilon^{\varepsilon} \to \upsilon & \text{weakly} & \text{in the } W_2^1(\Omega), & \\
\upsilon^{\varepsilon} \to \upsilon & \text{strongly} & \text{in the } L_p(\Omega), & 1 < p \leqslant 6, \\
\varepsilon p^{\varepsilon} \to 0 & \text{strongly} & \text{in the } L_2(\Omega). &
\end{array}$$

By passing to the limit at $\varepsilon \to 0$ in the corresponding identities, it is easy to establish that limit functions υ, p, ρ are a strongly generalized solution of the problem (1)–(4).

The Theorem 2 is proved. □

3. $\varepsilon^2 + \varepsilon h^2$ approximation of the temperature model of nonhomogeneous melts with given energy dissipation

The section presents a study of the initial boundary value problem for the non-stationary Navier–Stokes equations. Let's consider the temperature model of nonhomogeneous melt in the area $\Omega \subset R^2$:

$$\rho\left(\frac{\partial \upsilon}{\partial t} + (\upsilon \cdot \nabla)\,\upsilon\right) = \mu\Delta\upsilon - \nabla p + e\theta\rho + \rho f, \tag{22}$$

$$\frac{\partial \rho}{\partial t} + (\upsilon \cdot \nabla)\,\rho = 0, \tag{23}$$

$$div\,\upsilon = 0, \tag{24}$$

$$\rho\left(\frac{\partial \theta}{\partial t} + (\upsilon \cdot \nabla)\,\theta\right) = div\,(\lambda(\theta)\nabla\theta) + \mu\sigma, \tag{25}$$

$$\sigma = \sum_{i,j=1}^{2}\left(\frac{\partial \upsilon_i}{\partial x_j} + \frac{\partial \upsilon_j}{\partial x_i}\right)^2, \tag{26}$$

with the initial boundary conditions:

$$\begin{gathered} \upsilon|_{t=0} = \upsilon_0(x), \qquad \rho|_{t=0} = \rho_0(x), \qquad \theta|_{t=0} = \theta_0(x), \\ \upsilon|_S = 0, \qquad \left.\frac{\partial \theta}{\partial n}\right|_S = 0, \quad t \in [0, T], \end{gathered} \tag{27}$$

where σ – energy dissipation, $\upsilon(x,t)$ – velocities' vector function, $\theta(x,t)$ – temperature field, $\rho(x,t)$ – density field, $p(x,t)$ – pressure field, $f(x,t)$ – mass force vector, μ – melt viscosity, $\lambda(\theta)$ – thermal conductivity coefficient, n – external normal to the boundary of S, $e = \{0, 1\}$.

(22)–(27) problem's solvability is researched in the work [17].

The system of equations (22)–(26) is non-evolutionary, so the fractional steps method is difficult to apply directly. In this regard, given section unveils the research of the approximation of system (22)–(26) as an evolutionary system type and follows the existence theorem for solving an auxiliary problem. Let the melt move in a limited area $\Omega \subset R^2$ with a fairly smooth border S. For simplicity, we assume that the boundary S is impermeable and mass transfer between the melt and the external environment is absent.

Let's consider a system of equations with a small parameter, approximating the system of equations (22)–(26):

$$\rho^{\varepsilon}\left(\frac{\partial \upsilon^{\varepsilon}}{\partial t}+\left(\upsilon^{\varepsilon}\cdot\nabla\right)\upsilon^{\varepsilon}\right)=\mu\Delta\upsilon^{\varepsilon}-\nabla p^{\varepsilon}+e\theta^{\varepsilon}\rho^{\varepsilon}+\rho^{\varepsilon}f-\rho^{\varepsilon}\frac{\upsilon^{\varepsilon}}{2}div\,\upsilon^{\varepsilon}, \tag{28}$$

$$\frac{\partial \rho^{\varepsilon}}{\partial t}+\left(\upsilon^{\varepsilon}\cdot\nabla\right)\rho^{\varepsilon}=0, \tag{29}$$

$$\varepsilon p^{\varepsilon}+div\,\upsilon^{\varepsilon}=0, \tag{30}$$

$$\rho^{\varepsilon}\left(\frac{\partial \theta^{\varepsilon}}{\partial t}+\left(\upsilon^{\varepsilon}\cdot\nabla\right)\theta^{\varepsilon}\right)=div\left(\lambda\left(\theta^{\varepsilon}\right)\nabla^{\varepsilon}\theta^{\varepsilon}\right)+\mu\sigma^{\varepsilon}, \tag{31}$$

$$\sigma^{\varepsilon}=\sum_{i,j=1}^{2}\left(\frac{\partial \upsilon_i^{\varepsilon}}{\partial x_j}+\frac{\partial \upsilon_j^{\varepsilon}}{\partial x_i}\right)^2, \tag{32}$$

with the initial boundary conditions:

$$\begin{gathered}\upsilon^{\varepsilon}\big|_{t=0}=\upsilon_0(x),\qquad \rho^{\varepsilon}\big|_{t=0}=\rho_0(x),\qquad \theta^{\varepsilon}\big|_{t=0}=\theta_0(x),\\ \upsilon^{\varepsilon}\big|_{S}=0,\qquad \left.\frac{\partial\theta^{\varepsilon}}{\partial n}\right|_{S}=0,\quad t\in[0,T],\end{gathered} \tag{33}$$

Before proceeding to the proof of the theorem, let's formulate an important definition.

Definition 2 *A strong solution of the problem (22)–(27) is called a function $(\upsilon, p, \rho, \theta)$, summed together with derivatives included in the system of equations (22)–(26), that are satisfying (22)–(27) almost everywhere in the possible measure.*

Definition of a problem's strong solution is set similarly (28)–(33).

Theorem 3 *Let $f\in L_p(Q)$, $\Omega\subset E^2$, $\upsilon_0(x)\in W_p^1(\Omega)$, $0<m\leqslant\rho_0(x)\leqslant M<\infty$, $\lambda(\theta)$ be continuously differentiable by θ, $\rho_0(x)\in W_p^1(\Omega)$, $p>2$, $\lambda(\theta)\sim\theta^2$, at $\theta\to\infty$, $\theta_0(x)\in L_\infty(\Omega)$, $\theta_0\in L_p(\Omega)$, $\varepsilon>0$, $S\in C^2$, $\mu>0$.*

Then there is a unique strong solution to the problem (28)–(33) and for the solution the assessment takes place:

$$\left\|\frac{\partial \upsilon^{\varepsilon}}{\partial t}\right\|_{L_p\left(0,T,L_p(\Omega)\right)}+\left\|\upsilon^{\varepsilon}\right\|_{L_p\left(0,T,W_p^2(\Omega)\right)}+\frac{1}{\varepsilon}\left\|div\,\upsilon^{\varepsilon}\right\|_{L_p(0,T,L_p(\Omega))}+$$
$$+\left\|\rho^{\varepsilon}\right\|_{W_p^{1,1}(Q)}+\left\|\theta^{\varepsilon}\right\|_{W_p^{2,1}(Q)}\leqslant C<\infty,$$

where C – constant, independent of ε.

The proof of the theorem consists of three stages: obtaining a priori assessments, applying the Galerkin method for constructing approximate solutions and passing to the limit.

A priori assessments. By virtue of the maximum principle, we have:

$$0 < m \leqslant \rho_0^\varepsilon(x) \leqslant M < \infty.$$

Let's multiply Eq. (28) to $\upsilon^\varepsilon(x,t)$ scalar in the space $L_2(\Omega)$, and integrate the result by parts. By applying Cauchy inequality we have the following:

$$\left|\int\limits_\Omega \rho^\varepsilon\,(f,\upsilon^\varepsilon)\,\mathrm{d}x\right| \leqslant \left(\int\limits_\Omega \rho^\varepsilon\,|\upsilon^\varepsilon|^2\,\mathrm{d}x\right)^{\frac12}\left(\int\limits_\Omega \rho^\varepsilon|f|^2\mathrm{d}x\right)^{\frac12},$$

based on embedding theorems, we have an assessment:

$$\|\upsilon_x^\varepsilon\|_{L_p(0,T,L_p(\Omega))} + \frac{1}{\varepsilon}\,\|div\,\upsilon^\varepsilon\|_{L_p(0,T,L_p(\Omega))} \leqslant C < \infty. \tag{34}$$

Further, by multiplying (31) by θ_t^ε and integrating by area Ω by parts, we get:

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega \lambda\,(\theta^\varepsilon)\,\theta_x^{\varepsilon 2}\mathrm{d}x + \int\limits_\Omega \rho^\varepsilon\theta_t^{\varepsilon 2}\mathrm{d}x = \int\limits_\Omega \frac12\left\{\lambda'\,(\theta^\varepsilon)\,\theta_x^{\varepsilon 2}\cdot\theta_t^\varepsilon\right\}\mathrm{d}x +$$
$$+\int\limits_\Omega \mu\sigma^\varepsilon\theta_t^\varepsilon\mathrm{d}x - \int\limits_\Omega \rho^\varepsilon\,(\upsilon^\varepsilon\cdot\nabla)\,\theta^\varepsilon\cdot\theta_t^\varepsilon\mathrm{d}x.$$

We assess the integrals on the right-hand side and integrate them by variable t:

$$\max_{0\leqslant t\leqslant T}\|\theta_x^\varepsilon\|_{2,\Omega}^2 + \|\theta_t^\varepsilon\|_{2,Q}^2 \leqslant C.$$

Let us write the energy equation:

$$\rho^\varepsilon\theta_t^\varepsilon - (\lambda\,(\theta^\varepsilon)\,\Delta\theta^\varepsilon) = \mu\sigma^\varepsilon - \rho^\varepsilon\,(\upsilon^\varepsilon\cdot\nabla)\,\theta^\varepsilon + \lambda'\,(\theta^\varepsilon)\cdot\theta_x^{\varepsilon 2}$$

and multiply it by $\dfrac{1}{\rho}\Delta\theta^\varepsilon$. After integration by Ω we have:

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega \theta_x^{\varepsilon 2}\mathrm{d}x + \int\limits_\Omega \lambda\,(\theta^\varepsilon)\,\frac{1}{\rho}\,(\Delta\theta^\varepsilon)^2\,\mathrm{d}x =$$
$$= \int\limits_\Omega \left\{\rho\,(\upsilon^\varepsilon\cdot\nabla)\,\theta^\varepsilon - \mu\sigma - \lambda'\,(\theta^\varepsilon)\,\theta_x^{\varepsilon 2}\right\}\cdot\frac{1}{\rho}\Delta\theta^\varepsilon\mathrm{d}x.$$

Assessing the integrals on the right side, after integrating them by variable t, we find:

$$\max_{0\leqslant t\leqslant T} \|\theta_x^\varepsilon\|_{2,\Omega}^2 + \|\Delta\theta_t^\varepsilon\|_{2,Q}^2 \leqslant C.$$

As a result, we conclude:

$$\|\theta^\varepsilon\|_{W_p^{2,1}(Q)} \leqslant C < \infty. \tag{35}$$

By multiplying Eq. (29) by $\Delta\rho^\varepsilon$ and integrating by area Ω by parts, we get:

$$\frac{1}{2}\frac{d}{dt}\|\nabla\rho^\varepsilon\|_{2,\Omega}^2 + \int\limits_\Omega (\upsilon^\varepsilon\cdot\nabla)\,\rho^\varepsilon\Delta\rho^\varepsilon \,\mathrm{d}x = 0.$$

By virtue of the maximum principle, taking into account the assessment (34) it implies:

$$\|\rho^\varepsilon\|_{W_p^{1,1}(Q)} \leqslant C < \infty. \tag{36}$$

Estimating p^ε according to the negative norm, as in the work [16], we have:

$$\|p^\varepsilon\| \leqslant C\,\|\nabla p^\varepsilon\| < \infty. \tag{37}$$

It is known that if υ^ε, p^ε are solution of the following Stokes linear problem:

$$\begin{aligned} \mu\Delta\upsilon^\varepsilon - \nabla p^\varepsilon &= f,\\ \varepsilon p^\varepsilon + div\,\upsilon^\varepsilon &= 0,\\ \upsilon^\varepsilon|_{S=0} = 0, \qquad \left.\frac{\partial\rho^\varepsilon}{\partial n}\right|_S &= 0, \end{aligned} \tag{38}$$

then under condition that $f\in L_p(\Omega)$ the following inequality holds:

$$\|\upsilon^\varepsilon\|_{W_p^2\cap \overset{0}{W}{}_p^1} + \|p^\varepsilon\|_{W_p^1} \leqslant C\|f\|_{L_p}. \tag{39}$$

As a function f in the problem (38) we take the function:

$$f = -\rho^\varepsilon\left(\frac{\partial\upsilon^\varepsilon}{\partial t} + (\upsilon^\varepsilon\cdot\nabla)\,\upsilon^\varepsilon\right) - \ell\theta^\varepsilon\rho^\varepsilon - \rho^\varepsilon f - \rho^\varepsilon\frac{\upsilon^\varepsilon}{2}div\,\upsilon^\varepsilon.$$

Let's assess the right-hand side according to the Cauchy inequality by using the maximum principle:

$$\|f\|_{L_p(\Omega)}^2 \leqslant CM\left(\|\upsilon_t^\varepsilon\|^2 + \int\limits_\Omega \left(|\upsilon^\varepsilon|^2\,|\nabla\upsilon^\varepsilon|^2 + |f|^2 + \ell\,|\theta^\varepsilon|^2\right)\mathrm{d}x\right). \tag{40}$$

With taking into account the inequality of embedding and the assessments obtained from (34), (35), we have the following estimate:

$$\begin{aligned}\int\limits_{\Omega} |\upsilon^{\varepsilon}|^2 \, |\nabla \upsilon^{\varepsilon}|^2 \, \mathrm{d}x \leqslant \max_{\Omega} |\upsilon^{\varepsilon}|^2 \int\limits_{\Omega} |\nabla \upsilon^{\varepsilon}|^2 \, \mathrm{d}x \leqslant \\ \leqslant \|\upsilon^{\varepsilon}\|_{L_p(\Omega)} \, \|\upsilon^{\varepsilon}\|_{W_p^2(\Omega)\cap \overset{0}{W}{}_p^1(\Omega)} \, \|\nabla \upsilon^{\varepsilon}\|^2 \leqslant \\ \leqslant \delta \, \|\upsilon^{\varepsilon}\|^2_{W_p^2(\Omega)\cap \overset{0}{W}{}_p^1(\Omega)} + C_\delta \, \|\upsilon^{\varepsilon}\|^2_{\overset{0}{W}{}_p^1(\Omega)} .\end{aligned} \tag{41}$$

Let's multiply Eq. (28) by scalar $\upsilon^{\varepsilon}(t)$ in the space $L_2(\Omega)$, then assess the integrals from above in absolute value and by applying inequality of embedding theorems, we obtain following estimate:

$$\begin{aligned}\int\limits_{\Omega} |\nabla \upsilon^{\varepsilon}| \, |\upsilon^{\varepsilon}| \, |\upsilon^{\varepsilon}_t| \, \mathrm{d}x \leqslant C \, \|\nabla \upsilon^{\varepsilon}\|_{L_p(\Omega)} \max_{\Omega} |\upsilon^{\varepsilon}| \, \|\upsilon^{\varepsilon}_t\|_{L_p(\Omega)} \leqslant \\ \leqslant C \, \|\upsilon^{\varepsilon}\|^{\frac{1}{2}}_{L_p(\Omega)} \, \|\upsilon^{\varepsilon}\|^{\frac{1}{2}}_{W_p^2(\Omega)\cap \overset{0}{W}{}_p^1(\Omega)} \, \|\upsilon^{\varepsilon}\|_{\overset{0}{W}{}_p^1(\Omega)} \, \|\upsilon^{\varepsilon}_t\|_{L_p(\Omega)} \leqslant \\ \leqslant \delta \, \|\upsilon^{\varepsilon}_t\|^2_{L_p(\Omega)} + \delta \, \|\upsilon^{\varepsilon}\|^2_{W_p^2(\Omega)\cap \overset{0}{W}{}_p^1(\Omega)} + C_\delta \, \|\upsilon^{\varepsilon}\|^2_{\overset{0}{W}{}_p^1(\Omega)} .\end{aligned} \tag{42}$$

By following the method of assessment from the work [18], at the end we get an assessment:

$$\|\upsilon^{\varepsilon}_t\|_{L_p(0,T,L_p(\Omega))} + \|\upsilon^{\varepsilon}\|_{L_p(0,T,W_p^2(\Omega))} + \|\nabla p^{\varepsilon}\|_{L_p(0,T,L_p(\Omega))} \leqslant C < \infty, \tag{43}$$

where C does not depend on the small parameter value ε.

Let us establish one more assessment by a global time feature that is a constant which only depends on the problem's data. Further this assessment guarantees compactness in the space $L_2(Q)$ of sequences of approximate solutions that are constructed according to the Galerkin method.

Lemma 1 *For any δ, such that the condition is fulfilled $0 < \delta < T$, the following inequality is true:*

$$\int\limits_0^{T-\delta} \|\upsilon^{\varepsilon}(t+\delta) - \upsilon^{\varepsilon}(t)\|^2 \, \mathrm{d}t \leqslant C\delta^{\frac{1}{2}}.$$

Proof. Let us fix δ, t so that inequality held $0 \leqslant t \leqslant T - \delta$. Equations (28)–(32) on the time interval $\tau \in (t, t+\delta)$. Let's multiply Ee. (28) by scalar on an arbitrary function Φ in space $L_2(\Omega)$.

Then, after simple transformations, we arrive at the inequality:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\rho^{\varepsilon}\upsilon^{\varepsilon},\Phi\right)_{L_2(\Omega)} = \left(\rho^{\varepsilon}\left(\upsilon^{\varepsilon}\cdot\nabla\right)\Phi,\upsilon^{\varepsilon}\right)_{L_2(\Omega)} + \frac{1}{2}\left(\rho^{\varepsilon} div\,\upsilon^{\varepsilon}\cdot\upsilon^{\varepsilon},\Phi\right)_{L_2(\Omega)} + \\ + \left(\rho^{\varepsilon} f,\Phi\right)_{L_2(\Omega)} - \ell\left(\theta^{\varepsilon}\rho^{\varepsilon},\Phi\right)_{L_2(\Omega)} - \mu\left(\upsilon^{\varepsilon},\Phi\right)_{L_2(\Omega)} + \left(p^{\varepsilon}, div\,\Phi\right)_{L_2(\Omega)},$$

where $\Phi = \upsilon(t+\delta) - \upsilon(t)$.

Let's integrate obtained identity by a variable τ in the limits from t to $t+\delta$, and then put $\Phi = \upsilon^{\varepsilon}(t+\delta) - \upsilon^{\varepsilon}(t)$.

An expression $\rho^{\varepsilon}(t+\delta)\upsilon^{\varepsilon}(t+\delta) - \rho^{\varepsilon}(t)\upsilon^{\varepsilon}(t)$ we can write as follows $\rho^{\varepsilon}(t+\delta)\left(\upsilon^{\varepsilon}(t+\delta) - \upsilon^{\varepsilon}(t)\right) + \left(\rho^{\varepsilon}(t+\delta) - \rho^{\varepsilon}(t)\right)\upsilon^{\varepsilon}(t)$, and then the difference between $\rho^{\varepsilon}(t+\delta) - \rho^{\varepsilon}(t)$ was found by integrating Eq. (29) in the limits from t to $t+\delta$. Obtained ratio we will integrate by a variable t from 0 to $t+\delta$, and for each term on the right-hand side, we can use the assessments from the work [18], on the basis of which we derive the assessment of the Lemma 1.

Let us proceed to the construction of approximate solutions by the Galerkin method [16, 19].

Let $\Omega_{1\alpha} = u_{\alpha}\dfrac{\partial}{\partial x_{\alpha}}$I, be an orthonormal basis in the space $L_2(\Omega)$ of $W_p^2(\Omega) \cap \overset{0}{W}{}_p^1(\Omega)$. And the following ratio to be true:

$$\left(\phi_{jx},\omega_x\right)_{L_2(\Omega)} = \lambda_j\left(\phi_j,\omega_j\right)_{L_2(\Omega)}.$$

Approximated solution $\upsilon^{N,\varepsilon}(t)$ we will look for in the form of:

$$\upsilon^{N,\varepsilon}(t) = \sum_{k=1}^{N} C_k^N(t)\phi_k,$$

where $C_k^N(t) \in C^1[0,T]$.

Density $\rho^{N,\varepsilon}(t)$ is a classic solution to the problem:

$$\frac{\partial\rho^{N,\varepsilon}(t)}{\partial t} + \left(\upsilon^{N,\varepsilon}(t)\cdot\nabla\right)\rho^{N,\varepsilon}(t) = 0, \\ \rho^{N,\varepsilon}|_{t=0} = \rho_0^M(x), \tag{44}$$

where $\rho_0^M(x)$ is a smooth initial function.

The sequence $\rho_0^M(x)$, $M = 1,2,\ldots$ converges to the $\rho_0(x)$ in norms $L_p(\Omega)$, $W_p^1(\Omega)$, $\rho_0^M(x) \in C^2(\Omega)$. The pressure $p^{N,\varepsilon}(t)$ is a classic solution to the problem:

$$div\,\upsilon^{N,\varepsilon} = \varepsilon p^{N,\varepsilon}, \\ \int\limits_{\Omega} p^{N,\varepsilon}\,\mathrm{d}x = 0. \tag{45}$$

The temperature $\theta^{N,\varepsilon}(t)$ is defined as a classic solution to the problem:

$$\rho^{N,\varepsilon}\left(\frac{\partial\theta^{N,\varepsilon}(t)}{\partial t}+\left(\upsilon^{N,\varepsilon}(t)\cdot\nabla\right)\theta^{N,\varepsilon}(t)\right)=div\left(\lambda\left(\theta^{N,\varepsilon}(t)\right)\nabla\theta^{N,\varepsilon}(t)\right)+\mu\sigma^{N,\varepsilon},$$
$$\theta^{N,\varepsilon}|_{S=0}=\theta_0^M(x),\qquad \left.\frac{\partial\theta^{N,\varepsilon}(t)}{\partial n}\right|_S=0,\quad t\in[0,\,T], \tag{46}$$

where $\theta_0^M(x)-$ initial smooth function satisfying to the equation:

$$\left.\frac{\partial\theta_0^M(t)}{\partial n}\right|_S=0,\quad t\in[0,\,T].$$

Functions $C_k^N(t)$, $k=1,2,\ldots,N$, are determined by a system of ordinary differential equations with coefficients that are operably dependent on $\rho^{N,\varepsilon}(t)$, $p^{N,\varepsilon}(t)$:

$$\left(\rho^{N,\varepsilon}(t)\left(\frac{\partial\upsilon^{N,\varepsilon}(t)}{\partial t}+\left(\upsilon^{N,\varepsilon}(t)\cdot\nabla\right)\upsilon^{N,\varepsilon}(t)+\frac{1}{2}\upsilon^{N,\varepsilon}(t)div\,\upsilon^{N,\varepsilon}\right)-\mu\Delta\upsilon^{N,\varepsilon}(t)+\right.$$
$$\left.+\nabla p^{N,\varepsilon}(t)-\ell\theta^{N,\varepsilon}(t)\rho^{N,\varepsilon}(t)+\rho^{N,\varepsilon}(t)f,\phi_j\right)_{L_2(\Omega)}=0.$$

Based on the Schauder principle, using the obtained a priori assessments, there can be distinguished subsequences from sequences $\left\{\upsilon^{N,\varepsilon}\right\}$, $\left\{\rho^{N,\varepsilon}\right\}$, $\left\{p^{N,\varepsilon}\right\}$, $\left\{\theta^{N,\varepsilon}\right\}$ for which we have:

$$\begin{array}{llll}
\upsilon^{N,\varepsilon}\to\upsilon^{\varepsilon} & \text{weakly} & \text{in the} & L_p\left(0,T,W_p^2(\Omega)\right),\\
\theta^{N,\varepsilon}\to\theta^{\varepsilon} & \text{weakly} & \text{in the} & W_p^{2,1}(Q),\\
\rho^{N,\varepsilon}\to\rho^{\varepsilon} & *\,\text{weakly} & \text{in the} & W_p^{1,1}(Q),\\
\upsilon^{N,\varepsilon}\to\upsilon^{\varepsilon} & \text{strongly} & \text{in the} & L_p\left(0,T,L_p(\Omega)\right),\\
\theta^{N,\varepsilon}\to\theta^{\varepsilon} & \text{strongly} & \text{in the} & L_p\left(0,T,L_p(\Omega)\right),\\
\upsilon_t^{N,\varepsilon}\to\upsilon_t^{\varepsilon} & \text{weakly} & \text{in the} & L_p\left(0,T,L_p(\Omega)\right),\\
p^{N,\varepsilon}\to p^{\varepsilon} & \text{weakly} & \text{in the} & L_p\left(0,T,W_p^1(\Omega)\right).
\end{array}$$

Thus, the Theorem 3 is proved. □

The following is true.

Theorem 4 *Let all conditions of the Theorem 3 be fulfilled. Then the (28)–(33) problem's strong solution converges to a (22)–(27) problem's strong solution at* $\varepsilon\to 0$.

Proof. By virtue of the prior assessments obtained earlier, we have:

$$\begin{array}{llll}
\upsilon^\varepsilon \to \upsilon & \text{weakly} & \text{in the} & L_p\left(0, T, W_p^2(\Omega)\right), \\
\theta^\varepsilon \to \theta & \text{weakly} & \text{in the} & W_p^{2,1}(Q), \\
\rho^\varepsilon \to \rho & *\,\text{weakly} & \text{in the} & W_p^{1,1}(Q), \\
\upsilon^\varepsilon \to \upsilon & \text{strongly} & \text{in the} & L_p\left(0, T, L_p(\Omega)\right), \\
\theta^\varepsilon \to \theta & \text{strongly} & \text{in the} & L_p\left(0, T, L_p(\Omega)\right), \\
\upsilon_t^\varepsilon \to \upsilon_t & \text{weakly} & \text{in the} & L_p\left(0, T, L_p(\Omega)\right), \\
p^\varepsilon \to p & \text{weakly} & \text{in the} & L_p\left(0, T, W_p^1(\Omega)\right).
\end{array}$$

By going to the limit at $\varepsilon \to 0$ in the corresponding identities, we establish that the limit functions υ, p, ρ, θ are a (22)–(27) problem's strong solution.

The Theorem 4 is proved. □

4. Finite difference method implementation for the numerical solution hydrodynamic equations melts

There is needed a hydrodynamic equations' numerical solution implemented by finite difference method for computer modeling of melt's flow. One way of implementing numerical solutions is described in seections 2 and 3. We have developed specific algorithms for computer programming.

Let's consider a flat flow. Let Ω be an area of Euclidean space R^n, and $x = (x_1, x_2)$. We divide whole space $R^n(x,t)$ on elementary cells, the area of which will be equal to following:

$$x_i = k_i h, \qquad h > 0, \qquad k_i = 0, \pm 1, \pm 2, \ldots,$$

where $t = k\Delta t$; $k = 1, 2, \ldots, n$; h is a step.

Let's form difference ratios by x_i:

$$\upsilon_{x_i}(x,t) = \frac{1}{h}\left[\upsilon\left(x + he^j, t\right) - \upsilon(x,t)\right], \quad \upsilon_{\bar{x}_i}(x,t) = \frac{1}{h}\left[\upsilon(x,t) - \upsilon\left(x - he^j, t\right)\right].$$

The shift by x_i is defined as:

$$\overset{\pm i}{\upsilon}(x,t) = \upsilon\left(x \pm he^j, t\right).$$

Vectors e^j are the unit vectors along the axes x_i by itself. According to the work [6] velocity vectors are expressed as the ratios:

$$\upsilon_h^2 = \sum_{i=1}^n \upsilon_{ih}\upsilon_{ih}, \qquad \upsilon_{hx}^2 = \sum_{k=1}^n \upsilon_{hx_k}^2 = \sum_{i,k=1}^n \left(\upsilon_{ihx_k}\right)^2,$$

$$\upsilon_{h\bar{x}}^2 = \sum_{k=1}^{n} \upsilon_{h\bar{x}_k}^2 = \sum_{i,k=1}^{n} (\upsilon_{ih\bar{x}_k})^2 .$$

Summation over i, k are conducted from 1 to 2 for two-dimensional case, and from 1 to 3 for three-dimensional case. Then for arbitrary functions u_h, υ_h given on a lattice, we get the following expressions:

$$(u_h\upsilon_h)_{x_i} = u_{hx_i}\upsilon_h + \overset{+i}{u_h}\, \upsilon_{hx_i} = u_{hx_i}\, \overset{+i}{\upsilon_h} + u_h\upsilon_{hx_i}, \tag{47}$$

$$(u_h\upsilon_h)_{\bar{x}_i} = u_{h\bar{x}_i}\upsilon_h + \overset{-i}{u_h}\, \upsilon_{h\bar{x}_i} = u_{h\bar{x}_i}\, \overset{-i}{\upsilon_h} + u_h\upsilon_{h\bar{x}_i}, \tag{48}$$

$$u_{\bar{t}}^k u^k = \frac{\left(u^k\right)^2 - \left(u^{k-1}\right)^2 + (\Delta t)^2 \left(u_{\bar{t}}^k\right)^2}{2\Delta t}, \tag{49}$$

$$\sum_{\ell=0}^{m-1} u_{hx}(\ell)\upsilon_h(\ell) = \frac{-h\displaystyle\sum_{\ell=1}^{m} u_h(\ell)\upsilon_{h\bar{x}}(\ell) + u_h(m)\upsilon_h(m) + u_h(0)\upsilon_h(0)}{h} . \tag{50}$$

Above it is assumed that:

$$u_{\bar{t}}^k = \frac{1}{\Delta t}\left(u^k - u^{k-1}\right), \qquad u_{hx}(\ell) = \frac{1}{h}\left[u_h(\ell+1) - u_h(\ell)\right],$$
$$u_{h\bar{x}}(\ell) = \frac{1}{h}\left[u_h(\ell) - u_h(\ell-1)\right].$$

Thus, formulas (47)–(50) are a difference analogues of the product differentiation and interpolation formulas u_h, given on a lattice. To demonstrate given method after appropriate transformations, let's rewrite Eq. (22) in the form:

$$\frac{\partial \upsilon}{\partial t} + \sum_{k=1}^{n} Z_k(\upsilon) - \frac{1}{\varepsilon}\nabla div\, \upsilon = f, \tag{51}$$

where:

$$Z_k(w) = -\gamma\frac{\partial^2 w}{\partial x_k^2} + \upsilon_k\frac{\partial w}{\partial x_k} + \frac{1}{2}\frac{\partial \upsilon_k}{\partial x_k}w.$$

For simplicity, let's consider the case when $n = 2$. In order to do this, it is obvious that we need to divide time interval $[0, T]$ pointwise:

$$t_m = m\Delta t, \qquad t_{m-1/2} = \left(m - \frac{1}{2}\right)\Delta t,$$

where $m = 1, 2, \ldots, N$.

The above will allow us to consider layers t_m and t_{m-1}. Denote the indices so that υ, and also p are pointed to the layer's number on which they are calculated. There are various approximations of difference operator Z_k. Let's take he operator in the form proposed in [20, 21]:

$$Z_k^m(w) = -\gamma w_{x_k\bar{x}_k} + \frac{1}{2}\overset{+k}{\upsilon}{}_k^{m-\frac{k}{2}} w_{x_k} + \frac{1}{2}\upsilon_k^{m-\frac{k}{2}} w_{\bar{x}_k} + \frac{1}{2}\upsilon_{kx_k}^{m-\frac{k}{2}} w_{x_k}.$$

Then Eq. (51) can be represented by the following difference scheme:

$$\frac{1}{\Delta t}\left(\upsilon_1^{m-\frac{1}{2}} - \upsilon_1^{m-1}\right) + \tau_2^m\left(\upsilon_1^{m-\frac{1}{2}}\right) = \frac{1}{2}f_1^{m-\frac{1}{2}}, \tag{52}$$

$$\frac{1}{\Delta t}\left(\upsilon_1^m - \upsilon_1^{m-\frac{1}{2}}\right) + \tau_1^m\left(\upsilon_1^m\right) - \frac{1}{\varepsilon}\left(\upsilon_{1x_1}^m + \upsilon_{2x_2}^{m-\frac{1}{2}}\right)_{\bar{x}_1} = \frac{1}{2}f_1^m, \tag{53}$$

$$\frac{1}{\Delta t}\left(\upsilon_2^{m-\frac{1}{2}} - \upsilon_2^m\right) + \tau_2^m\left(\upsilon_2^{m-\frac{1}{2}}\right) - \frac{1}{\varepsilon}\left(\upsilon_{1x_1}^{m-1} + \upsilon_{2x_2}^{m-\frac{1}{2}}\right)_{\bar{x}_2} = \frac{1}{2}f_2^{m-\frac{1}{2}}, \tag{54}$$

$$\frac{1}{\Delta t}\left(\upsilon_2^m - \upsilon_2^{m-\frac{1}{2}}\right) + \tau_1^m\left(\upsilon_2^m\right) = \frac{1}{2}f_2^m, \tag{55}$$

where $m = 1, 2, \ldots, N$.

To complete construction of the difference scheme, the initial and boundary conditions should be added to presented equations. Without deriving formulas for boundary conditions, we will deduce:

$$\left\|\upsilon_1^{m-\frac{1}{2}}\right\|^2 - \left\|\upsilon_1^{m-1}\right\|^2 + \left\|\upsilon_1^{m-\frac{1}{2}} - \upsilon_1^{m-1}\right\|^2 + 2\gamma\Delta t\left\|\upsilon_{1x_2}^{m-\frac{1}{2}}\right\|^2 = \Delta t\left(f_1^{m-\frac{1}{2}}, \upsilon_1^{m-\frac{1}{2}}\right), \tag{56}$$

$$\begin{aligned}\left\|\upsilon_1^m\right\|^2 - \left\|\upsilon_1^{m-\frac{1}{2}}\right\|^2 + \left\|\upsilon_1^m - \upsilon_1^{m-\frac{1}{2}}\right\|^2 + 2\gamma\Delta t\left\|\upsilon_{1x_1}^m\right\|^2 \\ + \frac{2\Delta t}{\varepsilon}\left[\left\|\upsilon_{1x_1}^m\right\|^2 + \left(\upsilon_{1x_1}^m, \upsilon_{2x_2}^{m-\frac{1}{2}}\right)\right] = \Delta t\left(f_1^m, \upsilon_1^m\right),\end{aligned} \tag{57}$$

$$\begin{aligned}\left\|\upsilon_2^{m-\frac{1}{2}}\right\|^2 - \left\|\upsilon_2^{m-1}\right\|^2 + \left\|\upsilon_2^{m-\frac{1}{2}} - \upsilon_2^{m-1}\right\|^2 + 2\gamma\Delta t\left\|\upsilon_{2x_2}^{m-\frac{1}{2}}\right\|^2 \\ + \frac{2\Delta t}{\varepsilon}\left[\left\|\upsilon_{2x_2}^{m-\frac{1}{2}}\right\|^2 + \left(\upsilon_{1x_1}^{m-1}, \upsilon_{2x_2}^{m-\frac{1}{2}}\right)\right] = \Delta t\left(f_2^{m-\frac{1}{2}}, \upsilon_2^{m-\frac{1}{2}}\right),\end{aligned} \tag{58}$$

$$\left\|\upsilon_2^m\right\|^2 - \left\|\upsilon_2^{m-\frac{1}{2}}\right\|^2 + \left\|\upsilon_2^m - \upsilon_2^{m-\frac{1}{2}}\right\|^2 + 2\gamma\Delta t\left\|\upsilon_{2x_1}^m\right\|^2 = \Delta t\left(f_2^m, \upsilon_2^m\right). \tag{59}$$

Thus, we will obtain Eqs. (52)–(55) that are solved separately. The result allows to write machine programs for the numerical finite-difference methods implementation.

At this point it is suitable to consider application of proposed method on Dirichlet problem's example for the Poisson equation given in [22]. Integration is performed in a rectangular lattice in accordance with Fig. 1. Asterisk indicates internal nodes, boundary nodes are denoted by ∘.

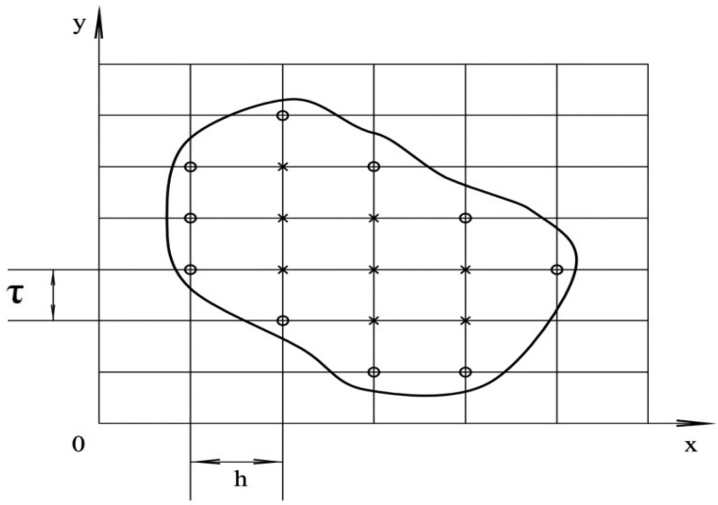

Figure 1: Integration area

According to the reference data, the solution of the Poisson equation is given in Table 1.

Table 1: First Dirichlet boundary value problem's solution for the Poisson equation from reference sources

Y	X					
	0.00	0.40	0.80	1.20	1.60	2.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.20	0.08	0.32	0.51	0.72	0.99	0.84
0.40	0.32	0.72	1.07	1.41	1.78	1.76
0.60	0.72	1.23	1.68	2.12	2.56	2.76
0.80	1.28	1.82	2.65	3.22	3.82	3.84
1.00	2.00	2.44	2.96	3.56	4.24	5.00

For the control example in Table 2 we gave the Dirichlet problem's solution already with different boundary conditions from same reference sources. By comparing first and second Dirichlet boundary value problems' solutions from reference sources presented in Tables 1 and 2 with program results for solving boundary value problems presented in Tables 3 and 4, we obtain a satisfactory coincidence of solutions for a given accuracy $\varepsilon = 10^{-1}$.

Table 2: Second Dirichlet boundary value problem's solution for the Poisson equation from reference sources

Y	X					
	0.00	0.40	0.80	1.20	1.60	2.00
0.00	1.00	1.40	1.80	2.20	2.60	3.00
0.20	2.00	1.05	0.95	1.08	1.44	2.96
0.40	2.00	1.02	0.60	0.59	0.93	2.84
0.60	4.00	1.36	0.78	0.63	0.93	2.64
0.80	5.00	2.78	2.12	1.81	1.64	2.36
1.00	6.00	5.84	5.36	4.56	3.44	2.00

Table 3: First Dirichlet boundary value problem's solution for the Poisson equation with a given accuracy $\varepsilon = 10^{-1}$.

Y	X										
	0.000	0.200	0.400	0.600	0.800	1.000	1.200	1.400	1.600	1.800	2.000
0.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.2	0.080	0.241	0.262	0.264	0.266	0.269	0.272	0.276	0.280	0.330	0.840
0.4	0.320	0.303	0.301	0.303	0.305	0.308	0.311	0.315	0.320	0.447	1.760
0.6	0.720	0.356	0.305	0.306	0.308	0.310	0.314	0.318	0.323	0.538	2.760
0.8	1.280	0.429	0.310	0.308	0.310	0.313	0.316	0.320	0.325	0.636	3.840
1.0	2.000	0.523	0.315	0.311	0.313	0.315	0.319	0.323	0.329	0.741	5.000
1.2	2.880	0.639	0.322	0.314	0.316	0.319	0.322	0.326	0.332	0.854	6.240
1.4	3.920	0.776	0.330	0.318	0.320	0.323	0.326	0.330	0.337	0.974	7.560
1.6	5.120	0.935	0.341	0.323	0.326	0.329	0.332	0.336	0.343	1.105	8.960
1.8	6.480	1.248	0.581	0.613	0.674	0.744	0.821	0.906	1.002	1.946	10.44
2.0	2.000	2.440	2.960	3.560	4.240	5.000	5.840	6.760	7.760	8.840	10.00

And when the accuracy increases to $\varepsilon = 10^{-4}$ our results presented in Table 5, in fact, coincide with the standard reference data results.

Table 4: Second Dirichlet boundary value problem's solution for the Poisson equation with a given accuracy $\varepsilon = 10^{-1}$

Y	X										
	0.000	0.200	0.400	0.600	0.800	1.000	1.200	1.400	1.600	1.800	2.000
0.0	1.000	1.200	1.400	1.600	1.800	2.000	2.200	2.400	2.600	2.800	3.000
0.2	2.000	1.109	1.011	1.012	1.016	1.019	1.023	1.026	1.029	1.138	2.960
0.4	3.000	1.215	1.004	0.997	0.996	0.994	0.992	0.990	0.988	1.087	2.840
0.6	4.000	1.325	1.007	0.997	0.995	0.993	0.991	0.988	0.985	1.074	2.640
0.8	5.000	1.436	1.009	0.996	0.994	0.992	0.989	0.986	0.983	1.057	2.360
1.0	6.000	1.546	1.012	0.995	0.993	0.990	0.987	0.984	0.981	1.035	2.000
1.2	7.000	1.656	1.015	0.994	0.992	0.989	0.986	0.983	0.979	1.008	1.560
1.4	8.000	1.767	1.017	0.994	0.991	0.988	0.984	0.981	0.977	0.977	1.040
1.6	9.000	1.877	1.020	0.993	0.990	0.986	0.983	0.979	0.975	0.942	0.440
1.8	10.00	2.018	1.059	1.027	1.022	1.016	1.009	1.000	0.992	0.917	-0.24
2.0	6.000	5.960	5.840	5.640	5.360	5.000	4.560	4.040	3.440	2.760	2.000

Table 5: First Dirichlet boundary value problem's solution for the Poisson equation with a given accuracy $\varepsilon = 10^{-4}$

Y	X					
	0.000	0.400	0.800	1.200	1.600	2.000
0.00	0.000	0.000	0.000	0.000	0.000	0.000
0.20	0.080	0.301	0.508	0.750	1.001	0.800
0.40	0.320	0.730	1.055	1.430	1.851	1.710
0.60	0.720	1.221	1.666	2.101	2.590	2.732
0.80	1.280	1.790	2.599	3.202	3.798	3.884
1.00	2.000	2.490	2.981	3.549	4.290	5.001

Obtained results show compiled program's correctness, as well as correctness of the stated boundary value problems for hydrodynamic equations that were considered by us above.

For clarity, let's present isolines and surfaces of solutions obtained in correspondence with Fig. 2 and Fig. 3.

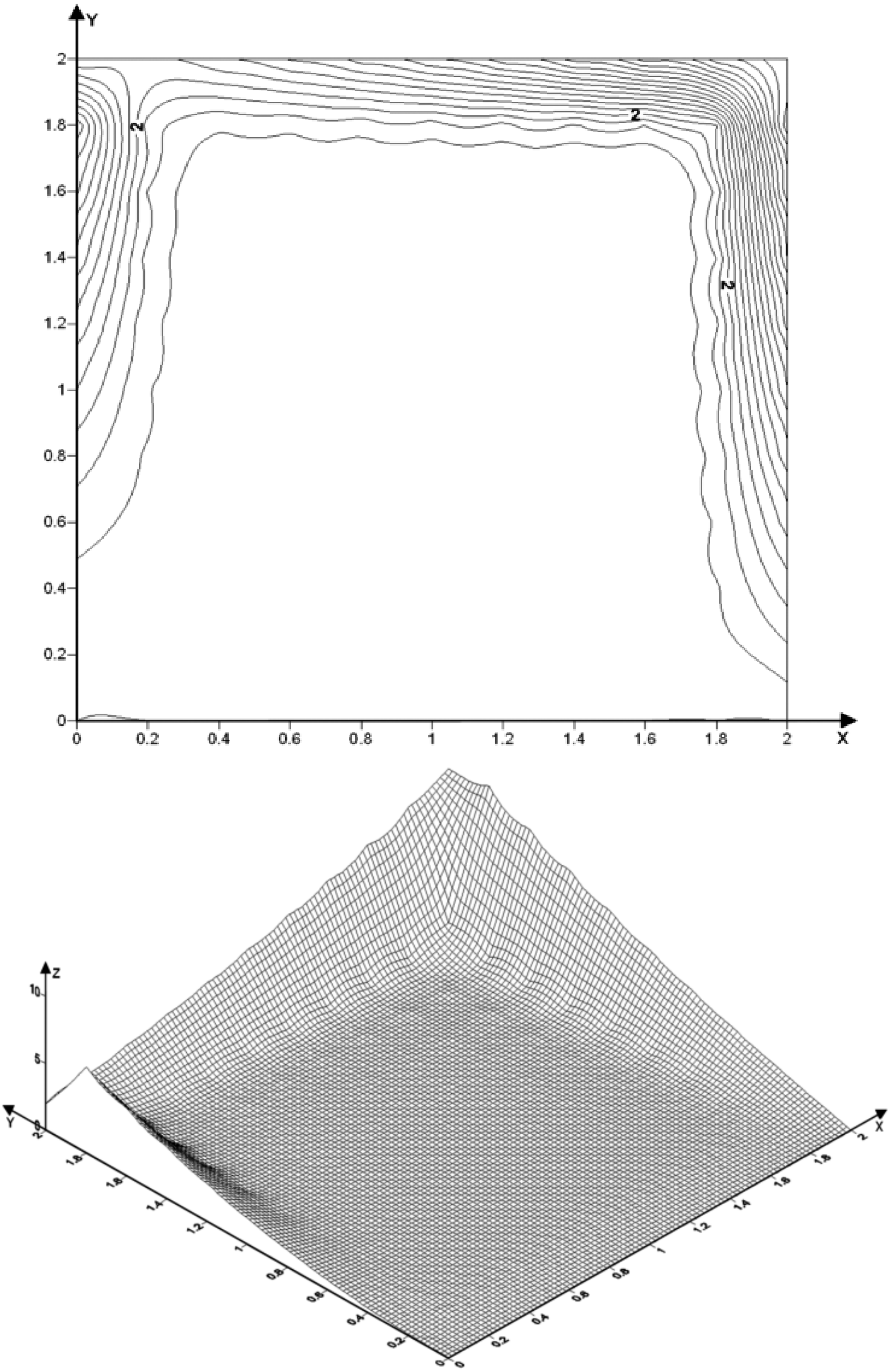

Figure 2: Isolines and surface for the first Dirichlet boundary value problem

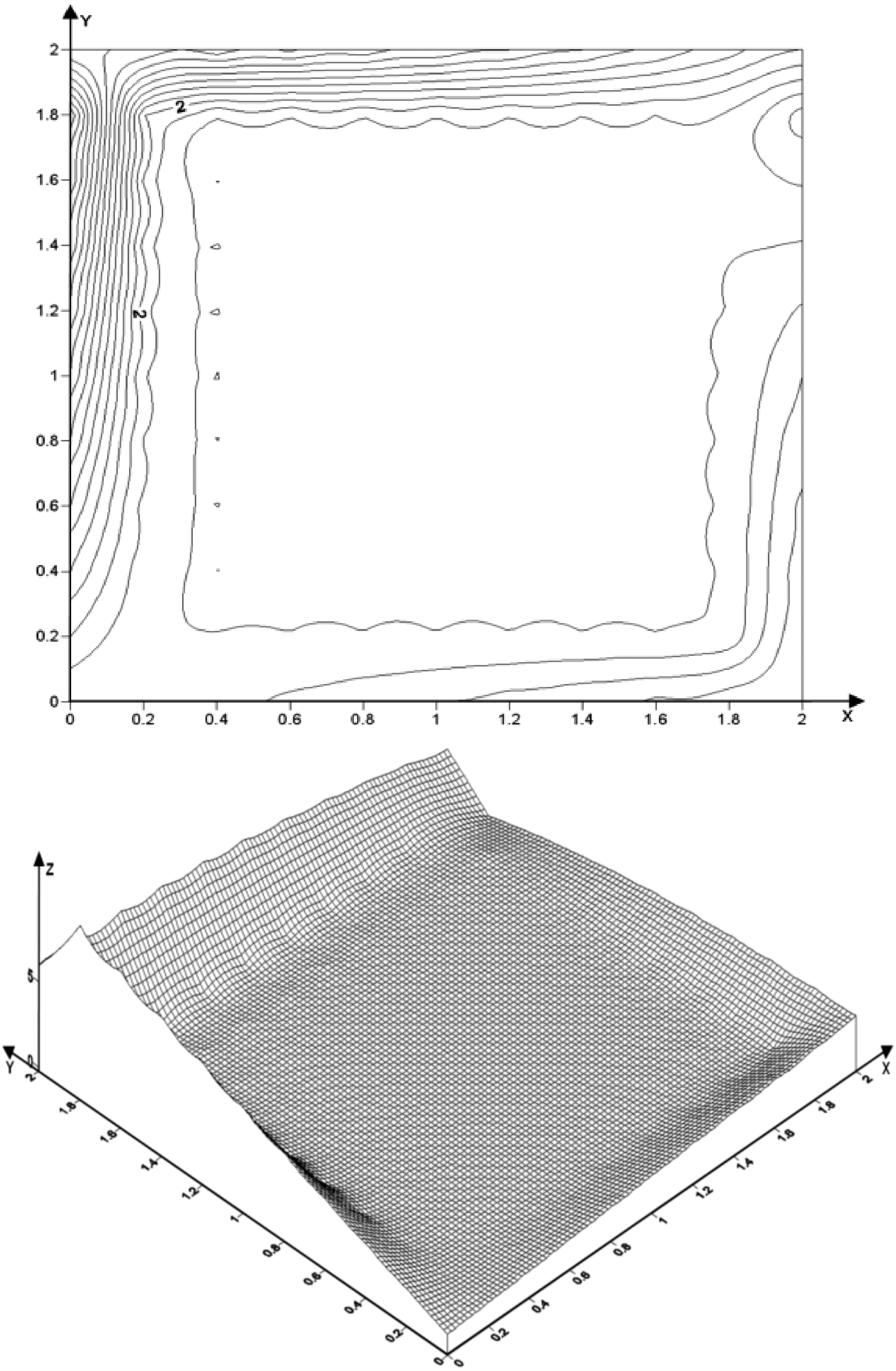

Figure 3: Isolines and surface for the second Dirichlet boundary value problem

5. Conclusion

The problems under consideration determine the set of interrelated differential and integral conditions necessary for successful numerical solution of Navier–Stokes equations.

This article is intended to acquaint both mathematicians and hydromechanics with what has been done so far respectively the study of the questions related to solvability and stability of boundary value problems for the Navier–Stokes equations. The article is based on the latest scientific results in the melts structure field, their movement mechanics and the use of modern hydrodynamics methods.

The article proposes a solution to the hydrodynamics equations. For computer modeling of melt's flow, a numerical solution to the equations is needed to be done by the finite difference method. The authors considered application of proposed method on Dirichlet problem's example for the Poisson equation. The results show the correctness of the program as well as the correctness of the boundary value problems for the hydrodynamic equations we considered above.

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