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Different linear control laws for fractional chaotic maps using Lyapunov functional

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Dynamics and control of discrete chaotic systems of fractional-order have received considerable attention over the last few years. So far, nonlinear control laws have been mainly used for stabilizing at zero the chaotic dynamics of fractional maps. This article provides a further contribution to such research field by presenting simple linear control laws for stabilizing three fractional chaotic maps in regard to their dynamics. Specifically, a one-dimensional linear control law and a scalar control law are proposed for stabilizing at the origin the chaotic dynamics of the Zeraoulia-Sprott rational map and the Ikeda map, respectively. Additionally, a two-dimensional linear control law is developed to stabilize the chaotic fractional flow map. All the results have been achieved by exploiting new theorems based on the Lyapunov method as well as on the properties of the Caputo h-difference operator. The relevant simulation findings are implemented to confirm the validity of the established linear control scheme.

Key words: discrete fractional calculus, chaotic maps, linear control, Lyapunov method

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1. Introduction

Discrete Fractional Calculus (DFC) has turned out to be a hotspot research topic in recent years [1–4]. The primary definition of the fractional-order operator in its discrete sense was first presented in [5], where the operator was derived by discretizing a continuous-time fractional operator. Successively, several types of difference operators have been proposed, including some *h-difference* operators of fractional-order, which represent further general extensions of difference operators of fractional-order [6–10].

The introduction of different discrete fractional operators has led to the publication of several papers regarding the nonlinear Fractional-order Discrete Systems (FoDSs) and their chaotic behaviors [11–18]. For example, in [11] the standard and the fractional sine maps have been analyzed in light of investigating their chaotic dynamics, whereas in [12] the presence of chaos in the logistic map of fractional-order has been addressed. In [13] chaotic attractors in the fractional Hénon map have been found, whereas in [14] the chaotic behavior of the delayed logistic map of fractional-order has been studied. In [15] a novel model of the generalized hyperchaotic Hénon map of fractional-order has been presented, while in [16] the hyperchaotic dynamics of the double-scroll map of fractional-order have been addressed. In [17], several chaotic attractors for a new generalized version for the Hénon map of three dimensions have been reported, whereas in [18] the symmetry properties of fractional maps with fixed points located on closed curves have been investigated. With the discovery of chaos in fractional maps, numerous endeavours have been dedicated to inspect many control methods proposed for effectively stabilizing the chaotic dynamics at the origin [19]. Some interesting results have been recently published regarding this challenging topic [20–27]. For example, in [20] control methods for three fractional chaotic maps (i.e., the Wang map, the Rössler map and the Stefanski map) have been studied. In [21] the fractional form of the Grassi-Miller map has been presented, along with a control law for stabilizing its dynamics. In [22] the control properties of a three-dimensional fractional map without equilibria have been investigated. Similarly, in [23] bifurcations and control of a quadratic fractional map without equilibrium points have been studied. In [24] the chaotic dynamics of the fractional Zeraoulia-Sprott rational map have been investigated. Additionally, in [24] a stabilization scheme based on the Lyapunov method has been illustrated. In [25], novel control laws for stabilizing three different maps of fractional-order have been established. Namely, the three maps considered in [25] are the fractional flow map, the fractional Lorenz map and the fractional Lozi map. In [26], the Ikeda map of fractional-order has been studied in term of its chaotic behavior, along with a stabilization method that exploits the stability properties of linear FoDSs.

It is worth noting that all the stabilization methods developed so far for fractional maps have exploited *nonlinear* control laws [20–26]. This article aims to provide a contribution to the research field by offering very simple *linear* control laws for stabilizing at the origin the chaotic dynamics of some well-known FoDSs. These maps, defined through using the Caputo h-difference operator, include the fractional Ikeda map, the fractional Zeraoulia-Sprott rational map and the fractional flow map. However, the remaining of this article is ordered in the following manner. In the next section, certain fundamental notions on the Caputo h-difference operator are reported. Afterward in Section 3, with the aim of stabilizing the dynamics of the fractional Ikeda map at zero, a novel theorem is proved by a *scalar* control law, while in order to stabilize the fractional chaotic Zeraoulia-Sprott map at zero, a novel linear control law of one dimension is established in Section 4. The objective is achieved by exploiting a novel theorem based on a suitable Lyapunov function. In Section 5, another novel theorem is proved, which assures the stabilization of the fractional flow map via establishing a new linear control law two dimensions. Note that, by virtue of the linearity of the control laws proposed herein, the conceived control schemes require less control effort with respect to the nonlinear techniques developed to date. Finally, simulation results are reported through the paper for the purpose of showing the correctness of the established scheme.

2. Basic tools

This part summarizes briefly some fundamental notions related to the Caputo h-difference operator.

Definition 1 *[7] The δ^{th}-order h-sum of the function $\Psi\colon (h\mathbb{N})_r \to \mathbb{R}$ is outlined as:*

$$ {}_h\Delta_r^{-\delta}\Psi(t) = \frac{h}{\Gamma(\delta)} \sum_{\tau=\frac{r}{h}}^{\frac{t}{h}-\delta} (t - \Upsilon(\tau h))_h^{(\delta-1)}\, \Psi(\tau h), \quad \Upsilon(\tau h) = (\tau+1)h, \qquad (1) $$

where $r \in \mathbb{R}$, $\delta > 0$, $t \in (h\mathbb{N})_{r+\delta h}$, and the h-falling factorial function can be expressed as:

$$ t_h^{(\delta)} = h^{\delta} \frac{\Gamma\left(\frac{t}{h}+1\right)}{\Gamma\left(\frac{t}{h}+1-\delta\right)}, $$

where $t \in \mathbb{R}$ and $(h\mathbb{N})_{r+(1-\delta)h} = \{r + (1-\delta)h, r + (2-\delta)h, \cdots\}$.

Definition 2 *[9] The Caputo-like difference operator of order $\delta > 0$ of the function $\Psi\colon (h\mathbb{N})_r \to \mathbb{R}$ is outlined as:*

$$ {}_h^C\Delta_r^\delta \Psi(t) = \Delta_r^{-(n-\delta)} \Delta^n \Psi(t), \tag{2}$$

where $\delta \notin \mathbb{N}$, $t \in (h\mathbb{N})_{r+(n-\delta)h}$, and where $\Delta\Psi(t) = \dfrac{\Psi(t+h) - \Psi(t)}{h}$ and $n = \lceil\delta\rceil + 1$.

Next, an effective theorem, reported in [19], will be briefly illustrated with the aim to identify the stability conditions of the zero equilibrium point for the FoDS written in the form:

$$ {}_h^C\Delta_r^\delta \Psi(t) = g\left(t + \delta h, \Psi(t + \delta h)\right), \tag{3}$$

where g is a nonlinear function, $t \in (h\mathbb{N})_{r+(1-\delta)h}$, and $\Psi(t) = (\psi_1(t), \psi_2(t), \ldots, \psi_n(t))^T$.

Theorem 1 *Suppose that $\psi = 0$ is an equilibrium point of the nonlinear FoDS given in (3), then this point will be asymptotically stable if $\exists$ a positive definite and decrescent scalar function $V(t, \Psi(t))$, in which ${}_h^C\Delta_r^\delta V(t, \Psi(t)) \leqslant 0$ and $t \in (h\mathbb{N})_{r+(1-\delta)h}$.*

In what follows, a useful inequality associated with the Lyapunov functions is introduced for completeness.

Lemma 1 *[19] The following inequality:*

$$ {}_h^C\Delta_r^\delta \left(\Psi^T(t)\Psi(t)\right) \leqslant 2\Psi^T(t + \delta h)\, {}_h^C\Delta_r^\delta \Psi(t), \tag{4}$$

holds $\forall t \in (h\mathbb{N})_{r+(1-\delta)h}$, where $0 < \delta \leqslant 1$.

3. Scalar control law

This part intends to prove a novel theorem established for stabilizing the dynamics of the fractional Ikeda map at zero through establishing a scalar control law. Referring to the fractional Ikeda map, it was introduced in [26] using the δ-Caputo delta difference operator. Herein, by exploiting the Caputo h-difference operator, the following fractional model is proposed:

$$\begin{aligned} {}_h^C\Delta_r^\delta u(t) &= 1 + \left[u(t+\delta h)\cos\theta(t+\delta h) - v(t+\delta h)\sin\theta(t+\delta h)\right]\eta \\ &\quad - u(t+\delta h), \\ {}_h^C\Delta_r^\delta v(t) &= \left[u(t+\delta h)\sin\theta(t+\delta h) + v(t+\delta h)\cos\theta(t+\delta h)\right]\eta \\ &\quad - v(t+\delta h), \end{aligned} \tag{5}$$

in which $(h\mathbb{N})_{r+(1-\delta)h}$, $0 < \delta \leqslant 1$, and η is an arbitrary parameter. For the purpose of revealing the dynamic characteristic of the intended map, the following formulas are numerically set out:

$$\begin{aligned} u_n &= u_0 + \frac{h^\delta}{\Gamma(\delta)} \sum_{\ell=0}^{\ell=n} \frac{\Gamma(n-\ell+\delta)}{\Gamma(n-\ell+1)} (1 + (u(\ell+1)\cos\theta(\ell+1) \\ &\quad - v(\ell+1)\sin\theta(\ell+1))\eta - u(\ell+1)), \\ v_n &= v_0 + \frac{h^\delta}{\Gamma(\delta)} \sum_{\ell=0}^{\ell=n} \frac{\Gamma(n-\ell+\delta)}{\Gamma(n-\ell+1)} ((u(\ell+1)\sin\theta(\ell+1) \\ &\quad + v(\ell+1)\cos\theta(\ell+1))\eta - v(\ell+1)), \end{aligned} \tag{6}$$

where u_0, v_0 are the initial states. Here, the two implicit equations given in (6) are employed to explore the chaotic behavior of the Ikeda map in its fractional-order. When $\eta = 0.9$, $h = 0.1$, and $u_0 = 0$, $v_0 = 0$, then the fractional-order Ikeda map will show chaotic behaviour. Figure 1, however, shows the chaotic

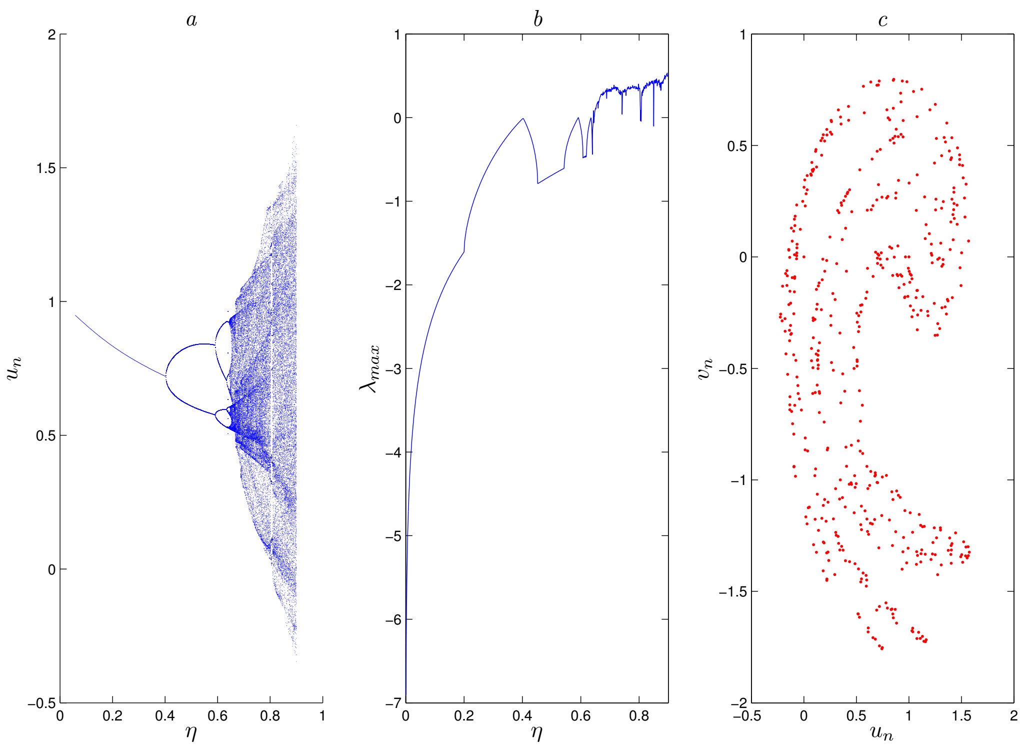

Figure 1: The diagram of bifurcation and the LLEs plots vs. the parameter η. a) The diagram of bifurcation. b) The LLEs. c) Chaotic attractor for $\eta = 0.9$

attractor obtained by simulating the two implicit equations given in (6) with the predictor corrector method proposed in [28], along with the Largest Lyapunov Exponents (LLEs) and the bifurcation diagram that are obtained through changing the parameter η from 0 to 0.9. In view of such figure, the chaotic behavior of the fractional Ikeda map given in (5) can be highlighted obviously by taking $\eta = 0.9$.

In the same context, we intend immediately to propose a controller for stabilizing the chaotic trajectories of the state-variables at zero in the Ikeda map (5) in its fractional order. This objective can be achieved by adding one linear term in the first state of the proposed fractional-order map. In other words, the fractional Ikeda map (5) can be controlled by tracking Theorem 2.

Theorem 2 *The following scalar control law:*

$$\mathbf{C} = -1, \tag{7}$$

can control the fractional Ikeda map reported in (5).

Proof. The controlled fractional Ikeda chaotic map is described by:

$$\begin{aligned} {}_{h}^{C}\Delta_{r}^{\delta}u(t) &= 1 + [u(t+\delta h)\cos\theta(t+\delta h) - v(t+\delta h)\sin\theta(t+\delta h)]\eta \\ &\quad - u(t+\delta h) + \mathbf{C}, \\ {}_{h}^{C}\Delta_{r}^{\delta}v(t) &= [u(t+\delta h)\sin\theta(t+\delta h) + v(t+\delta h)\cos\theta(t+\delta h)]\eta \\ &\quad - v(t+\delta h), \end{aligned} \tag{8}$$

where $\mathbf{C}$ is the proposed controller. Subtracting system (7) from system (5) yields following FoDS:

$$\begin{aligned} {}_{h}^{C}\Delta_{r}^{\delta}u(t) &= [u(t+\delta h)\cos\theta(t+\delta h) - v(t+\delta h)\sin\theta(t+\delta h)]\eta \\ &\quad - u(t+\delta h), \\ {}_{h}^{C}\Delta_{r}^{\delta}v(t) &= [u(t+\delta h)\sin\theta(t+\delta h) + v(t+\delta h)\cos\theta(t+\delta h)]\eta \\ &\quad - v(t+\delta h). \end{aligned} \tag{9}$$

Now, we should prove that the trivial solution of (9) is globally asymptotically stable. If so, we will deduce immediately that all the states of the controlled system given in (8) will definitely converge towards zero. Actually, this task can be performed using Lyapunov method that summarized before by Theorem 1. To see this, the following Lyapunov function has to be considered:

$$V = \frac{1}{2}\left(u^2(t) + v^2(t)\right), \quad t \in (h\mathbb{N})_{r+(1-\delta)h}. \tag{10}$$

Consequently, applying the fractional Caputo difference operator on (10) leads us to the following assertion:

$${}_{h}^{C}\Delta_{r}^{\delta}V = \frac{1}{2}{}_{h}^{C}\Delta_{r}^{\delta}u^2(t) + \frac{1}{2}{}_{h}^{C}\Delta_{r}^{\delta}v^2(t). \tag{11}$$

Using Lemma 1 yields:

$$\begin{aligned}
{}^{C}_{h}\Delta^{\delta}_{r} V &\leqslant u(t+\delta h)^{C}_{h}\Delta^{\delta}_{r} u(t) + v(t+\delta h)^{C}_{h}\Delta^{\delta}_{r} v(t) \\
&= \Big[u^2(t+\delta h)\cos\theta(t+\delta h) - u(t+\delta h)v(t+\delta h)\sin\theta(t+\delta h)\Big]\,\eta - u^2(t+\delta h) \\
&\quad + \Big[v(t+\delta h)u(t+\delta h)\sin\theta(t+\delta h) + v^2(t+\delta h)\cos\theta(t+\delta h)\Big]\,\eta - v^2(t+\delta h) \\
&= \eta u^2(t+\delta h)\cos\theta(t+\delta h) - u^2(t+\delta h) + \eta v^2(t+\delta h)\cos\theta(t+\delta h) - v^2(t+\delta h) \\
&\leqslant (\eta-1)u^2(t+\delta h) + (\eta-1)v^2(t+\delta h) < 0, \quad \text{(That is because } \eta = 0.9\text{)}.
\end{aligned}$$

This means that an efficient stabilization for all states of system (5) is occurred at the origin using the scalar control law (7).

For the purpose of confirming the validity of the established controller, the phase-space and the evolution of all states of the controlled system (8) are plotted as shown in Fig. 2. Such plots clearly show a stabilization at zero occurred for all chaotic dynamics of the fractional Ikeda map given in (5) by using the scalar control law given in (7).

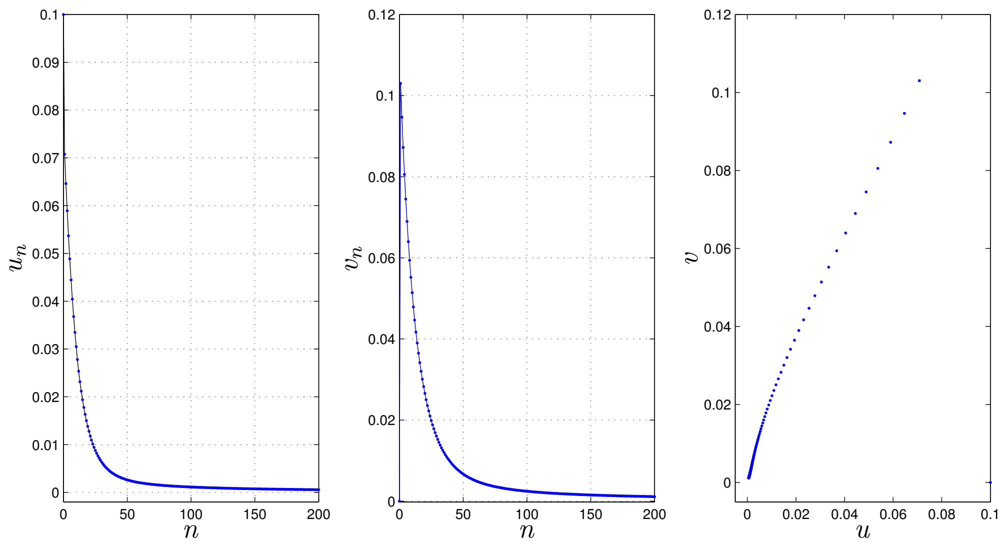

Figure 2: A stabilization of all states of the fractional Ikeda map (5) using the control law (7) with $\eta = 0.9$ and $\delta = 0.9$

4. One-dimensional linear control law

In what follows, an efficient linear control law of one dimension is established through setting up a novel theorem relies on suitable picking of Lyapunov function for the purpose of stabilizing the chaotic dynamics of the fractional Zeraoulia–Sprott rational map at zero. This two-dimensional map has been introduced in [24] using the δ-Caputo delta difference operator. Herein, through enacting the Caputo h-difference operator, the dynamics of such map will be described as:

$$\begin{aligned} {}^{C}_{h}\Delta^{\delta}_{r} u(t) &= \frac{-\rho u(t+\delta h)}{1+v^2(t+\delta h)} - u(t+\delta h), \\ {}^{C}_{h}\Delta^{\delta}_{r} v(t) &= u(t+\delta h) + (\beta - 1) v(t+\delta h), \end{aligned} \tag{12}$$

where $t \in (h\mathbb{N})_{r+(1-\delta)h}$. Actually, due to the presence of numerous chaotic attractors of this map that are typically generated through the quasi periodic route to chaos, it can be classified as more rich in it dynamics than that of the previous maps. However, in the light of theorem proposed in [29], the equivalent implicit discrete formulas of system (12) can be written in the form:

$$\begin{aligned} u_n &= u_0 + \frac{h^{\delta}}{\Gamma(\delta)} \sum_{\ell=0}^{n} \frac{\Gamma(n-\ell+\delta)}{\Gamma(n-\ell+1)} \left(\frac{-\rho u(\ell+1)}{1+v^2(\ell+1)} - u(\ell+1) \right), \\ v_n &= v_0 + \frac{h^{\delta}}{\Gamma(\delta)} \sum_{\ell=0}^{n} \frac{\Gamma(n-\ell+\delta)}{\Gamma(n-\ell+1)} (u(\ell+1) + (\beta-1) v(\ell+1)), \end{aligned} \tag{13}$$

where u_0, v_0 are initial states. Considering parameter's value $\beta = 0.6$ and varying ρ from 0 to 4 generate the diagram of bifurcation together with the LLEs depicted in Fig. 3. Different dynamic behaviors, including chaos periodic windows, can be ascertained in map (12). From which it can be seen that the system under consideration has a positive LLE when ρ takes highest values, indicating that the system has indeed a chaotic attractor, as shown in Fig. 3c for $\rho = 3.8$. Now, we intend to present a new theorem demonstrates a one-dimensional controller proposed for controlling the fractional Zeraoulia–Sprott map given in (12).

Theorem 3 *The linear control law:*

$$\mathbf{L}(t) = -v(t) - \rho u(t), \tag{14}$$

can control the fractional Zeraoulia–Sprott chaotic map reported in (12).

Proof. Combining map (12) together with the time-varying control parameter $\mathbf{L}$ implies the following controlled map:

$$\begin{aligned} {}^{C}_{h}\Delta^{\delta}_{r} u(t) &= \frac{-\rho u(t+\delta h)}{1+v^2(t+\delta h)} - u(t+\delta h) + \mathbf{L}(t+\delta h), \\ {}^{C}_{h}\Delta^{\delta}_{r} v(t) &= u(t+\delta h) + (\beta - 1) v(t+\delta h). \end{aligned} \tag{15}$$

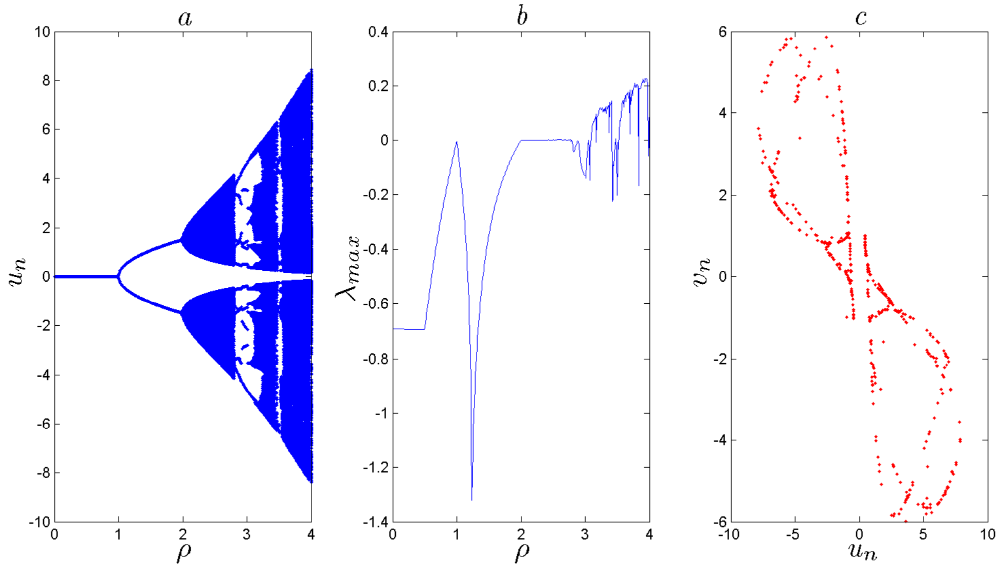

Figure 3: The diagram of bifurcation and the LLE plots vs. the parameter ρ. a) The diagram of bifurcation. b) The LLE. c) Chaotic attractor for $\rho = 3.8$

If one substitutes law (14) into (15), a simplified form of dynamics will be obtained. That is;

$$\begin{aligned} {}_h^C\Delta_r^\delta u(t) &= \frac{-\rho u(t+\delta h)}{1+v^2(t+\delta h)} - (\rho+1)u(t+\delta h) - v(t+\delta h), \\ {}_h^C\Delta_r^\delta v(t) &= u(t+\delta h) + (\beta-1)v(t+\delta h). \end{aligned} \tag{16}$$

By taking the Lyapunov function V in which $V = \frac{1}{2}\left(u^2(t)+v^2(t)\right)$, and then by exploiting Lemma 1, it follows that ${}_h^C\Delta_r^\delta V = \frac{1}{2}{}_h^C\Delta_r^\delta u^2(t) + \frac{1}{2}{}_h^C\Delta_r^\delta v^2(t)$. This consequently leads to the following assertions:

$$\begin{aligned} {}_h^C\Delta_r^\delta V &\leqslant u(t+\delta h){}_h^C\Delta_r^\delta u(t) + v(t+\delta h){}_h^C\Delta_r^\delta v(t) \\ &= \frac{-\rho u^2(t+\delta h)}{1+v^2(t+\delta h)} - (\rho+1)\,u^2(t+\delta h) - u(t+\delta h)v(t+\delta h) \\ &\quad + v(t+\delta h)u(t+\delta h) + (\beta-1)\,v^2(t+\delta h) \\ &\leqslant \frac{\rho u^2(t+\delta h)}{1+v^2(t+\delta h)} - (\rho+1)\,u^2(t+\delta h) + (\beta-1)\,v^2(t+\delta h) \\ &\leqslant \rho u^2(t+\delta h) - (\rho+1)\,u^2(t+\delta h) + (\beta-1)\,v^2(t+\delta h) \\ &= -u^2(t+\delta h) + (\beta-1)\,v^2(t+\delta h). \end{aligned}$$

Since $\beta-1$ is negative, it simply follows that ${}_h^C\Delta_r^\delta V < 0$. Hence, it can be concluded that the controlled states of the fractional map (12) are indeed stabilized at the origin by the linear control law (14).

For the purpose of highlighting the validity of the conceived scheme, the plots of the phase-space together with the states' evolution for the controlled map (15) are demonstrated in Fig. 4. These plots clearly highlight that all chaotic states of the fractional Zeraoulia–Sprott map (12) are stabilized at zero using very simple one-dimensional linear control law reported in (14).

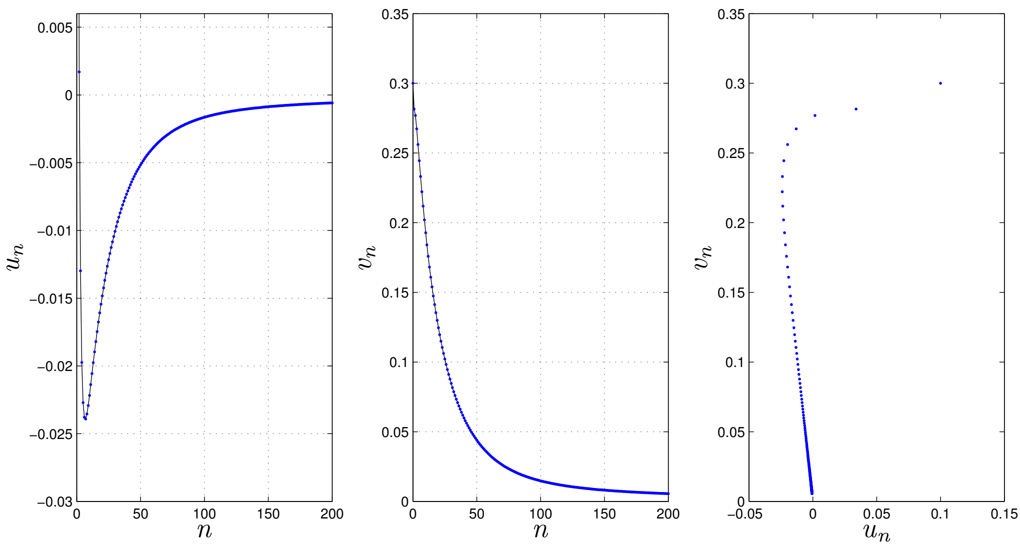

Figure 4: A stabilization of all states of the fractional Zeraoulia-Sprott map (12) using the control law (14) with $\rho = 3.8$, $\beta = 0.6$ and $\delta = 0.9$

5. Two-dimensional linear control law

In this section, a simple two-dimensional linear control law is proposed for stabilizing the dynamics of a fractional flow map. The objective will be achieved by developing a novel theorem based on the Lyapunov method. Herein, differently from the δ-Caputo delta difference operator that has been used in [25], the Caputo h-difference operator is adopted for obtaining the following fractional model of the flow map:

$$\begin{aligned} {}_h^C\Delta_r^\delta u(t) &= v(t+\delta h) + (\lambda - 1)u(t+\delta h), \\ {}_h^C\Delta_r^\delta v(t) &= \mu + u^2(t+\delta h) - v(t+\delta h), \end{aligned} \tag{17}$$

where $t \in (h\mathbb{N})_{r+(1-\delta)h}$. In particular, when $r = 0$, the equivalent implicit formulas will be in the form:

$$\begin{aligned} u_n &= u_0 + \frac{h^\delta}{\Gamma(\delta)} \sum_{\ell=0}^{n} \frac{\Gamma(n-\ell+\delta)}{\Gamma(n-\ell+1)} (v(\ell+1) + (\lambda-1)u(\ell+1)), \\ v_n &= v_0 + \frac{h^\delta}{\Gamma(\delta)} \sum_{\ell=0}^{n} \frac{\Gamma(n-\ell+\delta)}{\Gamma(n-\ell+1)} (\mu + u^2(\ell+1) - v(\ell+1)). \end{aligned} \tag{18}$$

Figure 5, however, shows the bifurcation diagram obtained by simulating the two equations given in (18) on λv_n-plane, along with the chaotic attractor. From such figure, one can see that the flow map of fractional-order given in (17) exhibits a chaotic behaviour over most of the range $[-0.12, 0.02[$ for $\mu = -1.17$. Note that the states' evolution of such map, which adopts the Caputo h-difference operator in its construction, are absolutely different from those of the map reported in [25], being the latter has been established on the basis of the δ-Caputo delta difference operator. In consequence of this development, the next task focuses on stabilizing all states of the fractional flow map and hence eliminating its chaotic motion by adding two linear terms to the first and the second equations of such map. In fact, this controller will force all trajectories, generated by system (17), to be tended to the zero equilibrium point. This target can be achieved through next theorem.

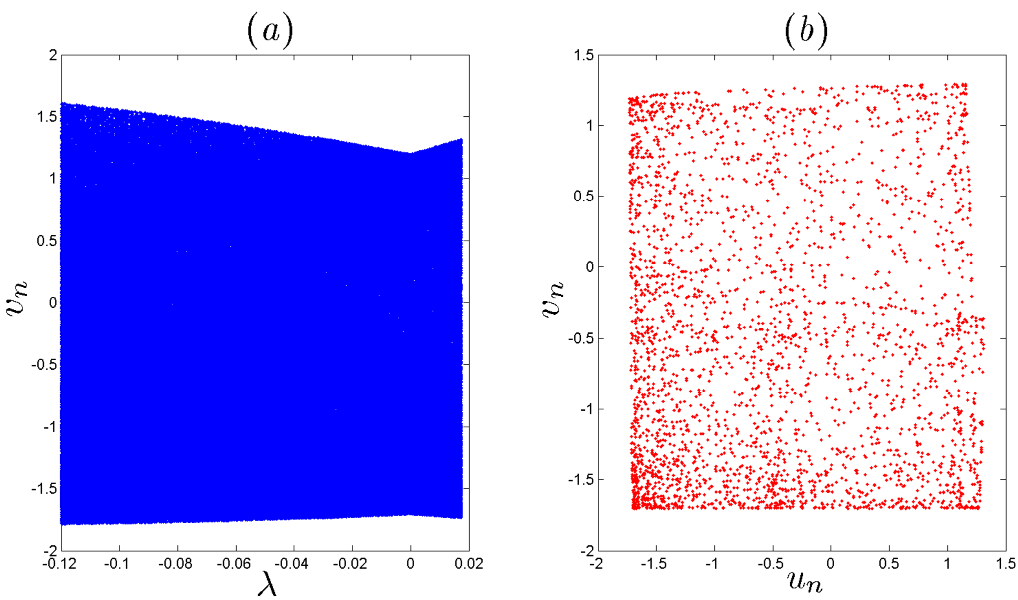

Figure 5: (a) Bifurcation versus the system's parameter λ. (b) Chaotic attractor of the fractional-order flow map

Theorem 4 *The dynamics of the fractional flow map converge asymptotically to the origin in view of the following control law:*

$$\begin{aligned} \chi_1(t) &= -(b+\lambda)u(t), \\ \chi_2(t) &= -\mu - u(t), \end{aligned} \tag{19}$$

where $|v(t)| \leqslant b, \quad \forall t \in (h\mathbb{N})_{r+(1-\delta)h}$.

Proof. The fractional flow map under controlled can be outlined as:

$$\begin{aligned} {}_h^C\Delta_r^\delta u(t) &= v(t+\delta h) + (\lambda - 1)u(t+\delta h) + \chi_1(t+\delta h), \\ {}_h^C\Delta_r^\delta v(t) &= \mu + u^2(t+\delta h) - v(t+\delta h) + \chi_2(t+\delta h). \end{aligned} \tag{20}$$

Consequently, system (20) will be as:

$$\begin{aligned} {}_h^C\Delta_r^\delta u(t) &= v(t+\delta h) - (b+1)u(t+\delta h), \\ {}_h^C\Delta_r^\delta v(t) &= u^2(t+\delta h) - v(t+\delta h). \end{aligned} \tag{21}$$

As a result of using the Lyapunov function $V = \dfrac{1}{2}\left(u^2(t) + v^2(t)\right)$ together with Lemma 1, we obtain the following assertions:

$$\begin{aligned} {}_h^C\Delta_r^\delta V &\leqslant u(t+\delta h){}_h^C\Delta_r^\delta u(t) + v(t+\delta h){}_h^C\Delta_r^\delta v(t) \\ &= u(t+\delta h)v(t+\delta h) - (b_2+1)u^2(t+\delta h) \\ &+ v(t+\delta h)u^2(t+\delta h) - v^2(t+\delta h) - u(t+\delta h)v(t+\delta h) \\ &= -(b+1)u^2(t+\delta h) + v(t+\delta h)u^2(t+\delta h) - v^2(t+\delta h) \\ &\leqslant -(b+1)u^2(t+\delta h) + |v(t+\delta h)|\, u^2(t+\delta h) - v^2(t+\delta h) \\ &\leqslant -(b+1)u^2(t+\delta h) + bu^2(t+\delta h) - v^2(t+\delta h) \\ &= -u^2(t+\delta h) - v^2(t+\delta h) < 0. \end{aligned}$$

Thus, in view of Theorem 1, one might deduce that the trivial solution of system (21) is globally asymptotically stable. Consequently, the controlled system given in (20) has been indeed stabilized at zero by the two-dimensional linear control law (19).

Remark 1 *The existence of b is justified by the property of the boundness of the states of chaotic maps. This constant can be found numerically easily.*

For the purpose of showing the validity of the established controller, the plots of the phase space together with the states' evolution of the controlled map (20) are demonstrated in Fig. 6.

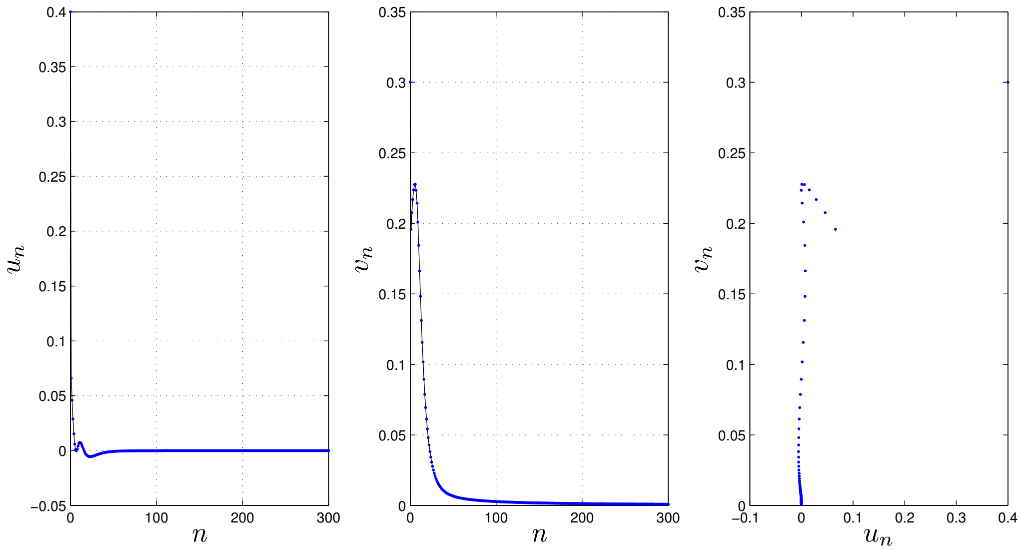

Figure 6: A stabilization of all states of the fractional flow map (17) using the control law (19) with $\lambda = -0.1$ and $\delta = 0.9$

6. Conclusion

So far, some nonlinear control laws have been mainly used for stabilizing the chaotic dynamics of fractional maps at zero. This work has made a contribution in this research field by proposing simple linear control laws for stabilizing the dynamics of some types of those fractional maps which have been established in view of the Caputo h-difference operator, particulary the Ikeda map, the Zeraoulia-Sprott rational map and the flow map. The objective has been achieved by proving three new theorems based on assuming suitable Lyapunov functions. By virtue of the linearity of the control laws proposed herein, the conceived methods for stabilizing the chaotic dynamics at zero require less control effort than that of those nonlinear techniques developed in literature to date. Finally, some simulation findings have been implemented with the aim of highlighting the validity of all proposed schemes.

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