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On theoretical and practical aspects of Duhamel's integral

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The paper is a new approach to the Duhamel integral. It contains an overview of formulas and applications of Duhamel's integral as well as a number of new results on the Duhamel integral and principle. Basic definitions are recalled and formulas for Duhamel's integral are derived via Laplace transformation and Leibniz integral rule. Applications of Duhamel's integral for solving certain types of differential and integral equations are presented. Moreover, an interpretation of Duhamel's formula in the theory of operator semigroups is given. Some applications of Duhamel's formula in control systems analysis are discussed. The work is also devoted to the usage of Duhamel's integral for differential equations with fractional order derivative.

Key words: Duhamel's integral, Duhamel's principle, Duhamel's formula, Laplace transformation, semigroup of operators, Leibniz integral rule, Volterra integral equation, Caputo fractional derivative

1. Introduction

Duhamel's integral owes its name to a French mathematician Jean-Marie Constant Duhamel who continued the ideas of Fourier and Laplace. While developing the theory of thermal conductivity, around the year 1830 he introduced a procedure known today as *Duhamel's principle*. It is a general method for obtaining solutions to inhomogeneous linear evolution equations like the heat equation, wave equation, or vibrating plate equation. The classical Duhamel principle allows to reduce the Cauchy problem for an inhomogeneous partial differential equation to Cauchy problem for corresponding homogeneous equation, which is more simpler to handle. In the solution procedures *Duhamel's integral* is usually involved. Therefore, despite the fact that the Duhamel integral belongs to the

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classical dependencies encountered in the Laplace transformation theory, its role and mathematical importance have significantly strengthened the use of in the partial differential theory and the theory of semigroups of operators.

In the linear systems theory, the Duhamel convolution integral describes the relation between a system's output (response) and its output with the use of the transfer function (kernel function). An application of Duhamel's integral for developing solution to heat conduction problems with time-dependent boundary conditions by utilizing the solution to the same problem with time-independent boundary conditions is presented in [31]. Usage of Duhamel's integral in some special load cases to single degree of freedom (SDOF) mass system is presented in [23]. Duhamel's integral with rooted kernel as the solution of heat equation inside oscillating gas bubble with moving boundary is studied in [24]. Applications of Duhamel's integral can also be found in the field of fractional calculus [1, 7, 17, 26, 48–50].

It seems that this rapid development of the area of application of the formula known as Duhamel's integral was certainly not smooth from the mathematical point of view. Hence the main purpose of this study is to review the more or less known information about Duhamel's integral on many levels of its applications. There is no such study that would fill some "understatements" and "gaps" in the literature.

The paper is organized as follows. In Section 2, we recall basic definitions and derive formulas for Duhamel's integral via Laplace transform and Leibniz integral rule. Section 3 contains analytic applications of the Duhamel integral for solving certain types of differential and integral equations. The method similar to the classical Duhamel principle, but with different auxiliary equation is proposed. Section 4 presents the adaptation of Duhamel's formula in the theory of operator semigroups. The basic concepts and facts are briefly outlined. Finally, after an appropriate introduction, the semigroup version of Duhamel's formula given by S. Mischler was presented. Section 5 contains some examples of control systems for which Duhamel's integral can be applied. The usage of Duhamel's integral to equations with fractional order derivative ends our article.

2. Formulas for Duhamel's integral

Let functions $f, g\colon [0,\infty) \to \mathbb{R}$ be Riemann-integrable on any closed interval $[0,t]$, $t \geqslant 0$. A *convolution* (one-sided) of functions f and g on the interval $[0,+\infty)$ is the function defined as

$$(f * g)(t) = f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau.$$

It is easy to verify that the convolution is commutative. Moreover, if f and g are originals, the convolution $f * g$ is also an original.

Let the convolution $f * g$ be differentiable. The derivative of convolution is called *Duhamel's integral*, i.e. Duhamel's integral is defined by the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}(f * g)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau. \tag{1}$$

Remark 1 *It is enough that one of the functions f or g is differentiable, then the convolution is differentiable.*

Assume that f is differentiable on $[0, \infty)$, f' and g are Riemann-integrable on any interval $[0, t], t \geqslant 0$, $f * g$ is differentiable on $[0, \infty)$, and $f, g, f' * g$ and $(f * g)'$ are originals. Applying the Laplace transform, the following formula for Duhamel's integral can be derived [10, 27, 37].

$$\frac{\mathrm{d}}{\mathrm{d}t}(f * g)(t) = (f' * g)(t) + f(0)g(t), \tag{2}$$

where $f(0) := \lim\limits_{t\to 0^+} f(t)$.

Moreover, if we additionally assume that g is differentiable on $[0, \infty)$, g' is Riemann-integrable on any interval $[0, t]$, $t \geqslant 0$, and $f * g'$ is the originals the following alternative formula for Duhamel's integral also holds

$$\frac{\mathrm{d}}{\mathrm{d}t}(f * g)(t) = (f * g')\,(t) + f(t)g(0), \tag{3}$$

where $g(0) := \lim\limits_{t\to 0^+} g(t)$.

Both equalities (2) and (3) are called *Duhamel's formula* and if all assumptions mentioned after Remark 1 are satisfied, they can be used interchangeably.

Remark 2 *The fact that the limits*

$$\lim_{t\to 0^+} f(t) \quad \textit{and} \quad \lim_{t\to 0^+} g(t)$$

exist and are finite follows from the assumptions that functions f' and g', respectively, are Riemann-integrable on any interval $[0, t], t \geqslant 0$.

We will derive formula (2) via the Laplace transform. We calculate the Laplace transform of the convolution $f' * g$ using the Convolution Theorem for the Laplace transform and the theorem about original differentiation [9, 32] we have

$$\begin{aligned}\mathcal{L}[(f' * g)(t)] &= \mathcal{L}[f'(t)]\mathcal{L}[g(t)] = \left[s\mathcal{L}[f(t)] - f(0)\right]\mathcal{L}[g(t)]\\ &= s\mathcal{L}[f(t)]\mathcal{L}[g(t)] - f(0)\mathcal{L}[g(t)].\end{aligned} \tag{4}$$

Moreover,

$$\mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}(f * g)(t)\right] = s\mathcal{L}[f(t)]\mathcal{L}[g(t)] - (f * g)(0) = s\mathcal{L}[f(t)]\mathcal{L}[g(t)], \tag{5}$$

since, under the assumptions for (2), $\lim\limits_{t\to 0^+}(f * g)(t) = 0$.

Combining formulas (4) and (5), we obtain the Laplace transform of Duhamel's integral

$$\mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}(f * g)(t)\right] = \mathcal{L}[(f' * g)(t)] + f(0)\mathcal{L}[g(t)].$$

Using the inverse Laplace transformation, on the basis of linearity and uniqueness, we obtain the formula (2) on $[0, \infty)$.

Formula (3) can be derived analogously.

Another method of deriving the formula (2) is based on the Leibniz integral rule. Below we present the theorem together with our authors' proof.

Theorem 1 *(Leibniz integral rule) If $f, g, h(\cdot, x) \in C^1(I)$, $I \subset \mathbb{R}$ is a compact interval, $f(t) < g(t)$ for all $t \in I$, $h(t, \cdot) \in C([f(t), g(t)])$ for all $t \in I$, then for $t \in I$ the following formula holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{f(t)}^{g(t)} h(t, x)\,\mathrm{d}x = \int\limits_{f(t)}^{g(t)} \frac{\partial}{\partial t}(h(t, x))\,\mathrm{d}x + h(t, g(t))g'(t) - h(t, f(t))f'(t). \tag{6}$$

Proof. Since $h(t, \cdot) \in C([f(t), g(t)])$ for all $t \in I$, the function $h(t, \cdot)$ has the primitive $H(t, \cdot)$ over the range of variable t. Therefore

$$\int\limits_{f(t)}^{g(t)} h(t, x)\,\mathrm{d}x = H(t, x)\Big|_{x=f(t)}^{x=g(t)} = H(t, g(t)) - H(t, f(t))$$

and, applying differentiating rules for functions of several variables, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{f(t)}^{g(t)} h(t, x)\,\mathrm{d}x &= \frac{\partial}{\partial t}H(t, u)\Big|_{u=g(t)} - \frac{\partial}{\partial t}H(t, v)\Big|_{v=f(t)}\\ &\quad + h(t, g(t)) \cdot g'(t) - h(t, f(t)) \cdot f'(t).\end{aligned} \tag{7}$$

Now we use the theorem of Hermann Amandus Schwarz that gives sufficient conditions for equality of mixed partial derivatives [33, 34]. If the following partial derivatives exist and they are continuous

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial u}H(t,u)\right) \qquad \text{and} \qquad \frac{\partial}{\partial u}\left(\frac{\partial}{\partial t}H(t,u)\right),$$

over some region $D \subset \mathbb{R}^2$, then the partial derivatives are equal on D. It follows that, if functions

$$\frac{\partial}{\partial t}h(t,u) \qquad \text{and} \qquad \frac{\partial}{\partial u}\left(\frac{\partial}{\partial t}H(t,u)\right) \tag{8}$$

exist and are continuous, then they are equal on D. Hence

$$\int\limits_{f(t)}^{g(t)} \frac{\partial}{\partial t}h(t,x)\,\mathrm{d}x = \int\limits_{f(t)}^{g(t)} \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}H(t,x)\right)\mathrm{d}x = \frac{\partial}{\partial t}H(t,x)\Big|_{x=f(t)}^{x=g(t)}$$

and, based on (7), we have (6). □

Remark 3 *Different proof of the Leibniz integral rule can be found, among others, in monograph [19, 20, 29], where the Mean Value Theorem and the limit transition theorem for the Riemann integral are applied [3].*

Remark 4 *Theorem 1 can be generalized to the case of vector functions. Nicolas Bourbaki in [4] describes such generalization.*

Let $a, b, A, B \in \mathbb{R}$, $a < b$, $A < B$ and let $\mathbb{Y}$ be a real Banach space.

Theorem 2 *Assume that a function $f\colon [a,b]\times[A,B] \to \mathbb{Y}$ is continuous, then a function $g\colon [A,B] \to \mathbb{R}$ defined by the formula*

$$g(\alpha) := \int\limits_a^b f(x,\alpha)\,\mathrm{d}x$$

is continuous. Let be given $\alpha_0 \in (A,B)$. If the differential quotient

$$\frac{f(x,\alpha)-f(x,\alpha_0)}{\alpha-\alpha_0}$$

is uniformly convergent on $[a, b]$ *for* $\alpha \to \alpha_0$*, then its limit is a continuous function. The limit is the partial derivative of* f *at* (x, α_0)*, which we denote by* $f'_\alpha(x, \alpha_0)$*. It folows that, if*

$$\frac{g(\alpha) - g(\alpha_0)}{\alpha - \alpha_0} = \int\limits_a^b \frac{f(x, \alpha) - f(x, \alpha_0)}{\alpha - \alpha_0} \mathrm{d}x,$$

then

$$g'(\alpha_0) = \int\limits_a^b f'_\alpha(x, \alpha) \mathrm{d}x. \tag{9}$$

Remark 5 *The formula* (9) *is a simplified version of* (6)*. The complete vector equivalent of* (6) *is presented below by formulating stronger, but also more general additional assumptions [4, 14].*

Let $\eta, \xi \in \mathbb{R}$, $\eta < \xi$, and $\mathbb{X}$, $\mathbb{Y}$ be real Banach spaces. Moreover, let $U \subset \mathbb{X}$ be non-empty open subset and let $f : [\eta, \xi] \times U \to \mathbb{Y}$ be a continuous function such that the partial derivative $f'_\alpha(x, \alpha)$ – the partial Fréchet differential of f – exists and is continuous on $[\eta, \xi] \times U$. Assume that two auxiliary functions $a, b : U \to [\eta, \xi]$ are of the class C^1 and

$$G(\alpha) := \int\limits_{a(\alpha)}^{b(\alpha)} f(x, \alpha) \mathrm{d}x, \qquad \alpha \in U.$$

Then $G \in C^1(U)$ and for any $\alpha \in U$, $h \in \mathbb{X}$, the following equality holds

$$\begin{aligned} dG(\alpha)h = (db(\alpha)h) f(b(\alpha), \alpha) - (da(\alpha)h) f(a(\alpha), \alpha) \\ + \left(\int\limits_{a(\alpha)}^{b(\alpha)} f'_\alpha(x, \alpha) \mathrm{d}x \right) h, \end{aligned} \tag{10}$$

where $dG(\alpha)$, $db(\alpha)$ and $da(\alpha)$ are the partial Fréchet differentials of the respective functions. Under weaker assumptions, for $\mathbb{X} = \mathbb{R}$ and when we consider the derivative only in the point$\alpha_0 \in U$, we obtain the form (6) of the formula (10), where $f'_\alpha(x, \alpha)$ is the partial Fréchet differential of f.

Remark 6 *Formula* (10) *is the vector equivalent of formula* (6)*.*

Now, we can derive formula (3) with the aid of the Leibniz integral rule. For this purpose we make different assumptions for functions f and g. Let f' and g are continuous on $[0, +\infty)$. Then, based on (6), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(f * g)(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau \\ &= \int_0^t \frac{\partial}{\partial t}[f(\tau)g(t-\tau)]\,\mathrm{d}\tau + f(\tau)g(t-\tau)\big|_{\tau=0}\cdot 0' + f(\tau)g(t-\tau)\big|_{\tau=t}\cdot t' \\ &= \int_0^t f(\tau)g'(t-\tau)\,\mathrm{d}\tau + f(t)g(0) = (f * g')(t) + f(t)g(0).\end{aligned}$$

In the same way one can derive the formula (3).

In the following numerical example we present an application of Duhamel's integral and formula (5) for finding the inverse Laplace transform of a given function.

Example 1 Consider the function $\frac{s^3}{(s^2+1)^2}$. Its inverse Laplace transform is

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s^3}{(s^2+1)^2}\right] &= \mathcal{L}^{-1}\left[s\cdot\frac{s}{s^2+1}\cdot\frac{s}{s^2+1}\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \cos(t-\tau)\cos\tau d\tau = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (\cos t + \cos(2\tau - t))d\tau \qquad (11) \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(t\cos t + \sin t) = \cos t - \frac{1}{2}t\sin t, \quad t \geqslant 0.\end{aligned}$$

Let us use other method to verify the result.

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s^3}{(s^2+1)^2}\right] &= \mathcal{L}^{-1}\left[\frac{s(s^2+1)-s}{(s^2+1)^2}\right] = \mathcal{L}^{-1}\left[\frac{s}{s^2+1} - \frac{s}{(s^2+1)^2}\right] \\ &= \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + \mathcal{L}^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}\cdot\frac{1}{s^2+1}\right)\right] = \cos t - \frac{1}{2}t\sin t, \quad t \geqslant 0.\end{aligned}$$

Calculation made in (11) also allow us to calculate the inverse Laplace transforms of the following rational functions

$$F_k(s) = \frac{s^k}{(s^2+1)^4}, \quad \text{where } k = 1, 2, 3, 4, 5, 6, 7.$$

Taking $g(t) = \cos t - \frac{1}{2} t \sin t$, $h(t) = \frac{1}{2}(t \cos t + \sin t)$, for $t \geqslant 0$, we have

$$\begin{aligned}
\mathcal{L}^{-1}[F_7(s)] &= \mathcal{L}^{-1}\left[s \cdot \frac{s^3}{(s^2+1)^2} \cdot \frac{s^3}{(s^2+1)^2}\right] = \frac{\mathrm{d}}{\mathrm{d}t}(g * g)(t) \\
&= \frac{1}{48}\Big((48 - 15t^2)\cos t + t(t^2 - 57)\sin t\Big), \\
\mathcal{L}^{-1}[F_6(s)] &= \mathcal{L}^{-1}\left[\frac{s^3}{(s^2+1)^2} \cdot \frac{s^3}{(s^2+1)^2}\right] = (g * g)(t) \\
&= \frac{1}{48}\Big(-t(t^2 - 33)\cos t + 3(5 - 4t^2)\sin t\Big), \\
\mathcal{L}^{-1}[F_5(s)] &= \mathcal{L}^{-1}\left[s \cdot \frac{s^2}{(s^2+1)^2} \cdot \frac{s^2}{(s^2+1)^2}\right] = \frac{\mathrm{d}}{\mathrm{d}t}(h * h)(t) \\
&= -\frac{1}{48} t\Big(-9t\cos t + (t^2 - 15)\sin t\Big), \\
\mathcal{L}^{-1}[F_4(s)] &= \mathcal{L}^{-1}\left[\frac{s^2}{(s^2+1)^2} \cdot \frac{s^2}{(s^2+1)^2}\right] = (h * h)(t) \\
&= \frac{1}{48}\Big(t(t^2 - 3)\cos t + 3(1 + 2t^2)\sin t\Big), \\
\mathcal{L}^{-1}[F_3(s)] &= \mathcal{L}^{-1}\left[\frac{s^2}{(s^2+1)^2} \cdot \frac{s}{(s^2+1)^2}\right] \\
&= \mathcal{L}^{-1}\left[\frac{s^2}{(s^2+1)^2} \cdot \left(-\frac{1}{2} \cdot \frac{1}{s^2+1}\right)'\right](t) \\
&= \int_0^t \frac{1}{2}(\tau \cos \tau + \sin \tau)\frac{(t - \tau)}{2}\sin(t - \tau)\,\mathrm{d}\tau \\
&= \frac{1}{48} t\Big(-3t\cos t + (3 + t^2)\sin t\Big), \\
\mathcal{L}^{-1}[F_2(s)] &= \mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2} \cdot \frac{s}{(s^2+1)^2}\right] \\
&= \mathcal{L}^{-1}\left[\left(-\frac{1}{2}\frac{1}{s^2+1}\right)' \cdot \left(-\frac{1}{2}\frac{1}{s^2+1}\right)'\right](t) \\
&= \frac{1}{4}\int_0^t \tau(\sin \tau)(t - \tau)\sin(t - \tau)\,\mathrm{d}\tau = \frac{1}{48}\Big(-t(3 + t^2)\cos t + 3\sin t\Big),
\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}^{-1}[F_1(s)] &= \mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^4}\right] = \mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\cdot s\left(\frac{1}{s(s^2+1)} - \frac{s}{(s^2+1)^2}\right)\right] \\ &= \mathcal{L}^{-1}\left[s\cdot\frac{s}{(s^2+1)^2}\left(\frac{1}{s} - \frac{s}{s^2+1} - \frac{s}{(s^2+1)^2}\right)\right] \\ &\overset{(5)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{1}{2}(t-\tau)\sin(t-\tau)\left(1-\cos\tau - \frac{1}{2}\tau\sin\tau\right)\mathrm{d}\tau \\ &= -\frac{1}{48}t\left(3t\cos t + (t^2-3)\sin t\right).\end{aligned}$$

It is worth noting that a general formula for $\mathcal{L}^{-1}\left[\frac{s^m}{(s^2+1)^n}\right]$, where $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $m < 2n$, requires the use of the Bessel functions of half integer order (see [40]) and there are not simple calculations by hand. For example, we have

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^n}\right] = \frac{\sqrt{\pi}}{(n-1)!}\left(\frac{x}{2}\right)^{n-\frac{1}{2}} J_{n-\frac{1}{2}}(x),$$

where $J_{n-\frac{1}{2}}(x)$ is the respective Bessel function (see [38]), which may be expressed in the form

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} \cos(x\sin\varphi)\cos^{2n-1}\varphi\,\mathrm{d}\varphi \\ &\qquad = \frac{1}{2}\int_{-1}^{1}(1-u^2)^{n-1}\cos(xu)\,\mathrm{d}u = \frac{\sqrt{\pi}(n-1)!}{2\left(\frac{x}{2}\right)^{n-\frac{1}{2}}} J_{n-\frac{1}{2}}(x).\end{aligned}$$

3. Analytic applications

3.1. Analytic solutions of linear differential equations

The Duhamel's integral can be used to solve linear differential equations with constant coefficients by solving an auxiliary differential equation with a function equal to 1 [28, 30]. More precisely, we have to solve the following linear ordinary differential equation

$$a_n x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \ldots + a_1 x(t) = f(t), \tag{12}$$

with initial conditions

$$x(0) = x'(0) = \ldots = x^{(n-1)}(0) = 0, \tag{13}$$

where $a_0, \ldots, a_{n-1}, a_n \in \mathbb{R}$ and f is continuous. Let us assume that we know the solution of equation

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \ldots + a_1 y(t) = 1 \tag{14}$$

with initial conditions

$$y(0) = y'(0) = \ldots = y^{(n-1)}(0) = 0. \tag{15}$$

If we go over to the Laplace transform, we obtain

$$P(t)\mathcal{L}[x(t)] = \mathcal{L}[f(t)] \tag{16}$$

for (12) and

$$P(t)\mathcal{L}[y(t)] = \frac{1}{t} \tag{17}$$

for (13), where P is a known polynomial of degree n in t. From (16) it follows that

$$\mathcal{L}[x(t)] = \frac{\mathcal{L}[f(t)]}{P(t)}$$

and from (17) that

$$P(t) = \frac{1}{t\mathcal{L}[y(t)]},$$

whence

$$\mathcal{L}[x(t)] = t\mathcal{L}[y(t)]\mathcal{L}[f(t)].$$

According to (5) we have

$$\mathcal{L}[x(t)] = \mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}(y * f)(t)\right],$$

so

$$x(t) = \frac{\mathrm{d}}{\mathrm{d}t}(y * f)(t).$$

Bearing in mind $y(0) = 0$, we find the solution $x(t)$ of (12) with initial condition (13) from the property (2) for Duhamel's integral on the form

$$x(t) = (y' * f)(t) = \int_0^t y'(t-\tau) f(\tau)\,\mathrm{d}\tau,$$

where $y(t)$ is the solution of problem (14) with initial conditions (15).

The above procedure is similar to the classical method of solving Bernoulli's differential equations

$$x'(t) + p(t)x = q(t)x^{\alpha}, \qquad \alpha \in \mathbb{R} \setminus \{0, 1\}. \tag{18}$$

Firstly we reduce this equation to a linear differential equation by substituting $y = x^{1-\alpha}$ and then we often use the variational method of solving. On the other hand, it can be solved directly by looking for a solution to differential equation (18) as the product of two functions $x(t) = u(t)v(t)$ regardless of the value of $\alpha \in \mathbb{R}$ (also for cases $\alpha = 0$ and $\alpha = 1$). In practice, the Duhamel's formula is only a trick and rather artificial in the technique of solving differential equations by hand calculations using the Laplace transform. It can be successfully replaced by direct proceedings with using the Convolution Theorem, which is neither typical nor obvious, as evidenced by examples from the known literature [8,10,18,28,37].

We demonstrate this in the following examples. We look for solutions to the considered differential equations in the class of originals.

The first example presents a method basing on the usage of Duhamel's formula.

Example 2 We find the solution of the following linear ordinary differential equation with zero initial conditions

$$x''(t) + x(t) = \frac{1}{2 + \cos t}, \qquad x(0) = x'(0) = 0.$$

First we solve the auxiliary differential equation

$$y''(t) + y(t) = 1, \qquad y(0) = y'(0) = 0.$$

We go over Laplace-operator equations

$$\begin{aligned} (s^2 + 1)\mathcal{L}[y(t)] &= \frac{1}{s}, \\ s\mathcal{L}[y(t)] &= \frac{1}{s^2 + 1} \end{aligned} \tag{19}$$

and

$$(s^2 + 1)\mathcal{L}[x(t)] = \mathcal{L}\left[\frac{1}{2 + \cos t}\right],$$

which implies

$$\begin{aligned} \mathcal{L}[x(t)] &= \frac{1}{s^2 + 1}\mathcal{L}\left[\frac{1}{2 + \cos t}\right] \\ &\overset{(19)}{=} s\mathcal{L}[y(t)]\mathcal{L}\left[\frac{1}{2 + \cos t}\right] \overset{(5)}{=} \mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(y(t) * \frac{1}{2 + \cos t}\right)\right] \end{aligned}$$

and

$$x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(y(t) * \frac{1}{2+\cos t}\right). \tag{20}$$

Hence, on the basis of Duhamel's formula (2), we obtain

$$x(t) = \frac{y(0)}{2+\cos t} + \int_0^t y'(t-\tau)\frac{1}{2+\cos\tau}\,\mathrm{d}\tau = \int_0^t y'(t-\tau)\frac{1}{2+\cos\tau}\,\mathrm{d}\tau. \tag{21}$$

From the formula (19) we get

$$\mathcal{L}[y(t)] = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

and

$$y(t) = 1 - \cos t,$$

which, based on formula (21), gives

$$\begin{aligned}
x(t) &= \int_0^t \frac{\sin(t-\tau)}{2+\cos\tau}\,\mathrm{d}\tau \\
&= \sin t \int_0^t \frac{\cos\tau}{2+\cos\tau}\,\mathrm{d}\tau + \cos t \int_0^t \frac{-\sin\tau}{2+\cos\tau}\,\mathrm{d}\tau \\
&= \sin t \int_0^t \left(1 - \frac{2}{2+\cos\tau}\right)\mathrm{d}\tau + \cos t\,\ln(2+\cos\tau)\Big|_0^t \\
&= t\sin t - 2\sin t \int_0^t \frac{1}{2+\cos\tau}\,\mathrm{d}\tau + \cos t\,\ln\frac{2+\cos t}{3}
\end{aligned} \tag{22}$$

but

$$\begin{aligned}
\int \frac{1}{2+\cos\tau}\,\mathrm{d}\tau &= \int \frac{1}{1+2\cos^2\frac{\tau}{2}}\,\mathrm{d}\tau = \int \frac{1}{\sin^2\frac{\tau}{2} + 3\cos^2\frac{\tau}{2}}\,\mathrm{d}\tau \\
&= \int \frac{1}{\cos^2\frac{\tau}{2}\left(\tan^2\frac{\tau}{2}+3\right)}\,\mathrm{d}\tau = \frac{2\sqrt{3}}{3}\arctan\frac{\tan\frac{\tau}{2}}{\sqrt{3}} + C, \quad C \in \mathbb{R},
\end{aligned}$$

which implies

$$x(t) = (\sin t)\Big(t - \frac{4\sqrt{3}}{3}\arctan\frac{\tan\frac{t}{2}}{\sqrt{3}}\Big) + \cos t \ln\frac{2+\cos t}{3}. \tag{23}$$

Remark 7 *If the Duhamel formula were not used for formula (20), the calculations would be significantly complicated by the final computation of the derivative of a function of a type such as (23). On the other hand, the direct application of the Laplace transform to the original equation immediately leads to formula (22), which significantly simplifies the calculations.*

The next example presents a solutions obtained by using of the Convolution Theorem only.

Example 3 We are looking for a solution to the following differential equation in the class of originals

$$x''(t) - 2x'(t) + x(t) = \frac{te^t}{(t^2+1)^2}, \tag{24}$$
$$x(0) = x'(0) = 0.$$

We assume that such solution $x(t)$, $t \geqslant 0$, exists. First we find the Laplace transforms of both sides of equation (24)

$$(s^2 - 2s + 1)\mathcal{L}[x(t)] = \mathcal{L}\left[\frac{te^t}{(t^2+1)^2}\right],$$

which implies

$$\mathcal{L}[x(t)] = \frac{1}{(s-1)^2}\mathcal{L}\left[\frac{te^t}{(t^2+1)^2}\right] = \mathcal{L}[te^t]\mathcal{L}\left[\frac{te^t}{(t^2+1)^2}\right].$$

Next, using the Convolution Theorem, we get

$$x(t) = \int_0^t (t-\tau)e^{t-\tau}\frac{\tau e^\tau}{(\tau^2+1)^2}\,\mathrm{d}\tau = e^t\int_0^t (t-\tau)\left(-\frac{1}{2(\tau^2+1)}\right)'\mathrm{d}\tau$$
$$= e^t\left[\frac{\tau - t}{2(\tau^2+1)}\Big|_0^t - \frac{1}{2}\int_0^t \frac{d\tau}{\tau^2+1}\right] = \frac{1}{2}e^t[t - \arctan t].$$

Note that $x(t)$, $t \geqslant 0$, is the original, which confirms the correctness of our assumption.

3.2. Estimating solutions of certain types of integral equations

The Duhamel formula can be successfully used to estimate the norms of solutions of LDE and selected integral equations. For example, let us consider the Volterra integral equation of the second kind with convolution kernel [36]

$$F(t) + \int_0^t K(t-\tau)F(\tau)\,\mathrm{d}\tau = G(t), \tag{25}$$

where K and G are given functions. Let $f(t)$ be a solution of the auxiliary equation

$$f(t) + \int_0^t K(t-\tau)f(\tau)\,\mathrm{d}\tau = 1. \tag{26}$$

Then for the solution $F(t)$ of (25) the following relation holds true by Duha-mel's formula

$$F(t) = \frac{\mathrm{d}}{\mathrm{d}t}(f * G(t)). \tag{27}$$

Using this formula, in [36] authors reported an interesting relationship between the respective norms of the functions F, f and G. They proved the following fact.

Theorem 3 *If $p > 1$, $f \in L_p(\rho)$ and $G \in W_p^1(\gamma)$ then we have $F \in L_p(\omega)$ and*

$$\|F\|_{L_p(\omega)} \leqslant \|f\|_{L_p(\rho)}\|G\|_{W_p^1(\gamma)}, \tag{28}$$

where

$$\omega(t) = \left[\int_0^t \rho^{1/(1-p)}(\tau)\gamma^{1/(1-p)}(t-\tau)\,\mathrm{d}\tau\right]^{1-p},$$

$t \in [a,b]$, ρ is a positive and continuous function on $[a,b]$, $L_p(\rho)$ denotes the Lebesgue space of complex valued measurable functions f on $[a,b]$ such that $\|f\|_{L_p(\rho)} < \infty$, where

$$\|f\|_{L_p(\rho)} := \left(\int_a^b |f(\tau)|^p\rho(\tau)\,\mathrm{d}\tau\right)^{1/p},$$

$\gamma\colon [a,b] \to \mathbb{R}_{\geqslant 0}$ and γ is absolutely continuous on $[a,b]$. The space $W_p^1(\gamma)$ consists of those and only those functions G which can be represented in the form

$G(t) = \int_a^t \varphi(\tau)d\tau$, $t \in [a,b]$ for some $\varphi \in L_p(\rho)$. We note that if we define on $W_p^1(\rho)$ a norm

$$\|G\|_{W_p^1(\rho)} := \left(\int_a^b \left|\frac{\mathrm{d}}{\mathrm{d}\tau}G(\tau)\right|^p \rho(\tau)dt\right)^{1/p}$$

then the pair $(W_p^1(\rho), \|\cdot\|_{W_p^1(\rho)})$ is a Banach space.

Remark 8 *In some interesting cases of integral equations (25) and (26), relations between functions F and f can be obtained in other, more specific form. For example, consider the following generalization of Duc and Nhan [11, Example 4] integral equation*

$$y(t) + \psi'(t)\int_0^t y(\tau)\mathrm{d}\tau = G(t)\left(\int_0^t y(\tau)\mathrm{d}\tau\right)^{\alpha}, \tag{29}$$

where $\psi\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, $\psi'(t) > 0$ for all $t \in \mathbb{R}_{\geqslant 0}$ and $\alpha \in \mathbb{R}$, $\alpha \neq 1$. After substitution

$$z(t) = \int_0^t y(\tau)\mathrm{d}\tau, \qquad t \in \mathbb{R}_{\geqslant 0},$$

the original integral equation (29) is reduced to the first-order linear nonhomogeneous ordinary differential equation (the Bernoulli differential equation)

$$z'(t) + \psi'(t)z(t) = G(t)(z(t))^{\alpha}. \tag{30}$$

The solution of this one has the form

$$z(t) = e^{\psi(0)-\psi(t)}\sqrt[1-\alpha]{(1-\alpha)\int_0^t G(\tau)e^{(1-\alpha)(\psi(\tau)-\psi(0))}\mathrm{d}\tau},$$

which implies the relation

$$\frac{z(t)}{\zeta(t)} = \sqrt[1-\alpha]{\frac{\int_0^t G(\tau)e^{(1-\alpha)(\psi(\tau)-\psi(0))}\mathrm{d}\tau}{\int_0^t e^{(1-\alpha)(\psi(\tau)-\psi(0))}\mathrm{d}\tau}} \tag{31}$$

if $\zeta(t)$ denotes the solution of (30) *for $G(t) = 1$. Identity* (31) *possesses the form corresponding to relation* (27) *and is even more adequate to get an inequality of the form* (28)*. It should be remembered that for the appropriate solutions $Y(t)$ and $y(t)$ of the integral equation* (29)*, we have the dependencies*

$$z(t) = \int_0^t Y(\tau)\,\mathrm{d}\tau \qquad \text{and} \qquad \zeta(t) = \int_0^t y(\tau)\,\mathrm{d}\tau.$$

Furthermore, if $z_1(t)$ and $z_2(t)$ are the solutions of equation (29) *for $G(t) = G_1(t)$ and $G(t) = G_2(t)$, respectively, then by* (30)*, the following relations hold*

$$(z_1(t))^{1-\alpha} - (z_2(t))^{1-\alpha} = (\zeta(t))^{1-\alpha} \frac{\int_0^t (G_1(\tau) - G_2(\tau)) e^{(1-\alpha)(\psi(\tau)-\psi(0))}\,\mathrm{d}\tau}{\int_0^t e^{(1-\alpha)(\psi(\tau)-\psi(0))}\,\mathrm{d}\tau}.$$

3.3. Linear systems approach

In this subsection, we propose the application of Duhamel's integral to the analysis of linear systems. Consider a linear system described by the following n-th order linear differential equation

$$\begin{aligned} &x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \cdots + a_1 x'(t) + a_0 x(t) \\ &\qquad = b_m u^{(m)}(t) + b_{n-1}u^{(m-1)}(t) + \cdots + b_1 u'(t) + b_0 u(t) \end{aligned} \tag{32}$$

for $t \geqslant 0$, with initial conditions

$$x(0) = \cdots = x^{(n-1)}(0) = u(0) = \cdots = u^{(m)}(0) = 0.$$

The function u is a given input (control) and x is unknown output (the response of the system) that is, the solution to be found, $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_m$ are real constants. Let x and u be originals.

Applying the Laplace transform to equation (32), we have

$$\begin{aligned} &\left(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right) X(s) \\ &\qquad = \left(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0\right) U(s), \end{aligned}$$

where $X(s) = \mathcal{L}[x(t)]$ and $U(s) = \mathcal{L}[u(t)]$.

Let us denote

$$K_x(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0,$$
$$K_u(s) = b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0.$$

Hence $K_x(s)X(s) = K_u(s)U(s)$ and the transfer function (kernel function) for the system (32) takes the form

$$K(s) = \frac{K_u(s)}{K_x(s)} = \frac{X(s)}{U(s)}.$$

Now we write an auxiliary equation

$$x_H^{(n)}(t) + a_{n-1}x_H^{(n-1)}(t) + \cdots + a_1 x'_H(t) + a_0 x_H(t) = H(t), \tag{33}$$

with zero initial conditions, where H is the Heaviside function

$$H(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geqslant 0. \end{cases}$$

Applying the Laplace transform to equation (33), we have

$$\left(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right) X_H = 1 \cdot \mathcal{L}[H(t)],$$

and the transfer function for the system (33) takes the form

$$K_H(s) = \frac{X_H(s)}{\mathcal{L}[H(t)]} = sX_H(s).$$

Hence

$$X_H(s) = \frac{K_H(s)}{s}.$$

The solution $x_H(t) = \mathcal{L}^{-1}[X_H(t)]$ to equation (33) is called the *indicial admittance*. Thus the solution to (32) is the following Duhamel's integral

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left[sX_H(s)(b_m s^m + \cdots + b_1 s + b_0)U(s)\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t x_H(\tau)v(t-\tau)\,\mathrm{d}\tau, \end{aligned} \tag{34}$$

where $v(t) = \mathcal{L}^{-1}[(b_m s^m + \cdots + b_1 s + b_0)U(s)]$.

Finally, using Duhamel's formula (3), the solution to (32) takes the form

$$x(t) = \int_0^t x'_H(\tau)v(t-\tau)\,\mathrm{d}\tau + x_H(0)v(t) = \int_0^t x'_H(\tau)v(t-\tau)\,\mathrm{d}\tau, \tag{35}$$

taking into account zero initial conditions.

Alternatively,

$$x(t) = \int_0^t x_H(\tau)v'(t-\tau)\,\mathrm{d}\tau + x_H(t)v(0)\int_0^t x_H(\tau)v'(t-\tau)\,\mathrm{d}\tau. \tag{36}$$

The following example illustrates the procedure.

Example 4 Solve the initial value problem

$$x''(t) - x'(t) = u'(t) + u(t)$$

$$x(0) = x'(0) = u(0) = u'(0) = 0$$

for

(a) $u(t) = \sin t$,

(b) $u(t) = e^t \sinh 2t$.

The auxiliary equation is

$$x''_H(t) - x'_H(t) = H(t).$$

Let $\mathcal{L}[x_H(t)] = X_H(s)$. Using the Laplace transformation method to the above equation, we obtain

$$X_H(s) = \frac{1}{s^2(s-1)}$$

and

$$x_H(t) = e^t - t - 1, \quad t \geqslant 0.$$

Hence $x'_H(t) = e^t - 1$.

Ad **(a)** We calculate

$$v(t) = \mathcal{L}^{-1}\left[(s+1)\frac{1}{s^2+1}\right] = \sin t + \cos t, \quad t \geqslant 0.$$

Hence, based on (35), the solution is

$$x(t) = \int_0^t (e^\tau - 1)(\sin(t-\tau) + \cos(t-\tau))\,\mathrm{d}\tau$$

$$= e^t \int_0^t (e^{-u} - e^{-t})(\sin u + \cos u)\,\mathrm{d}u = e^t - \sin t - 1, \quad t \geqslant 0,$$

since

$$\int e^{-u}(\sin u + \cos u)\,\mathrm{d}u = -e^{-u}\cos u + \text{const}\,.$$

Ad **(b)** Here

$$v(t) = \mathcal{L}^{-1}\left[(s+1)\frac{2}{(s-1)^2 - 4}\right] = 2e^{3t}, \quad t \geqslant 0.$$

Therefore, using formula (35), the solution for $t \geqslant 0$ takes the form

$$x(t) = 2\int_0^t (e^{t-\tau} - 1)e^{3\tau}\,\mathrm{d}\tau = \frac{1}{3}(e^{3t} - 3e^t + 2).$$

4. Semigroups of operators

Currently, Duhamel's formula has found an appropriate interpretation in the theory of operator semigroups [12, 13, 35, 51]. This formally advanced approach makes the Duhamel formula take on a new, fuller dimension.

Definition 1 *A family $\{T(t)\}_{t\geqslant 0}$ of linear operators bounded on a real or complex Banach space $\mathbb{X}$ is called a one-parameter strongly continuous semigroup or semigroup of C_0 class if it satisfies the conditions:*

1) $(\forall t, s \in \mathbb{R}_{\geqslant 0})\,(T(t+s) = T(t)T(s))$,

2) $T(0) = id_{\mathbb{X}}$,

3) $(\forall x \in \mathbb{X})\,($*function* $\mathbb{R}_{\geqslant 0} \ni t \mapsto T(t)x$ *is continuous with respect to usual topology on* $\mathbb{R})$.

In the last case we say that

$$\xi_x(t) := T(t)x$$

defined on $(x, t) \in \mathbb{X} \times \mathbb{R}_{\geqslant 0}$ *is strongly continuous.*

Families of operators of the form $\{e^{At}\}_{t\geqslant 0}$, where $A \in M_n(\mathbb{C})$, $n \in \mathbb{N}$, are standard examples of one-parameter strongly continuous semigroups.

Theorem 4 *Let $\{T(t)\}_{t\geqslant 0}$ be a family of linear operators bounded on a Banach space $\mathbb{X}$ over $\mathbb{R}$ or $\mathbb{C}$, satisfying the conditions 1) and 2) from definition (1). Then the following conditions are equivalent:*

1) the family $\{T(t)\}_{t\geqslant 0}$ is a semigroup of C_0 class,

2) $\lim_{t\to 0^+} T(t)x = x$ for all $x \in \mathbb{X}$,

3) there exist $\delta \in \mathbb{R}_{>0}$, $M \in \mathbb{R}$, $M \geqslant 1$ and a dense affine variety $D \subset \mathbb{X}$ such that

(a) $\|T(t)\| \leqslant M$ for all $t \in [0, \delta]$,

(b) $\lim_{t\to 0^+} T(t)x = x$ for all $x \in D$.

Theorem 5 *For each strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ there exist constants $M, \omega \in \mathbb{R}$, $M \geqslant 1$ such that*

$$\|T(t)\| \leqslant Me^{\omega t}$$

for all $t \geqslant 0$.

The fundamental concept related to each strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ is the concept of a generator (infinitesimal generator) of this semigroup.

Definition 2 *The operator*

$$Ax := \lim_{h\to 0^+} \frac{1}{h}(T(h)x - x)$$

defined for each $x \in \mathbb{X}$ for which the above right-hand limit exists is called a generator (infinitesimal generator) A of a given strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ defined on the real or complex Banach space $\mathbb{X}$. The set of all $x \in \mathbb{X}$ for which the limit exists is denoted by $D(A)$ and called the domain of operator A.

Theorem 6 *Let A be a generator of the strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$. Then the following statements are true:*

1) $D(A)$ is the affine variety dense in $\mathbb{X}$,

2) $A\colon D(A) \to \mathbb{X}$ is the linear operator, *3) for every $x \in D(A)$ and $t \in \mathbb{R}_{\geqslant 0}$ it holds $T(t)x \in D(A)$. While for every $x \in \mathbb{X}$ and $t \in \mathbb{R}_{\geqslant 0}$ it holds $\int_0^t T(s)xds \in D(A)$. Moreover, for every $x \in D(A)$ and $t \in \mathbb{R}_{\geqslant 0}$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}T(t)x = T(t)Ax = AT(t)x$$

and

$$T(t)x - x = \begin{cases} A\displaystyle\int_0^t T(s)x\mathrm{d}s, & \text{for } x \in \mathbb{X}, \\ \displaystyle\int_0^t T(s)Ax\mathrm{d}s, & \text{for } x \in D(A). \end{cases}$$

Theorem 7 *The generator of a strongly continuous semigroup is a closed operator and defines this semigroup uniquely.*

Since now we shall call denote each strongly contiguous semigroup $\{T(t)\}_{t\geqslant 0}$ by S_A, where A is the generator of this semigroup and write $S_A(t)$ instead of $T(t)$ for all $t \in \mathbb{R}_{\geqslant 0}$.

Duhamel's formula in the semigroup theory has been defined and proved by S. Mischler in [35] as follows.

Theorem 8 *(Duhamel's formula) Let S_A and S_B be two operator semigroups on the same Banach space $\mathbb{X}$. Let us suppose that domains of A and B are equal and*

$$\Lambda S_B,\ S_B\Lambda \in L^1([0,T];\ \mathcal{B}(\mathbb{X}))$$

for any $T \in \mathbb{R}_{>0}$, where $\Lambda := A - B$. Then we have in $\mathcal{B}(\mathbb{X})$

$$S_A = S_B + S_A * \Lambda S_B = S_B + S_B\Lambda * S_A.$$

Proof. Let us take $x \in D(A) = D(B)$, $t > 0$, and define the mapping

$$s \mapsto u(s) := S_A(s)S_B(t-s)x \in C^1([0,t];\ \mathbb{X}) \cap C([0,t];\ D(A)).$$

Then we obtain

$$u'(s) = S_A(s)AS_B(t-s)x - S_A(s)BS_B(t-s)x = S_A(s)\Lambda S_B(t-s)x$$

for any $s \in (0, t)$. Hence we deduce the equality

$$S_A(t)x - S_B(t)x = \int_0^t u'(s)\,\mathrm{d}s = \int_0^t S_A(t-s)\Lambda S_B(s)x\,\mathrm{d}s.$$

By density $C^1([0,t];\mathbb{X}) \cap C([0,t];D(A))$ in $C([0,t];D(A))$, and continuity of $S_A(t)$ and $S_B(t)$ it follows that the same formula holds for any $x \in \mathbb{X}$, which implies the equality $S_A = S_B + S_A * \Lambda S_B$. The second equality follows by reversing the role of S_A and S_B. □

5. Practical applications

In this section some examples of control systems for which Duhamel's integral has been applied are presented.

Example 5 Consider the LC electrical circuit consisting of an inductor (represented by L) and a capacitor (represented by C) connected together, shown in Fig. 1.

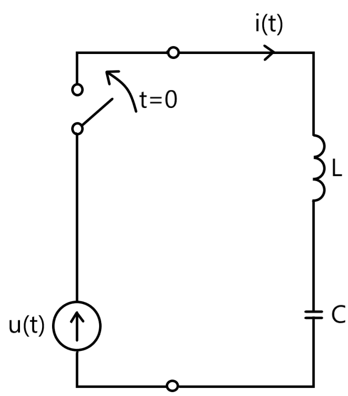

Figure 1: LC electrical circuit

The system is described by the following integro-differential equation [37]

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}\int_0^t i\,\mathrm{d}t = u(t), \qquad t \geqslant 0$$

with zero initial condition, where i is a current intensity, u is a voltage, L and C are positive constants.

In [37], rather rarely used the Laplace-Carlson transformation was applied to solve the equation. We propose to use the Laplace transformation method.

The auxiliary equation is

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}\int_0^t i\,\mathrm{d}t = H(t).$$

Let $\mathcal{L}[i(t)] = I(s)$ and $\mathcal{L}[u(t)] = U(s)$. Applying the Laplace transformation, we have

$$sLI(s) + \frac{1}{sC}I(s) = \frac{1}{s}.$$

We calculate

$$I_H(s) = \frac{C}{CLs^2 + 1}$$

and then

$$i_H(t) = \sqrt{\frac{C}{L}}\sin\frac{t}{\sqrt{CL}}.$$

In order to use formula (35), we differentiate

$$\frac{\mathrm{d}i_H}{\mathrm{d}t} = \frac{1}{L}\cos\frac{t}{\sqrt{CL}}.$$

Denoting $\omega_0^2 = \frac{t}{CL}$ (ω_0 represents frequency), we obtain the Duhamel integral

$$i(t) = \frac{1}{L}\int_0^t u(\tau)\cos\omega_0(t-\tau)\,\mathrm{d}\tau.$$

It is a formula that expresses the dependence of the current i on the applied voltage u in the circuit under consideration.

Example 6 *Consider the following equation of motion that govern the behaviour of a single-degree-of-freedom system [6]*

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\vartheta\omega_0\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x(t) = -\frac{\mathrm{d}^2u}{\mathrm{d}t^2}, \tag{37}$$

where x, $\frac{\mathrm{d}x}{\mathrm{d}t}$, $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ are the relative displacement, velocity and acceleration of the SDOF system, respectively, ω_0 is the angular frequency, ϑ is the viscous damping factor, $\frac{\mathrm{d}^2u}{\mathrm{d}t^2}$ represents the base acceleration.

If we assume zero initial conditions $x(0) = x'(0) = 0$, then the solution of equation (37) at fixed time $t \in (0, +\infty)$ is the following Duhamel integral

$$x(t) = -\int_0^t u''(\tau)h(t-\tau)\,\mathrm{d}\tau,$$

where $\omega_a = \omega_0\sqrt{1-\vartheta^2}$, h denotes the impulse response of system, and $h(t-\tau) = \frac{1}{\omega_a}e^{-\vartheta\omega_0(t-\tau)}\sin\omega_0(t-\tau)$.

Example 7 *Consider inhomogeneous wave equation on the infinite string*

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = F(t,x), \tag{38}$$

$$u(0,x) = f(x), \qquad \frac{\partial}{\partial t}u(0,x) = g(x),$$

where F, f, g are given sufficiently smooth functions. We decompose the system into two simpler problems, that is the initial value problem for the homogeneous wave equation and inhomogeneous problem with zero initial conditions:

$$\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} = 0, \tag{39}$$

$$u(0,x) = f(x), \qquad \frac{\partial u}{\partial t}(0,x) = g(x)$$

and

$$\frac{\partial^2 w}{\partial t^2} - c^2\frac{\partial^2 w}{\partial x^2} = F(t,x), \tag{40}$$

$$w(0,x) = 0, \qquad \frac{\partial w}{\partial t}(0,x) = 0.$$

Of course, if v is a solution to (39) and w is a solution to (40), then $u(t,x) = v(t,x) + w(t,x)$ solves (38). The problem (39) is easy to solve by the d'Alembert formula whereas solving the problem (40) is based on Duhamel's integral. To find the solution w let us consider an auxiliary problem

$$\frac{\partial^2 r}{\partial t^2} - c^2\frac{\partial^2 r}{\partial x^2} = 0, \qquad t > \tau, \tag{41}$$

$$r(\tau,\tau,x) = 0, \qquad \frac{\partial r}{\partial t}(\tau,\tau,x) = F(\tau,x),$$

which solution is the function $r = r(t,\tau,x)$.

We will show that

$$w(t,x) = \int_0^t r(t,\tau,x)\,\mathrm{d}\tau$$

is the solution to (40). Indeed, since

$$\int_0^t r(t,\tau,x)\,\mathrm{d}\tau = \int_0^t 1 \cdot r(t,\tau,x)\,\mathrm{d}\tau = H(t) * r(t,\tau,x),$$

we have Duhamel's integral

$$\frac{\partial w}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t r(t,\tau,x)\,\mathrm{d}\tau \overset{(3)}{=} r(t,\tau,x) + \int_0^t \frac{\partial}{\partial t} r(t,\tau,x)\,\mathrm{d}\tau.$$

It follows that

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t} r(t,\tau,x) + \int_0^t \frac{\partial^2}{\partial t^2} r(t,\tau,x)\,\mathrm{d}\tau = F(t,x) + \int_0^t \frac{\partial^2}{\partial t^2} r(t,\tau,x)\,\mathrm{d}\tau.$$

Moreover,

$$\frac{\partial^2 w}{\partial x^2} = \int_0^t \frac{\partial^2}{\partial x^2} r(t,\tau,x)\,\mathrm{d}\tau = \frac{1}{c^2}\int_0^t \frac{\partial^2}{\partial t^2} r(t,\tau,x)\,\mathrm{d}\tau.$$

Thus we have $\dfrac{\partial^2 w}{\partial t^2} - c^2\dfrac{\partial^2 w}{\partial x^2} = F(t,x)$ and $w(0,x) = \dfrac{\partial w}{\partial t}(0,x) = 0$, which needed to be shown.

6. Fractional-order case

In recent years, there has been a large increase in the number of publications on fractional order integro-differential calculus. Control systems described by differential equations with fractional order derivatives were discussed, among others, in [2, 5, 16, 21, 22, 25, 39, 41–47]. However, there are few publications on the usage of Duhamel's integral to equations with fractional order derivative. For example, S. Umarov and E. Saydamatov in [49] consider a generalization of Duhamel's principle in the case of fractional differential equations. Here the Duhamel principle for the Caputo fractional derivative is studied while in [48] a solution to equations with the Riemann-Liouville derivative is presented. Nevertheless, it has to be pointed out that initial value problems with the Riemann-Liouville derivatives has no unequivocal physical interpretation. In the case of the Caputo derivative, initial conditions can be interpreted analogously to the case of integer-order derivatives. Therefore, we present the Duhamel method for solving differential equations with the Caputo fractional derivative and pseudo-differential operator.

Fractional Duhamel's principle in the case of $0 < \alpha < 1$, the Caputo fractional derivarive

$$ {}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,\mathrm{d}\tau, $$

and the pseudo-differential operator $A(D_x)$ defined on $\mathbb{R}^n$, where $D_x = (D_1, \ldots, D_n)$, $D_j = \dfrac{\partial}{\partial x_j}$, $j = 1, \ldots, n$, can be formulated as the following theorem.

Theorem 9 *[49] Suppose that $V(t, \tau, x), 0 \leqslant \tau \leqslant t,\ x \in \mathbb{R}^n$ is a solution of the Cauchy problem for homogeneous equation*

$$ {}^{C}D^{\alpha} V(t, \tau, x) - A(D_x) V(t, \tau, x) = 0 \tag{42} $$

$$ V(\tau, \tau, x) = {}^{C}D^{1-\alpha} f(\tau, x), \tag{43} $$

where $f = f(t, x)$ is a given function satisfying the condition $f(0, x) = 0$. Then the convolution integral

$$ v(t, x) = \int_{0}^{t} V(t, \tau, x)\,\mathrm{d}\tau \tag{44} $$

is a solution of the inhomogeneous Cauchy problem

$$ {}^{C}D^{\alpha} v(t, x) - A(D_x) v(t, x) = f(t, x) \tag{45} $$

$$ v(0, x) = 0. $$

Moreover, for linear fractional-order differential equations it is possible to use the method applying Duhamel's integral discussed in Subsection 3.3. The following example presents the usage of the method for a linear system described by a fractional differential equation with the Caputo derivative and zero initial condition.

Example 8 Consider a damper-spring control system described by the following fractional-order differential equation (see for example: [15])

$$ {}^{C}D^{\alpha}x(t) + \eta x(t) = \frac{\eta}{k}u(t), \qquad x(0) = 0, \tag{46} $$

$t \geqslant 0$, where $\alpha \in (0, 1)$, $\eta = \frac{k\sigma^{1-\alpha}}{\beta}$, k is the spring constant, β is the damping coefficient, σ is an auxiliary parameter of the system with

(a) a constant source (control) $u(t) = f_0$,

(b) a periodic source $u(t) = f_0 \cos(\omega t)$,

f_0 and ω are given constants.

It is easy to verify that formula (34) holds also for linear systems with constant coefficients with the Caputo derivative. For the system (46) we obtain the solution in the form of Duhamel's integral

$$ x(t) = \mathcal{L}^{-1}\left[sX_H(s)\frac{\eta}{k}U(s)\right] = \frac{\eta}{k}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t x_H(\tau)u(t-\tau)\,\mathrm{d}\tau, \tag{47} $$

where $\mathcal{L}[u(t)] = U(s)$, $\mathcal{L}[x_H(t)] = X_H(s)$, and x_H is a solution to the auxiliary equation

$$ {}^{C}D^{\alpha}x_H(t) + \eta x_H(t) = H(t), \qquad x_H(0) = 0. \tag{48} $$

Next, using formula (35), we have

$$ x(t) = \frac{\eta}{k}\int_0^t x'_H(\tau)u(t-\tau)\,\mathrm{d}\tau. \tag{49} $$

Let us find the solution x_H to the auxiliary equation (48). Applying the Laplace transform method, we have

$$ s^{\alpha}X_H(s) + \eta X_H(s) = \frac{1}{s} $$

and

$$ X_H(s) = \frac{1}{s(s^{\alpha}+\eta)}. $$

Finally,

$$ \begin{aligned} x_H(t) &= \mathcal{L}^{-1}\left[\frac{1}{s}\right] * \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}+\eta}\right] = H(t) * t^{\alpha-1}E_{\alpha,\alpha}(-\eta t^{\alpha}) \\ &= \int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\eta\tau^{\alpha})\,\mathrm{d}\tau, \end{aligned} $$

where $E_{\alpha,\alpha}$ is the two-parameter Mittag-Leffler function, e.g.

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{50}$$

for $\alpha, \beta \in \mathbb{R}_+$, $z \in \mathbb{C}$.

Hence, substituting x_H into (49), the solution to (46) takes the form

$$x(t) = \frac{\eta}{k} \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(-\eta\tau^\alpha) u(t-\tau)\,\mathrm{d}\tau. \tag{51}$$

It follows that the solution of the Cauchy problem **(a)** is

$$x(t) = \frac{\eta f_0}{k} \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(-\eta\tau^\alpha)\,\mathrm{d}\tau$$

and for **(b)** we have

$$x(t) = \frac{\eta f_0}{k} \int_0^t \tau^{\alpha-1} \cos\omega(t-\tau) E_{\alpha,\alpha}(-\eta\tau^\alpha)\,\mathrm{d}\tau.$$

7. Concluding remarks

Properties and applications of Duhamel's integral have been discussed in the paper. Theoretical foundations on the ground of the theory of operator semigroups have been studied. Numerical examples that illustrate theoretical considerations, estimating the norms of solutions of selected integral equations as well as examples of practical applications for control systems have been presented. Application of Duhamel's integral and the Duhamel principle in the case of fractional-order systems has been also provided in the paper.

In addition to the summary of known facts about Duhamela's integral, the paper contains a number of original results. The new proof of Theorem 1 have been proposed. All the proofs of this theorem known to the authors have a completely different course, they are based on different assumptions. Subsection 3.1 contains original comments on the use of the Duhamel formula to solve linear differential equations with constant coefficients, which was presented from the practical side in Examples 2 and 3. Remark 8 contains new results on nonlinear integral equation, where formula (31) has been proposed as an equivalent to Duhamel's formula (28), and which allows for obtaining inequalities of type (28) more efficiently. New is also the procedure basing on Duhamel's integral for the analysis of linear systems presented in Subsection 3.3. The proof of Theorem 8 is partly original.

The strengths of the work are numerous examples. All numerical examples are invented by the authors. Especially interesting is Example 1. Moreover, in Section 5 (for integer order case) and Section 6 (for fractional order case) some new solutions of known problems have been proposed.

The study is a unique compendium of knowledge about Duhamel's integral.

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