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On optimal control problem subject to fractional order discrete time singular systems

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In this work, we present optimal control formulation and numerical algorithm for fractional order discrete time singular system (DTSS) for fixed terminal state and fixed terminal time endpoint condition. The performance index (PI) is in quadratic form, and the system dynamics is in the sense of Riemann-Liouville fractional derivative (RLFD). A coordinate transformation is used to convert the fractional-order DTSS into its equivalent non-singular form, and then the optimal control problem (OCP) is formulated. The Hamiltonian technique is used to derive the necessary conditions. A solution algorithm is presented for solving the OCP. To validate the formulation and the solution algorithm, an example for fixed terminal state and fixed terminal time case is presented.

Key words: fractional optimal control problem, discrete time singular system, fractional derivative, Hamiltonian technique

1. Introduction

Fractional derivatives (FDs) represent derivatives of arbitrary real order. Generalization of the calculus of integer order is the fractional calculus (FC). It is a branch of mathematics concerned with the study of FDs and integrals. FDs

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find its application in every branch of science and engineering. The concept of FC is as old as calculus of integer order. The origins of FDs theory may be tracked back to a remark in Leibniz's list to L'Hospital in which the meaning of half order derivative is discussed [1]. FD is not a point property, perhaps for this reason it is an outstanding tool for the depiction of memory and hereditary effects of distinct systems. This is the primary benefit of FD as in integer order representation aforesaid characteristics are ignored [2]. From the standpoint of applicability, FDs appear in electrochemistry, viscoelasticity, control theory, mechatronics, electrical engineering, image processing, biophysics, mechanics, rheology, signal processing, biology, bioengineering, economics, etc. [2–4].

This work deals with the application of FDs in optimal control problem (OCP). The calculation of optimal control subject to system constraints in order to optimize the provided PI is defined as OCP [5]. When PI, or dynamic constraints, or both in OCP has one FD term, the problem is referred to be fractional optimal control problem (FOCP) [6].

Notable work has been reported in literature regarding OCP of fractional order continuous time non-singular systems. In this context, Agrawal [6] formulated FOCPs described by Riemann-Liouville fractional derivative (RLFD) or Caputo FD (CFD). Numerical approaches are used to solve the optimal conditions. Chiranjeevi and Biswas [7] presented different case studies of FOCPs with constraints on control. Biswas and Sen [8,9], proposed formulation for fixed terminal time and free terminal time FOCPs. For the solution of state and control, shooting method and Grünwald-Letnikov approximation-based techniques have been used. Biswas and Sen [10] proposed a reflection operator based solution algorithm for FOCPs described by RLFD or CFD. Tricaud and Chen [11] introduced formulation of FOCP using Oustaloup recursive approximation. Ding et al. [12] proposed formulation and solution of FOCP of human immunodeficiency virus epidemic model. They showed that fractional optimal control can offer improved quality of treatment as compared to integer order optimal control. Analytical method for solving conformable FOCPs is proposed by Chiranjeevi and Biswas [13]. Dehghan et al. [14–16] proposed different numerical schemes based on modified Jacobi polynomials, semidefinite programming approach and collocation method, and properties of the Legendre multiwavelets for solving FOCPs. Gomoyunov [17], presented FOCP described by CFD for optimizing Bolza type PI.

All the above reported works in literature consider continuous time FOCPs. Discrete time FOCP works reported in literature are very limited. In this respect, discrete time FOCP for fixed terminal state and free terminal state systems have been discussed in [18–20]. In these papers, RLFD and CFD are considered, and PI is given in a quadratic form. Solution method presented is similar to integer order solution procedure of OCP. Czyronis [21] proposed dynamic programming approach for solving FOCP of discrete time system. Trujillo and Ungureanu [22] develop formulation and solution of FOCP of stochastic discrete time systems. Ruszewski [23] considered the discrete time fractional order system stability.

The above mentioned FOCPs are dealt with non-singular systems. Singular systems (SSs) are of some special interest because they have some special features like “noncausality, nonproperness of transfer matrix, consistent initial conditions, input derivatives in the state dynamics”, etc. [24]. We find SSs in several areas like social, engineering, economic and biological systems [24].

Reported work in literature on FOCP of continuous time SSs is not much. In this regard, ‘pseudo state space’ formulation for FOCP of continuous time SS is presented in [25, 26]. In literature, formulation and different solution techniques, e.g., Grünwald-Letnikov approximation based technique described by RLFD [27–29] or CFD [30–32] and reflection operator based technique described by RLFD or CFD [33, 34] for FOCP of continuous time SSs are presented. Kaczorek [35] proposed coordinate transformation to convert fractional order DTSS into its equivalent non-singular form. However, FOCPs of discrete time SSs is still an open research area.

In this paper, we present OCP of DTSS using the coordinate transformation. DTSS is transformed into an equivalent non-singular form, and then OCP is formulated. An example is considered to validate the formulation and solution algorithm. According to the state of the art, this is the first time that a formulation and simulation algorithm of OCP of fractional order DTSS with fixed terminal state and fixed terminal time end point condition are presented.

The rest of the article is structured as follows: OCP formulation for fractional order DTSS is presented in Section 2; In Section 3, solution algorithm is presented for solving FOCP; Numerical analysis is given in Section 4 for validating the effectiveness of formulation and solution algorithm; Finally, conclusions of the work are given in Section 5.

2. OCP formulation of fractional order DTSS

Consider the fractional order DTSS expressed in terms of FDEs as Eq. (1) [36]

$$E\Delta^{\alpha}x(k+1) = Ax(k) + Bu(k), \quad k \in \mathbb{Z}_{+} = \{0, 1, \ldots\}, \tag{1}$$

where Δ^{α} is the difference operator, described by Eq. (2) [36]

$$\Delta^{\alpha}x(k) = \sum_{\varsigma=0}^{k}(-1)^{\varsigma}\binom{\alpha}{\varsigma}x(k-\varsigma). \tag{2}$$

We consider quadratic PI as Eq. (3)

$$J = \sum_{k=0}^{N-1} \left[x^T(k) M x(k) + u^T(k) S u(k) \right], \tag{3}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $E \in \Re^{n\times n}$ is the singular matrix, $x \in \Re^{n\times 1}$, $u \in \Re^{m\times 1}$, $M \in \Re^{n\times n} > 0$ and $S \in \Re^{m\times m} > 0$.

The problem is defined as follows: the calculation of optimal control subject to system constraints in order to optimize the provided PI is defined as OCP.

We may describe feedback control law by considering the fractional order system's stabilizability and detectability properties [30] as Eq. (4)

$$u(k) = \mathrm{K}x(k) + w(k), \tag{4}$$

where $\mathrm{K} \in \Re^{m\times n}$ is the feedback gain matrix, and $w \in \Re^{m\times 1}$ is the new control vector.

We may construct K to satisfy the relationship given by Eq. (5)

$$\deg\left(|zE - (A + B\mathrm{K})|\right) = rank(E).$$

By using Eq. (4) in Eq. (1), we obtain Eq. (5)

$$E\Delta^{\alpha} x(k+1) = (A + BK)\, x(k) + Bw(k). \tag{5}$$

Two non-singular matrices F and G may be constructed to satisfy the Eq. (6) by considering the Lemma mentioned in Eq. (6) [24, 30]

$$FEG = diag(I_{n_1}, \mathrm{O}), \qquad F(A + BK)G = diag(\widetilde{\Lambda}, I_{n_2}) \tag{6}$$

where $\widetilde{\Lambda} \in \Re^{n_1\times n_1}$ is a new state matrix, $n_1 = rank(E)$, $\mathrm{O} \in \Re^{n_2\times n_2}$ is a nilpotent matrix and $n_1 + n_2 = n$.

We may choose coordinate transformation [24, 30]

$$x(k) = G \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \qquad x_1 \in \Re^{n_1}, \quad x_2 \in \Re^{n_2}. \tag{7}$$

By taking into account Eq. (6) and Eq. (7), the Eq. (5) is modified as Eq. (8)

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \Delta^{\alpha} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \widetilde{\Lambda} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} w(k). \tag{8}$$

By considering Eq. (2), the Eq. (8) is modified as Eq. (9)

$$\begin{gathered} x_1(k+1) = \sum_{i=0}^{k} d(i) x_1(k-i) + B_1 w(k), \\ 0 = x_2(k) + B_2 w(k). \end{gathered} \tag{9}$$

We obtain the following equation by considering Eq. (7) and Eq. (9) as Eq. (10)

$$\begin{aligned}\begin{bmatrix} x(k) \\ u(k) \end{bmatrix} = \begin{bmatrix} I & 0 \\ K & I \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} = \begin{bmatrix} G & 0 \\ KG & I \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ w(k) \end{bmatrix} \\ = \begin{bmatrix} G & 0 \\ KG & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -B_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1(k) \\ w(k) \end{bmatrix}.\end{aligned} \tag{10}$$

By using Eq. (10), we can modify Eq. (3) as

$$\begin{aligned} J &= \sum_{k=0}^{N-1} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} M & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \\ &= \sum_{k=0}^{N-1} \left\{ \begin{bmatrix} x_1(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -B_2 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} G & 0 \\ KG & I \end{bmatrix}^T \begin{bmatrix} M & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} G & 0 \\ KG & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -B_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1(k) \\ w(k) \end{bmatrix} \right\} \\ &= \sum_{k=0}^{N-1} \begin{bmatrix} x_1(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} \bar{M} & \Im \\ \Im^T & \bar{S} \end{bmatrix} \begin{bmatrix} x_1(k) \\ w(k) \end{bmatrix} \\ &= \sum_{k=0}^{N-1} \left[x_1^T(k)\bar{M}x_1(k) + x_1^T(k)\Im w(k) + w^T(k)\Im^T x_1(k) + w^T(k)\bar{S}w(k) \right]. \end{aligned}$$

Finally, PI becomes as Eq. (11)

$$J = \sum_{k=0}^{N-1} \left[x_1^T(k)\widetilde{M}x(k) + v^T(k)\bar{S}v(k) \right], \tag{11}$$

where $\widetilde{M} = \bar{M} - \Im\bar{S}^{-1}\Im^T$, $v(k) = w(k) + \bar{S}^{-1}\Im^T x_1(k)$.

Substitute $w(k) = v(k) - \bar{S}^{-1}\Im^T x_1(k)$ in Eq. (8), we get Eq. (12)

$$x_1(k+1) = \sum_{\varsigma=0}^{k} c(\varsigma)x_1(k-\varsigma) + B_1 v(k), \tag{12}$$

where $c(0) = \widetilde{\Lambda} + \alpha I - B_1\bar{S}^{-1}\Im^T$, $c(\varsigma) = (-1)^\varsigma \begin{pmatrix} \alpha \\ \varsigma + 1 \end{pmatrix})I$, $\varsigma = 1, 2, \ldots, k$.

Therefore, the fractional order DTSS given by Eq. (1) is transformed into its equivalent non-singular form given by Eq. (12). Now, we obtain the formulation of FOCP by considering the Eq. (11) and Eq. (12) as PI and system dynamics.

We can define augmented PI in terms of Lagrange multiplier λ as

$$J_a = \sum_{k=0}^{N-1}\Bigg(x_1^T(k)\widetilde{M}x_1(k) + v^T(k)\bar{S}v(k) + \left[\sum_{\varsigma=0}^{k} c(\varsigma)x_1(k-\varsigma) + B_1 v(k) - x_1(k+1)\right]^T \lambda(k+1)\Bigg).$$

Hamiltonian function is be defined as

$$H(k) = x_1^T(k)\widetilde{M}x_1(k) + v^T(k)\bar{S}v(k) + \left[\sum_{\varsigma=0}^{k} c(\varsigma)x_1(k-\varsigma) + B_1 v(k)\right]^T \lambda(k+1).$$

Augmented PI in terms of Hamiltonian is given by the following equation

$$J_a = x_1^T(0)\lambda(0) - x_1^T(N)\lambda(N) + \sum_{k=0}^{N-1}\left[H(k) - x_1^T(k)\lambda(k)\right].$$

We can write the first variation of J_a as

$$\delta J_a = -\lambda(N)\delta x_1^T(N) + \sum_{k=0}^{N-1}\Bigg[\left(\frac{\partial H(k)}{\partial x_1^T(k)} - \lambda(k)\right)\delta x_1^T(k) + \frac{\partial H(k)}{\partial v^T(k)}\delta v^T(k) + \left(\frac{\partial H(k-1)}{\partial \lambda^T(k)} - x_1(k)\right)\delta\lambda^T(k)\Bigg].$$

For optimum $\delta J_a = 0$ [5]. Therefore, the necessary conditions are given by Eq. (13)–(15)

$$x_1(k+1) = \frac{\partial H(k)}{\partial \lambda^T(k+1)} = \sum_{\varsigma=0}^{k} c(\varsigma)x_1(k-\varsigma) + B_1 v(k), \tag{13}$$

$$\lambda(k) = \sum_{k=0}^{N-1}\frac{\partial H(k)}{\partial x_1^T(k)} = \left[\widetilde{M} + \widetilde{M}^T\right]x_1(k) + \sum_{\varsigma=0}^{N-k-1} c^T(\varsigma)\lambda(k+\varsigma+1), \tag{14}$$

$$\frac{\partial H(k)}{\partial v^T(k)} = 0 \;\Rightarrow\; v(k) = -\left[\bar{S} + \bar{S}^T\right]^{-1} B_1^T \lambda(k+1). \tag{15}$$

Finally, δJ_a becomes $-\lambda(N)\delta x_1^T(N) = 0$.

3. Solution algorithm

In this section, we use a general solution technique [18–20] for the solution of fixed terminal state problem. Applying the z-transform to Eq. (13), assuming that initial condition $x(k=0)=x_0$ is given, and then using inverse z-transform, we can obtain the solution of state equation as Eq. (16)

$$x_1(k)=\varphi(k)x_1(0)-\sum_{\varsigma=0}^{k-1}\varphi(k-\varsigma-1)B_1\left[\bar{S}+\bar{S}^T\right]^{-1}B_1^T\lambda(\varsigma+1), \tag{16}$$

where $\varphi(0)=I_n$, $\varphi(k)=\sum_{\varsigma=0}^{k-1}c(\varsigma)\varphi(k-\varsigma-1)$.

The vector representation of Eq. (16) is given by Eq. (17)

$$\begin{bmatrix} x_1(1)\\ \vdots \\ x_1(N)\end{bmatrix}=\begin{bmatrix}\varphi(1)\\ \vdots\\ \varphi(N)\end{bmatrix}x_1(0) + \begin{bmatrix}\varphi(0) & \cdots & 0\\ \vdots & \ddots & \vdots\\ \varphi(N-1) & \cdots & \varphi(0)\end{bmatrix}\left[-B_1\left[\bar{S}+\bar{S}^T\right]^{-1}B_1^T\right]\begin{bmatrix}\lambda(1)\\ \vdots\\ \lambda(N)\end{bmatrix} \tag{17}$$

The vector representation of Eq. (14) is obtained by Eq. (18)

$$\begin{bmatrix}\lambda(1)\\ \vdots\\ \lambda(N)\end{bmatrix}=\begin{bmatrix}\varphi^T(N-1)\\ \vdots\\ \varphi^T(0)\end{bmatrix}\lambda(N) + \begin{bmatrix}0 & \varphi^T(0) & \cdots & \varphi^T(N-2)\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \varphi^T(0)\\ 0 & 0 & \cdots & 0\end{bmatrix}\left[\widetilde{M}+\widetilde{M}^T\right]\begin{bmatrix}x_1(0)\\ \vdots\\ x_1(N-1)\end{bmatrix}. \tag{18}$$

For fixed terminal state $\delta x_1^T(N)=0$, there is no transversality condition. This problem is solved with given boundary conditions $x_1(0)$ and $x_1(N)$. We can write Eq. (18) by eliminating $x_1(0)$ as Eq. (19)

$$\begin{bmatrix}\lambda(1)\\ \vdots\\ \lambda(N)\end{bmatrix}=\begin{bmatrix}\varphi^T(N-1)\\ \vdots\\ \varphi^T(0)\end{bmatrix}\lambda(N)+\begin{bmatrix}\varphi^T(0)\ell_1 & \dots & \varphi^T(N-2)\ell_1 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & \varphi^T(0)\ell_1 & 0\\ 0 & \dots & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(1)\\ \vdots\\ x_1(N)\end{bmatrix} \tag{19}$$

where $\ell_1=\left[\widetilde{M}+\widetilde{M}^T\right]$.

By using Eq. (19) in Eq. (17), and simplifying, we get Eq. (20)

$$\begin{bmatrix} x_1(1) \\ \vdots \\ x_1(N) \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1N} \\ \varepsilon_{21} & \varepsilon_{22} & \cdots & \varepsilon_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{N1} & \varepsilon_{N2} & \cdots & \varepsilon_{NN} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \varphi(1) \\ \vdots \\ \varphi(N) \end{bmatrix} x_1(0) \right.$$
$$\left. + \begin{bmatrix} \varphi(0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \varphi(N-1) & \cdots & \varphi(0) \end{bmatrix} \left[-B_1 \ell_2^{-1} B_1^T \right] \begin{bmatrix} \varphi^T(N-1) \\ \vdots \\ \varphi^T(0) \end{bmatrix} \lambda(N) \right\}$$
$$= \begin{bmatrix} \sigma(1) \\ \vdots \\ \sigma(N) \end{bmatrix} x_1(0) - \begin{bmatrix} \theta(1) \\ \vdots \\ \theta(N) \end{bmatrix} \lambda(N), \tag{20}$$

where

$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1N} \\ \varepsilon_{21} & \varepsilon_{22} & \cdots & \varepsilon_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{N1} & \varepsilon_{N2} & \cdots & \varepsilon_{NN} \end{bmatrix}^{-1} = \left[\begin{bmatrix} I & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & I \end{bmatrix} \right.$$
$$\left. + \begin{bmatrix} \varphi(0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \varphi(N-1) & \cdots & \varphi(0) \end{bmatrix} \left[B_1 \ell_2^{-1} B_1^T \right] \begin{bmatrix} \varphi^T(0)\ell_1 & \dots & \varphi^T(N-2)\ell_1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \varphi^T(0)\ell_1 & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix} \right]^{-1},$$

$$\begin{bmatrix} \sigma(1) \\ \vdots \\ \sigma(N) \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1N} \\ \varepsilon_{21} & \varepsilon_{22} & \cdots & \varepsilon_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{N1} & \varepsilon_{N2} & \cdots & \varepsilon_{NN} \end{bmatrix}^{-1} \begin{bmatrix} \varphi(1) \\ \vdots \\ \varphi(N) \end{bmatrix},$$

$$\begin{bmatrix} \theta(1) \\ \vdots \\ \theta(N) \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1N} \\ \varepsilon_{21} & \varepsilon_{22} & \cdots & \varepsilon_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{N1} & \varepsilon_{N2} & \cdots & \varepsilon_{NN} \end{bmatrix}^{-1} \begin{bmatrix} \varphi(0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \varphi(N-1) & \cdots & \varphi(0) \end{bmatrix}$$
$$\cdot \left[-B_1 \ell_2^{-1} B_1^T \right] \begin{bmatrix} \varphi^T(N-1) \\ \vdots \\ \varphi^T(0) \end{bmatrix}$$

and $\ell_2 = \left[\bar{S} + \bar{S}^T \right]$.

From Eq. (20), we get Eq. (21)

$$\lambda(N) = \theta^{-1}(N)\left[\sigma(N)x_1(0) - x_1(N)\right]. \tag{21}$$

By using $\lambda(N)$in Eq. (20), then we get the optimal state$x_1(k)$. We get the costate $\lambda(k)$ by substituting $x_1(k)$ and $\lambda(N)$ in Eq. (19). Once $\lambda(k)$ is known, we can obtain $v(k)$by using Eq. (15). We get $w(k)$ by using $v(k)$ from the relation $w(k) = v(k) - \bar{S}^{-1}\Im^T x_1(k)$. Thereafter, $u(k)$ and $x_2(k)$can be obtained by using the Eqs. (4) and (9).

4. Experiments and results

Consider a fractional order DTSS (1) with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

PI (9) with $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R = [2]$ and the given conditions as $x_1(0) = \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix}$, $x_1(10) = \begin{bmatrix} 0.0570 \\ -0.0322 \end{bmatrix}$, $N = 10$.

The matrices F, G and K can be constructed as [37] $F = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $K = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$.

Let $x_1(k) = \begin{bmatrix} x_{11}(k) \\ x_{12}(k) \end{bmatrix}$, $\begin{bmatrix} x_{11}(0) \\ x_{12}(0) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix}$.

By applying the solution algorithm discussed in previous section, we get the results shown in Figs. 1–5.

Numerical results are presented for distinct α and $N = 10$. Figures 1–5 show the optimal states $x_{11}(t)$, $x_{12}(t)$ and $x_2(t)$, control u_{opt} and minimum value of PI $J_{\min}$ for the fixed terminal state problem. From these results, we can observe that if α increases the amplitudes of both the states and control also increases like in references [18, 19]. These results also show that large values of αdemand more control effort. We can also observe that when α decreases minimum value of PI is decreased.

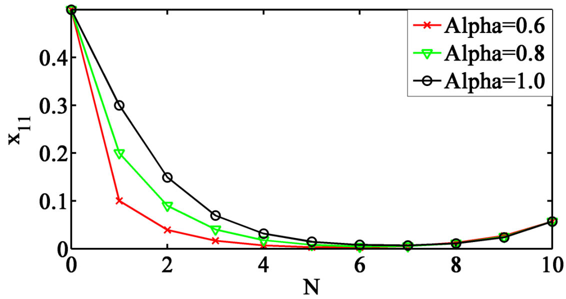

Figure 1: Optimal state x_{11} for $\alpha = 0.6,\ 0.8,\ 1.0$

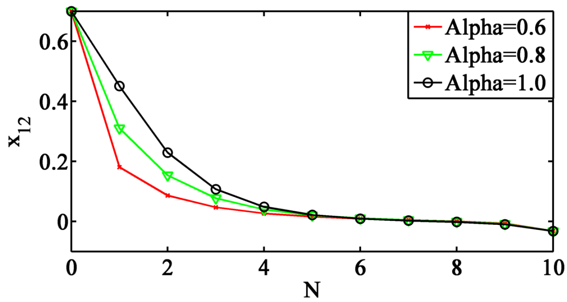

Figure 2: Optimal state x_{12} for $\alpha = 0.6,\ 0.8,\ 1.0$

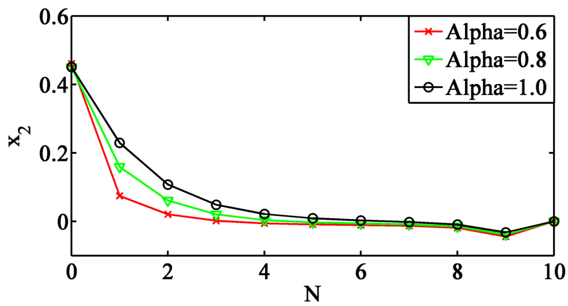

Figure 3: Optimal state x_2 for $\alpha = 0.6,\ 0.8,\ 1.0$

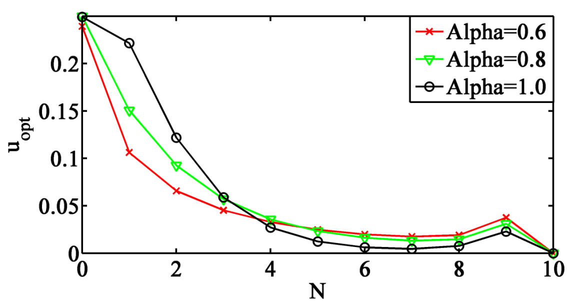

Figure 4: Optimal control u_{opt} for $\alpha = 0.6,\ 0.8,\ 1.0$

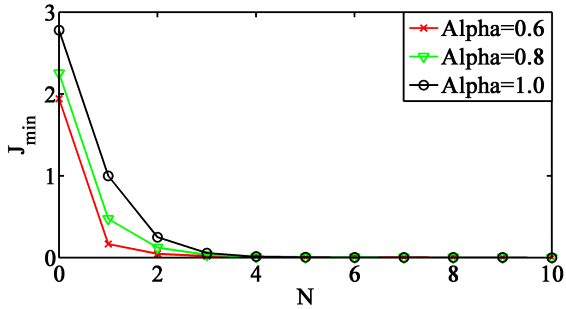

Figure 5: Minimum value of PI $J_{\min}$ for $\alpha = 0.6,\ 0.8, 1.0$

5. Conclusions

Formulation and solution algorithm for OCP of fractional order DTSS in terms of RLFD has been discussed in this paper. The general form of PI in terms of $x(t)$ and $u(t)$ is considered. By using transformation, we have converted fractional order DTSS into its equivalent non-singular form, and then applied optimal control theory for obtaining necessary conditions. A general solution method has been proposed for numerical simulation. An example is considered for validating the adequacy of formulation and numerical algorithm. Numerical results are produced at distinct α. From these results, we can observe that if α increases, then the amplitudes of both the states and control also increases. These results also show that large values of α demand more control effort. We can also observe that when α decreases, then the minimum value of PI is decreased. Therefore, we conclude that consideration of FOCP can give considerable benefits than equivalent integer order OCP. According to the state of the art, this is the first time a formulation and solution of FOCP of DTSS with fixed terminal state and fixed terminal time end point condition is presented.

Nomenclature

FD – Fractional derivative
DTSS – Discrete time singular system
RLFD – Riemann-Liouville fractional derivative
CFD – Caputo fractional derivative
FOCP – Fractional optimal control problem
OCP – Optimal control problem
PI – Performance index
FC – Fractional calculus
SS – Singular system

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