10.24425/acs.2021.139734

Archives of Control Sciences
Volume 31(LXVII), 2021
No. 4*, pages 865–881*

Application of maximum principle to optimization of production and storage costs

Liviu POPESCU and Ramona DIMITROV

A problem of optimization for production and storge costs is studied. The problem consists in manufacture of n types of products, with some given restrictions, so that the total production and storage costs are minimal. The mathematical model is built using the framework of driftless control affine systems. Controllability is studied using Lie geometric methods and the optimal solution is obtained with Pontryagin Maximum Principle. It is proved that the economical system is not controllable, in the sense that we can only produce a certain quantity of products. Finally, some numerical examples are given with graphical representation.

Key words: optimal control, Pontryagin Maximum Principle, controllability, production and storage

1. Introduction

The theory of optimal control has been used as a framework in different domains as economics, cybernetics, operations research or engineering. The development of computer science has helped with a better understanding of the phenomena studied, as well as the numerical solution of some equations, but also the simulation and graphical representation of numerical examples. Seierstad and Sydsater in their book [25] gave a notable contribution to the literature of the control theory of economic processes and Arrow [3] applied the optimal control theory to economic growth. Also, Sethi and Thompson [27] studied production and inventory problems, optimal consumption of natural resources and

L. Popescu (corresponding author), e-mail: liviu.popescu@edu.ucv.ro and R. Dimitrov, e-mail: ramona.dimitrov@yahoo.com, are with University of Craiova, Faculty of Economics and Business Administration Department of Statistics and Economic Informatics, Al. I. Cuza st., No. 13, Craiova 200585, Romania.

This work was supported by the grant POCU380/6/13/123990, co-financed by the European Social Fund within the Sectorial Operational Program Human Capital 2014–2020.

Received 10.05.2021.

applications of optimal control to management science. Weber [30] presented an introduction to the use of optimal control techniques for continues-time systems in economics and Caputo [7] added complementary methods for applications of optimal control to operational research.

An important method in the study of solutions to optimal control problems is given by Pontryagin's Maximum Principle [4], which generates first-order differential equations, which are necessary conditions for optimality. It is assumed that a curve $c(t) = (x(t), u(t))$ can be an optimal solution if there exists a lifting of $x(t)$ to the dual space $(x(t), p(t))$ that satisfies the Hamilton-Jacoby-Bellman equations.

One of the problems that is studied in the case of these control systems is the issue of controllability. Controllability is the ability to move the system from a given initial state to a final state, in finite time, using some available control variables. Lie geometric methods in the study of controllability problem are used by many authors. Thus, Brocket [6] showed the connection between control and Lie theory. In his book [13], Jurdjevic deals with control affine systems and LaValle [17] presented the cases of holonomic and nonholonomic control systems. Agrachev and Sachkov [1] studied the control theory from the geometric viewpoint. Also, Lie geometric methods and framework of Lie algebroids have been used by Popescu [22–24] in order to solve some problems involving inventory and production.

The purpose of this paper is to solve an economical problem involving production and storage costs and can be considered as a generalization of the results obtained by Kamien and Schwartz in the paper [14]. The novelty of the approach compared to the existing literature is given by the fact that we propose an optimal manufacturing plan for n products under certain manufacturing conditions. Also, the controllability is studied using Lie geometric methods. A mathematical model is proposed, using the framework of control systems. These types of problems are common in the literature and can be found in many research papers, with different constraints. Axsäter [3] and Ortega, Lin [20] gave an overview of earlier research concerning control theory applications in production and inventory control, existing up to that date. Sethi [26] studied the applications of the Maximum Principle to some type of production and inventory problems. In [12] Hermosilla, Vinter and Zidani, used Hamilton–Jacobi–Bellman equations for optimal control processes with convex state constraints. Benjaafar, Gayon and Tepe [5] applied optimal control theory for a production-inventory system with customer impatience. An optimal control model for continuous time production and setup scheduling is investigated by Kogan and Khmelnitsky in [15]. A new approach to maximize the profit/cost ratio in a stock-dependent demand inventory model is proposed by Pando and Sicilia in [21]. Qiu, Qiao and Pardalos studied in [16] the optimal production and inventory management policies for products with perishable inventory, while optimal control of a production-inventory system with product returns and two disposal options is investigated by Gayon, Vercraene and Flapper in [11]. Danahe, Chelbi and Rezg [8] proposed an optimal production plan for a multi-products manufacturing system with production rate dependent failure rate. Also, Gaimon [10] and Feichtinger, Hartl [9] studied, among others, optimal pricing and production, capacity decisions in an inventory model. In paper [18] Li and Wang proposed an integrated replenishment and production control policy under inventory inaccuracy and time-delay, while Li and Arreola-Risa investigated in [19] optimization of a production-inventory system under a cost target. Schwartz and Rivera [28] described a process control approach to tactical inventory management in production-inventory systems, and Towill, Evans and Cheema [29] proposed an analysis and design of an adaptive minimum reasonable inventory control system.

The paper is organized as follows. In the second section, a problem of production and storage optimization is proposed and the mathematical model is given, using the optimal control framework. In section three is presented the methodology, which will be applied in order to find the optimal solution. This consists in applying the Maximum Principle for the optimal control problem, while Lie geometric methods are used in order to study the controllability. In section four, the optimal solution of the economic problem is proved and some numerical examples are given. The novelty of the paper can be found in sections two and four. Finally, some conclusions are presented.

2. A problem of production and storage optimization

Let us consider that a company must manufacture n types of products in a fixed period of time T. It is known that certain percentages from quantities of $P_1, P_2, \ldots, P_{n-1}$ products are used in the manufacture of the P_n product, by a given law. Also, it is assumed that the unit production costs for $P_1, P_2, \ldots, P_{n-1}$ increase linearly with the production level and the cost of production operations for the product P_n is considered negligible (for example, P_n is a product packaged and unassembled). We have the unit storage costs of holding inventory given by $(\beta_1, \beta_2, \ldots, \beta_n)$ for each product. In these conditions we are looking for a plan of production to ensure the required quantity at specified delivery data at minimum costs of production and storage. We assume that we have no restrictions on production or storage capacity. A particular case with a single product is studied by Kamien and Schwartz [14].

We have the following notations: P_i = products, $i = 1, 2, \ldots, n$; T – period of time to ensure the quatities of products; $x^i(t)$ – the inventory accumulated by time t; s_i – final quantities required; $p^i(t)$ – the rate of production; c_i – unit production cost. Also, we have the assumptions: $x^i(0) = 0$; $x^i(T) = s_i$

and the production costs increase linearly with the production level $c_i = \alpha_i p^i$, $\alpha_1, \dots, \alpha_{n-1} \geqslant 0$.

2.1. Mathematical model

The inventory level is the cumulated past production $p^i = p^i(t)$, and considering $x^i(0) = 0$, we have

$$x^i(t) = \int_0^t p^i(s)\,\mathrm{d}s.$$

It results that the rate of change of inventory level $\dot{x}^i$ is the production and we have $\dot{x}^i = p^i$. Considering $\dot{x}^i = u^i$, $i = \overline{1, n-1}$ the control variables (the production rate can be controllable), it is assumed that the production rate for P_n is given by the law

$$\dot{x}^n = \sum_{i=1}^{n-1} k_i \dot{x}^i, \tag{1}$$

where $k_1, \dots, k_{n-1} \in [0, 1]$.

The unit production costs c_i increase linearly with the production level, $c_i = \alpha_i p^i$ where $\alpha_1, \dots, \alpha_{n-1} \geqslant 0$ are positive constants and we have that the total cost of production is

$$\sum_{i=1}^{n-1} c_i p^i = \sum_{i=1}^{n-1} \alpha_i (p^i)^2 = \sum_{i=1}^{n-1} \alpha_i \left(\dot{x}^i\right)^2.$$

We obtain that the total cost, including the costs of holding inventory, is given by

$$\sum_{i=1}^{n-1} \alpha_i \left(\dot{x}^i\right)^2 + \sum_{i=1}^{n-1} \beta_i (1 - k_i) x^i + \beta_n x^n,$$

where $k_1, \dots, k_{n-1} \in [0, 1]$ represent the percentages from the quantities of $P_1, \dots, P_{n-1}$ used in the manufacture of P_n. Finally, we obtain the following optimal control problem

$$\begin{gathered}
\dot{x}^1 = u^1, \\
\dots, \\
\dot{x}^{n-1} = u^{n-1} \\
\dot{x}^n = \sum_{i=1}^{n-1} k_i \dot{x}^i, \\
x^i(0) = 0, \qquad x^i(T) = s_i, \\
u^1, \dots, u^{n-1} \geqslant 0, \qquad k_1, \dots, k_{n-1} \in [0, 1].
\end{gathered} \tag{2}$$

We are looking for a plan of production with minimum cost

$$\min_u \int_0^T \left(\sum_{i=1}^{n-1} \alpha_i \left(\dot{x}^i\right)^2 + \sum_{i=1}^{n-1} \beta_i (1 - k_i) x^i + \beta_n x^n \right) \mathrm{d}t.$$

3. Methodology

The methodology consists in applying the Pontryagin's Maximum Principle for the optimal control problem in the case of driftless control affine systems. The controllability of the distributional system is studied using the Lie geometric methods and Frobenius theorem in the case of holonomic distributions.

3.1. Preliminaries on optimal control and maximum principle

In the following, we consider a smooth n-dimensional manifold M, in particular a subset of R^n. A continuous control system on the manifold M is given by a set of differential equations depending on some parameters, in the form

$$\frac{\mathrm{d}x^i(t)}{\mathrm{d}t} = f^i(x(t), u(t)), \qquad i = \overline{1, n},$$

where $x = (x^1, \ldots, x^n) \in M$ are the state variables of the control system and $u = (u^1, \ldots, u^m) \in U \subset R^m$ represents the control variables ($m \leqslant n$). Considering x_0 and x_1 two states of the system, an optimal control problem consists of finding the ways through the system is brought from the initial state x_0 to the final state x_1 and minimizing the functional cost

$$\min_u \int_0^T L(x(t), u(t))\,\mathrm{d}t, \qquad x(0) = x_0, \quad x(T) = x_1,$$

where L is the *Lagrangian* function (energy, cost, time, distance, etc.).

In other words, we must find the trajectories of our control system which connect two given points on M such that a certain optimality condition is satisfied. An important method for studying the optimal solutions in control theory is given by Pontryagin Maximum Principle. It generates the differential equations of first order, which are necessary for the optimal solutions. For each optimal trajectory $c(t) = (x(t), u(t))$, it gives a lift on the dual space $(x(t), p(t))$ satisfying the Hamilton–Jacobi–Bellman (HJB) equations. The Hamiltonian function on dual space is given by

$$H(x, p, u) = \sum_{i=1}^{n} p_i f^i(x, u) - L(x, u),$$

where $(p_1, \ldots, p_n)$ are momentum variables. The maximization condition with respect to control variables u, given by

$$H(x(t), p(t), u(t)) = \max_v H(x(t), p(t), v),$$

leads to $\dfrac{\partial H}{\partial u^i} = 0$($H$ is assumed to be smooth with respect to u) and the extreme trajectories satisfy the HJB equations

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x^i}. \tag{3}$$

Next, we will consider a driftless control affine system (distributional systems) in the form

$$\dot{x}(t) = \sum_{i=1}^{m} u^i(t) X_i(x(t)), \tag{4}$$

where $\dot{x}(t) = \dfrac{\mathrm{d}x(t)}{\mathrm{d}t}$ and $X_1, \ldots, X_m$ are smooth vector fields on the manifold M, called the input vector fields. A first problem that can be studied in the case of these systems is the issue of controllability. We say that the system is *controllable* if for any two states x_0 and x_1 there exists a solution curve of (4) connecting x_0 to x_1. Controllability is the ability to move a system from a given initial state to any final state, in finite time, using the available controls. It is interesting to note that the information about controllability of distributional systems is contained in the structure of the Lie algebra generated by the family of vector fields X_i, $i = \overline{1, m}$. A distribution Δ on the manifold M is a map which assigns to each point in M a subspace of the tangent space TM at this point $x \in M \to \Delta(x) \subset T_xM$. The distribution Δ is called locally finitely generated if there is a family of vector fields X_i, $i = \overline{1, m}$, called local generators of Δ which spans Δ, that is $\Delta(x) = \mathrm{span}\{X_1(x), \ldots, X_m(x)\} \subset T_xM$. The distribution Δ has constant dimension k if $\dim \Delta(x) = k$, for all points $x \in M$. The Lie bracket of two vector fields is given by $[X, Y](f) = X(Y(f)) - Y(X(f))$.

The distribution $\Delta = \mathrm{span}\{X_1, \ldots, X_m\}$ is called *involutive* if the Lie bracket of any two vectors from Δ belongs to Δ, that is, for $X, Y \in \Delta$ it results $[X, Y] \in \Delta$. It results that every Lie bracket can be expressed as a linear combination of the system vector fields, and therefore

$$\left[X_i, X_j\right] = \sum_{k=1}^{m} L_{ij}^k X_k.$$

We recall that a *foliation* $\{S_\alpha\}_{\alpha \in I}$ of M is a partition of $M = \bigcup_{\alpha \in I} S_\alpha$ of M into disjoint connected submanifolds S_α called leaves. A distribution Δ of constant dimension on M is called *integrable* (holonomic) if there exists a foliation $\{S_\alpha\}_{\alpha\in I}$ on M whose tangent bundle is Δ, that is $T_xS = \Delta(x)$, where S is the leaf passing through x. The well-known Frobenius theorem says that if Δ is a distribution with constant dimension on the manifold M, then Δ is integrable if and only if Δ is involutive.

If we return to the case of driftless control affine system (4) and the distribution Δgenerated by the input vectors X_i, $i = \overline{1, m}$ is integrable (holonomic) with constant dimension, then the system is not controllable and Δ determines a foliation on M with the property that any curve is contained in a single leaf of the foliation. With other words, any two points can be joined by an optimal trajectory if and only if they are situated on the same leaf.

4. Solution of the economic problem

We observe that the system (2) is a dritless control affine system on $M = R^3_+$ written in the form

$$\dot{x} = \sum_{i=1}^{n-1} u^i X_i, \qquad x = (x^1, \ldots, x^n)^t \in R^n_+$$

$$\min_u \int_0^T F(u(t), x(t))\,\mathrm{d}t,$$

where

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ k_1 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ k_2 \end{pmatrix}, \quad \ldots, \quad X_{n-1} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ k_{n-1} \end{pmatrix},$$

$$F(u(t), x(t)) = \sum_{i=1}^{n-1} \left(\alpha_i (u^i)^2 + \beta_i (1 - k_i) x^i\right) + \beta_n x^n.$$

We are looking for the optimal trajectories starting from the point $(0, 0, \ldots, 0)$ and endpoint $(s_1, s_2, \ldots, s_n)$. The distribution $\Delta = \mathrm{span}\{X_1, \ldots, X_{n-1}\}$ generated by the vector fields $X_1, \ldots, X_{n-1}$ has constant dimension, $\dim \Delta(x) = n - 1$, for all $x \in R^n$. Also, in natural basis $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}\right\}$ of R^n the vector fields have the expressions

$$X_1 = \frac{\partial}{\partial x^1} + k_1 \frac{\partial}{\partial x^n}, \quad \ldots, \quad X_{n-1} = \frac{\partial}{\partial x^{n-1}} + k_{n-1} \frac{\partial}{\partial x^n},$$

and using the Lie brackets formula

$$[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X,$$

we obtain that the Lie bracket of the vectors fields from distribution is given by

$$\left[X_i, X_j\right] = \left[\frac{\partial}{\partial x^i} + k_i \frac{\partial}{\partial x^n}, \ \frac{\partial}{\partial x^j} + k_j \frac{\partial}{\partial x^n}\right] = 0.$$

It results that the distribution Δ is involutive. Using the Frobenius theorem, it results that the distribution is integrable and as a consequence, it determines a foliation on state space R_+^n. Consequently, two points can be joined by an optimal trajectory if and only if they are situated on the same leaf. It results that the our economical system is not controllable, in the sense that we can not manufacture any quantity required. Indeed, by using the equation (1) we have, by integration

$$x^n = \sum_{i=1}^{n-1} k_i x^i + c,$$

which are hyperplanes of R^n, which determine a foliation. Using the condition $x^i(0) = 0$, it results $c = 0$, and from $x^i(T) = s_i$ we obtain that the system is controllable if and only if

$$s_n = \sum_{i=1}^{n-1} k_i s_i \, .$$

We will use the Pontryagin's Maximum Principle in order to find the optimal solution.

Theorem 1 *The optimal solution of the economic system (2) has the form*

a) if $0 < T \leqslant 2\sqrt{\dfrac{\alpha_i s_i}{\beta_i(1-k_i) + k_i\beta_n}}, \ i = \overline{1, n-1}$

then

$$x^i(t) = \frac{\beta_i(1-k_i) + k_i\beta_n}{4\alpha_i} t^2 + \left(\frac{s_i}{T} - \frac{\beta_i(1-k_i) + k_i\beta_n}{4\alpha_i} T\right) t,$$

$$u^i(t) = \frac{\beta_i(1-k_i) + k_i\beta_n}{2\alpha_i} t + \left(\frac{s_i}{T} - \frac{\beta_i(1-k_i) + k_i\beta_n}{4\alpha_i} T\right)$$

for $\leqslant t \leqslant T$
and

b) if $T > 2\sqrt{\dfrac{\alpha_i s_i}{\beta_i(1-k_i)+k_i\beta_n}}$, $i = \overline{1,\, n-1}$

then

$$x^i(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant T - \dfrac{4\alpha_i s_i}{(\beta_i(1-k_i)+k_i\beta_n)\,T}, \\ \dfrac{\beta_i(1-k_i)+k_i\beta_n}{4\alpha_i}t^2 + \left(\dfrac{s_i}{T} - \dfrac{\beta_i(1-k_i)+k_i\beta_n}{4\alpha_i}T\right)t & \\ & \text{if } T - \dfrac{4\alpha_i s_i}{(\beta_i(1-k_i)+k_i\beta_n)\,T} < t \leqslant T; \end{cases}$$

$$u^i(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant T - \dfrac{4\alpha_i s_i}{(\beta_i(1-k_i)+k_i\beta_n)\,T}, \\ \dfrac{\beta_i(1-k_i)+k_i\beta_n}{2\alpha_i}t + \left(\dfrac{s_i}{T} - \dfrac{\beta_i(1-k_i)+k_i\beta_n}{4\alpha_i}T\right) & \\ & \text{if } T - \dfrac{4\alpha_i s_i}{(\beta_i(1-k_i)+k_i\beta_n)\,T} < t \leqslant T. \end{cases}$$

where, in both cases,

$$x^n(t) = \sum_{i=1}^{n-1} k_i x^i(t).$$

Proof. The Hamiltonian function on dual space is given by

$$H = \sum_{i=1}^{n} p_i \dot{x}^i - F,$$

with $p_1, \ldots, p_n$, the momentum variables, which leads to

$$H = \sum_{i=1}^{n-1} p_i u^i + p_n \sum_{i=1}^{n-1} k_i u^i - \sum_{i=1}^{n-1}\left(\alpha_i (u^i)^2 + \beta_i(1-k_i)x^i\right) - \beta_n x^n.$$

The condition $\dfrac{\partial H}{\partial u^i} = 0$, $i = \overline{1,\, n-1}$ yields the following equations

$$p_i + p_n k_i - 2\alpha_i u^i = 0,$$

and it results

$$u^i = \frac{p_i + p_n k_i}{2\alpha_i}.$$

Next, we replace the expressions of the control variables $u^1, \ldots, u^{n-1}$ into the expression of the Hamiltonian and by straightforward computation it results

$$H = \sum_{i=1}^{n-1} \left(\frac{(p_i + p_n k_i)^2}{4\alpha_i} - \beta_i (1 - k_i) x^i \right) - \beta_n x^n.$$

Using the Hamilton-Jacobi-Bellman equations (3) we obtain the following system of first order differential equations:

$$\dot{x}^i = \frac{\partial H}{\partial p_i} = \frac{p_i + p_n k_i}{2\alpha_i}, \quad i = \overline{1, n-1}, \tag{5}$$

$$\dot{x}^n = \frac{\partial H}{\partial p_n} = \sum_{i=1}^{n-1} \frac{(p_i + p_n k_i) k_i}{2\alpha_i}, \tag{6}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x^i} = \beta_i (1 - k_i),$$
$$\dot{p}_n = -\frac{\partial H}{\partial x^n} = \beta_n .$$

From (5) we get

$$\ddot{x}^i = \frac{\dot{p}_i + \dot{p}_n k_i}{2\alpha_i} = \frac{\beta_i (1 - k_i) + k_i \beta_n}{2\alpha_i}$$

which leads through integration to

$$\dot{x}^i(t) = \frac{\beta_i (1 - k_i) + k_i \beta_n}{2\alpha_i} t + c_i ,$$

and finally

$$x^i(t) = \frac{\beta_i (1 - k_i) + k_i \beta_n}{4\alpha_i} t^2 + c_i t + d_i .$$

The condition $x^i(0) = 0$ leads to $d_i = 0$ and the condition $x^i(T) = s_i$ implies

$$c_i = \frac{s_i}{T} - \frac{\beta_i (1 - k_i) + k_i \beta_n}{4\alpha_i} T.$$

It results the solution

$$x^i(t) = \frac{\beta_i (1 - k_i) + k_i \beta_n}{4\alpha_i} t^2 + \left(\frac{s_i}{T} - \frac{\beta_i (1 - k_i) + k_i \beta_n}{4\alpha_i} T \right) t. \tag{7}$$

The solution is optimal, because the Hamiltonian function is convex. These are the polynomial functions of degree two and the equations $x^i(t) = 0$, have the solutions $t_1 = 0$ and $t_2 = T - \dfrac{4\alpha_i s_i}{(\beta_i(1-k_i) + k_i\beta_n)T}$. Moreover, the economical condition $x^i(t) \geqslant 0$, $u^i(t) \geqslant$ for $t \geqslant 0$ leads to the following cases:

a) $0 < T \leqslant 2\sqrt{\dfrac{\alpha_i s_i}{\beta_i(1-k_i) + k_i\beta_n}}$, with the optimal solution given by (7), for $0 \leqslant t \leqslant T$

and

b) $T > 2\sqrt{\frac{\alpha_i s_i}{\beta_i(1-k_i)+k_i\beta_n}}$,

which leads to $x^i(t) = 0$, for $0 \leqslant t \leqslant T - \dfrac{4\alpha_i s_i}{(\beta_i(1-k_i) + k_i\beta_n)T}$ and the optimal solution given by (7) for $T - \dfrac{4\alpha_i s_i}{(\beta_i(1-k_i) + k_i\beta_n)T} < t \leqslant T$ which ends the proof. □

4.1. Numerical examples

Example 1. We consider the following data for the case of three products P_1, P_2, P_3.

– the period of time is $T = 1$;

– the final quantities of products are:

$$s_1 = 4, \quad s_2 = 2, \quad s_3 = 3;$$

– storage costs are given by

$$\beta_1 = 2, \quad \beta_2 = 2, \quad \beta_3 = 2;$$

– the coeficients are

$$k_1 = 0.5, \quad k_2 = 0.5, \qquad \alpha_1 = 0.5, \quad \alpha_2 = 0.5.$$

The system is controllable, because $s_3 = k_1 s_1 + k_2 s_2$. It results that $0 < T < 2\sqrt{\dfrac{\alpha_1 s_1}{\beta_1(1-k_1) + k_1\beta_3}} = 2$ and $0 < T < 2\sqrt{\dfrac{\alpha_2 s_2}{\beta_2(1-k_2) + k_2\beta_3}} = \dfrac{2}{\sqrt{2}}$, which leads to the case a) for $x^1(t)$ and $x^2(t)$, with optimal solution given by

$$\begin{aligned} x^1(t) &= t^2 + 3t, & u^1(t) &= 2t + 3, \\ x^2(t) &= t^2 + t, & u^2(t) &= 2t + 1, \\ x^3(t) &= t^2 + 2t & &\text{for } 0 \leqslant t \leqslant 1. \end{aligned}$$

It results that the production for all products starts at time $t = 0$. The optimal solution $(x^1(t), x^2(t), x^3(t))$ is illustrated in Fig. 1.

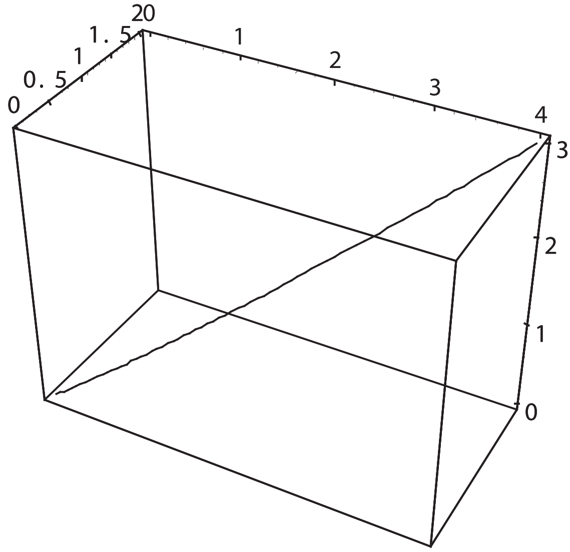

Figure 1: Optimal solution $(x^1(t), x^2(t), x^3(t))$ for $0 \leqslant t \leqslant 1$

Example 2. We consider the following data for the case of three products P_1, P_2, P_3:

– the period of time is $T = 2$;

– the final quantities of products are:

$$s_1 = 8, \quad s_2 = 4, \quad s_3 = 6;$$

– storage costs are given by

$$\beta_1 = 2, \quad \beta_2 = 2, \quad \beta_3 = 2;$$

– the coeficients:

$$k_1 = 0.5, \quad k_2 = 0.5, \qquad \alpha_1 = 0.5, \quad \alpha_2 = 0.25.$$

The system is controllable, because $s_3 = k_1 s_1 + k_2 s_2$. We obtain that $0 < T < 2\sqrt{\dfrac{\alpha_1 s_1}{\beta_1(1-k_1)+k_1\beta_3}} = 2\sqrt{2}$ and $T > 2\sqrt{\dfrac{\alpha_2 s_2}{\beta_2(1-k_2)+k_2\beta_3}} = \dfrac{2}{\sqrt{2}}$, which leads to the case a) for $x^1(t)$ and case b) for $x^2(t)$ with optimal solution given by

$$x^1(t) = t^2 + 2t, \quad \text{for } 0 \leqslant t \leqslant 2, \qquad u^1(t) = 2t + 2,$$

$$x^2(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant 1, \\ 2t^2 - 2t & \text{if } 1 < t \leqslant 2, \end{cases}, \qquad u^2(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant 1, \\ 4t - 2 & \text{if } 1 < t \leqslant 2, \end{cases}$$

$$x^3(t) = \begin{cases} \frac{t^2}{2} + t & \text{if } 0 \leqslant t \leqslant 1, \\ \frac{3t^2}{2} & \text{if } 1 < t \leqslant 2. \end{cases}$$

From an economic point of view, the production on the first product starts at time $t = 0$ and for the second product the production starts later at $t = 1$. Production on the third product, which depends on the first two, begins at time $t = 0$. The optimal solution $(x^1(t), x^2(t), x^3(t))$ is given in Fig. 2.

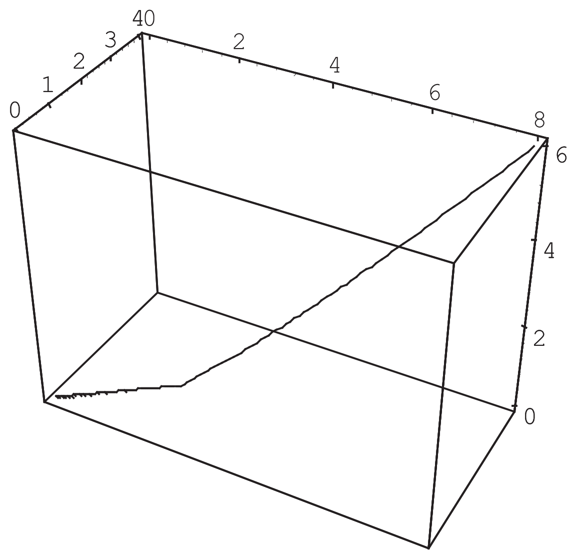

Figure 2: Optimal solution $(x^1(t), x^2(t), x^3(t))$ for $0 \leqslant t \leqslant 2$

Example 3. We consider the following data for the case of three products P_1, P_2, P_3:

– the period of time is $T = 4$;

– final quantities of products are:

$$s_1 = 12, \quad s_2 = 8, \quad s_3 = 10;$$

– storage costs are given by

$$\beta_1 = 2, \quad \beta_2 = 2, \quad \beta_3 = 2;$$

– the coeficients:

$$k_1 = 0.5, \quad k_2 = 0.5, \qquad \alpha_1 = 0.5, \quad \alpha_2 = 0.5.$$

The system is controllable, because $s_3 = k_1 s_1 + k_2 s_2$. We have that $T > 2\sqrt{\dfrac{\alpha_1 s_1}{\beta_1(1-k_1)+k_1\beta_3}} = 2\sqrt{3}$ and $T > 2\sqrt{\dfrac{\alpha_2 s_2}{\beta_2(1-k_2)+k_2\beta_3}} = 2\sqrt{2}$, which lead to the case b) for $x^1(t)$ and $x^2(t)$ with optimal solution given by

$$x^1(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant 1, \\ t^2 - t & \text{if } 1 < t \leqslant 4, \end{cases} \qquad u^1(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant 1, \\ 2t - 1 & \text{if } 1 < t \leqslant 4; \end{cases}$$

$$x^2(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant 2, \\ t^2 - 2t & \text{if } 2 < t \leqslant 4, \end{cases} \qquad u^2(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant 2, \\ 2t - 2 & \text{if } 2 < t \leqslant 4, \end{cases}$$

$$x^3(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant 1, \\ \dfrac{t^2 - t}{2} & \text{if } 1 < t \leqslant 2, \\ t^2 - \dfrac{3t}{2} & \text{if } 2 < t \leqslant 4. \end{cases}$$

It is found that the production for the first product starts at time $t = 1$ and for the second at time $t = 2$. Production on the third product, which depends on the first two, begins at time $t = 1$. The optimal solution $(x^1(t), x^2(t), x^3(t))$ is represented in Fig. 3.

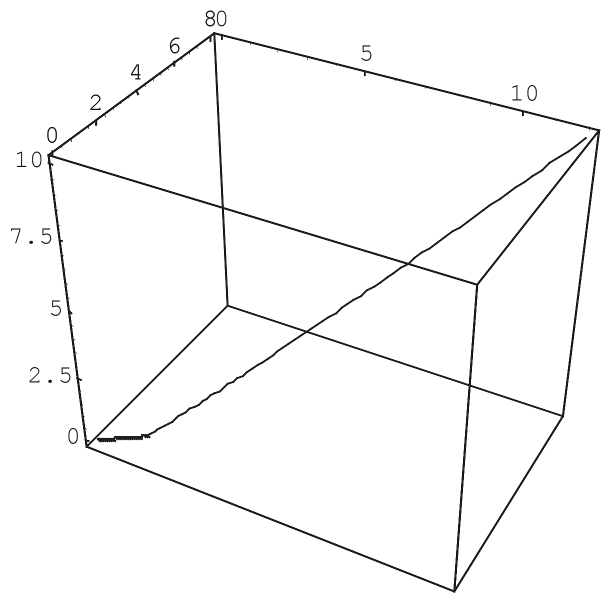

Figure 3: Optimal solution $(x^1(t), x^2(t), x^3(t))$ for $0 \leqslant t \leqslant 4$

5. Conclusions

In this paper a problem of minimizing the total costs of production and storage is studied. The problem consists in the manufacture of n types of products in certain economic conditions and in a fixed period of time. The mathematical model is proposed, using the framework of control affine systems and the optimal solutions is obtained with the Pontryagin Maximum Principle. The economic system is not controllable, in the sense that we cannot manufacture any quantity of products. The problem has a solution if and only if there is a certain connection between the final stocks quantities. Finally, some illustrative examples are given in the particular case $n = 3$ and the optimal solutions are graphically represented.

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