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Positivity and cyclicity of descriptor electrical circuits with chain structure

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Abstract. The positivity and cyclicity of descriptor linear electrical circuits with chain structure is considered. Two classes of descriptor linear electrical circuits are analyzed. Some new properties of these classes of electrical circuits are established. The results are extended to fractional descriptor linear electrical circuits.

Key words: cyclic matrix; descriptor; positive; electrical circuit; fractional; Metzler matrix.

1. INTRODUCTION

A dynamical system and an electrical circuit is called positive if its state variables take nonnegative values for all nonnegative inputs and nonnegative initial conditions. The positive linear systems have been investigated in [1–9]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Mathematical fundamentals of the fractional calculus are given in the monographs [5,7,10,11]. Fractional dynamical systems have been investigated in [4–7, 10, 12–16].

Positive linear systems with different fractional orders have been addressed in [4, 5, 7, 15].

An overview of state of the art in descriptor systems theory is given in [17–20]. Stability of this class of dynamical systems was investigated in [18, 19, 21–23].

In this paper the positivity and cyclicity of descriptor linear electrical circuits with chain structure are investigated. The paper is organized as follows. In Section 2 some definitions and theorems concerning positive descriptor systems and cyclic matrices are recalled. New results concerning descriptor electrical circuits with cyclic state matrices are presented in Section 3. An extension of these results to fractional descriptor electrical circuits is given in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\mathbb{R}$ – the set of real numbers, $\mathbb{R}^{n\times m}$ – the set of $n\times m$ real matrices, $\mathbb{M}_n$ – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), $\mathbb{I}_n$ – the $n\times n$ identity matrix.

2. PRELIMINARIES

Consider the descriptor continuous-time linear system

$$E\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t), \tag{1a}$$

$$y(t) = Cx(t), \tag{1b}$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^p$ are the state, input and output vectors and E, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{p\times n}$. It is assumed that $\det E \neq 0$ and

$$\det[E\lambda - A] \neq 0 \quad \text{for some } \lambda\in\mathbb{C}. \tag{2}$$

Assuming that for some chosen $c\in\mathbb{C}$ we have $\det[Ec-A]\neq 0$ and premultiplying (1a) by $[Ec-A]^{-1}$ we obtain

$$\bar{E}\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \bar{A}x(t) + \bar{B}u(t), \tag{3a}$$

where

$$\bar{E} = [Ec-A]^{-1}E, \quad \bar{A} = [Ec-A]^{-1}A, \quad \bar{B} = [Ec-A]^{-1}B. \tag{3b}$$

Note that equations (1a) and (3a) have the same solution $x(t)$.

Definition 1. [19, 20] The smallest nonnegative integer q satisfying

$$\operatorname{rank}\bar{E}^q = \operatorname{rank}\bar{E}^{q+1} \tag{4}$$

is called the index of the matrix $\bar{E}\in\mathbb{C}^{n\times n}$.

Definition 2. [19, 20] A matrix $\bar{E}^D$ is called the Drazin inverse of $\bar{E}$ if it satisfies the conditions:

$$\bar{E}\bar{E}^D = \bar{E}^D\bar{E}, \tag{5a}$$

$$\bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D, \tag{5b}$$

$$\bar{E}^D\bar{E}^{q+1} = \bar{E}^q. \tag{5c}$$

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The Drazin inverse of a square matrix always exists and is unique. If $\det\bar{E} \neq 0$ then $\bar{E}^D = \bar{E}^{-1}$. Some methods for computation of the Drazin inverse are given in [20].

Lemma 1. [19, 20] The matrices $\bar{E}$ and $\bar{A}$ defined by (3b) have the following properties:

$$\bar{A}\bar{E} = \bar{E}\bar{A},\ \ \bar{A}^D\bar{E} = \bar{E}\bar{A}^D,\ \ \bar{E}^D\bar{A} = \bar{A}\bar{E}^D,\ \ \bar{A}^D\bar{E}^D = \bar{E}^D\bar{A}^D, \tag{6a}$$

$$\ker\bar{A} \cap \ker\bar{E} = \{0\}, \tag{6b}$$

$$\left(\mathbb{I}_n - \bar{E}\bar{E}^D\right)\bar{A}\bar{A}^D = \mathbb{I}_n - \bar{E}\bar{E}^D, \quad \left(\mathbb{I}_n - \bar{E}\bar{E}^D\right)\left(\bar{E}\bar{A}^D\right)^q = 0. \tag{6c}$$

Also, the matrices $\bar{E}$, $\bar{E}^D$ and $\bar{A}$ can be written in the form

$$\begin{gathered}\bar{E} = T\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix}T^{-1}, \quad \bar{A} = T\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}T^{-1}, \\ \bar{E}^D = T\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}T^{-1},\end{gathered} \tag{6d}$$

where $T \in \mathbb{R}^{n\times n}$, $\det T \neq 0$, $J \in \mathbb{R}^{n_1\times n_1}$ is a nonsingular matrix, $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix with the nilpotency index q, i.e. $N^{q-1} \neq 0$, $N^q = 0$, $A_1 \in \mathbb{R}^{n_1\times n_1}$, $A_2 \in \mathbb{R}^{n_2\times n_2}$ and $n_1 + n_2 = n$.

Lemma 2. [20] If

$$x_1(t) = \bar{E}\bar{E}^D x(t), \quad x_2(t) = (\mathbb{I}_n - \bar{E}\bar{E}^D)x(t), \tag{7a}$$

$$x_1(t) + x_2(t) = x(t), \tag{7b}$$

then equation (3a) is equivalent to the following equations:

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \bar{A}_1 x_1(t) + \bar{B}_1 u(t), \tag{8a}$$

$$\bar{N}\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = x_2(t) + \bar{B}_2 u(t), \tag{8b}$$

where

$$\begin{gathered}\bar{A}_1 = \bar{E}^D\bar{A}, \quad \bar{B}_1 = \bar{E}^D\bar{B} \\ \bar{N} = (\mathbb{I}_n - \bar{E}\bar{E}^D)\bar{A}^D\bar{E}, \quad \bar{B}_2 = (\mathbb{I}_n - \bar{E}\bar{E}^D)\bar{A}^D\bar{B}.\end{gathered} \tag{8c}$$

In this class of dynamical systems the matrix $\bar{A}_1 = \bar{E}^D\bar{A}$ can not be considered as a typical state matrix. From the solution of equation (8a) it follows that the matrix $\bar{A}_1$ may contain unimportant entries that are further canceled through multiplication by $x_0 \in \operatorname{Im}\bar{E}\bar{E}^D$. Therefore, as a state matrix of the system we will assume

$$\bar{A}_G = \bar{A}_1 + G(\mathbb{I}_n - \bar{E}\bar{E}^D), \tag{9}$$

where $G \in \mathbb{R}^{n\times n}$ is arbitrary. The term $G(\mathbb{I}_n - \bar{E}\bar{E}^D)$ eliminates from the matrix $\bar{A}_1$ those unimportant entries [20, 21].

Definition 3. [20] The descriptor continuous-time linear system (3a), (1b) (or equivalently (1)) is called (internally) positive if $x(t) \in \mathbb{R}^n_+$ and $y(t) \in \mathbb{R}^p_+$, $t \geq 0$ for any consistent initial conditions $x(0) \subset \mathbb{R}^n_+$ and all admissible inputs $u(t) \subset \mathbb{R}^m_+$, $t \geq 0$ such that $\dfrac{\mathrm{d}^k u(t)}{\mathrm{d}t^k} \in \mathbb{R}^m_+$, $k = 1,\ldots,q-1$.

Theorem 1. [20] The descriptor continuous-time linear system (3a), (1b) (or equivalently (1)) is positive if and only if there exists a matrix $G \in \mathbb{R}^{n\times n}$ such that

$$\bar{A}_G \in \mathbb{M}_n \tag{10a}$$

and

$$\begin{gathered}\operatorname{Im}\bar{E}\bar{E}^D \subset \mathbb{R}^n_+, \quad \bar{B}_1 \in \mathbb{R}^{n\times m}_+, \quad C \in \mathbb{R}^{p\times n}_+, \quad D \in \mathbb{R}^{p\times m}_+ \\ -\bar{N}^k\bar{B}_2 \in \mathbb{R}^{n\times m}_+, \quad k = 1,\ldots,q-1.\end{gathered} \tag{10b}$$

Definition 4. The positive descriptor continuous-time linear system (3a), (1b) (or equivalently (1a)) is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \tag{11}$$

for all consistent initial conditions $x_0 \in \bar{E}\bar{E}^D v$ (where $v \in \mathbb{R}^n$ is an arbitrary vector) and $u(t) = 0$.

Theorem 2. [20] The positive descriptor continuous-time linear system (3a), (1b) (or equivalently (1a)) for $u(t) = 0$ is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. All coefficients of the characteristic equation

$$\det[Es - A] = a_r s^r + a_{r-1}s^{r-1} + \ldots + a_1 s + a_0 = 0 \tag{12}$$

   are positive, i.e. $a_i > 0$ for $i = 0,1,\ldots,r$, where $r < n$.
2. There exists strictly positive vector $\lambda^T = [\,\lambda_1 \ \ \ldots \ \ \lambda_n\,]$, $\lambda_k > 0$, $k = 1,\ldots,n$ such that

$$\bar{A}_G\lambda < 0 \quad \text{for an arbitrary matrix } G \in \mathbb{R}^{n\times n}. \tag{13}$$

Let

$$\varphi(s) = \det\left[\mathbb{I}_n s - \bar{A}_G\right] = s^n + \bar{a}_{n-1}s^{n-1} + \ldots + \bar{a}_1 s + \bar{a}_0 \tag{14}$$

be the characteristic polynomial of the matrix $\bar{A}_G$. The minimal polynomial $\psi(s)$ of the matrix $\bar{A}_G$ is related to the characteristic polynomial (14) by

$$\psi(s) = \frac{\varphi(s)}{D_{n-1}(s)}, \tag{15}$$

where $D_{n-1}(s)$ is the greatest common divisor of all $n-1$ order minors of the matrix $\left[\mathbb{I}_n s - \bar{A}_G\right]$ [3, 6, 24]. From (15) it follows that $\psi(s) = \varphi(s)$ if and only if $D_{n-1}(s) = 1$.

Definition 5. The matrix $\bar{A}_G$ is called cyclic if $\psi(s) = \varphi(s)$.

Theorem 3. [25] The real matrix

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & 0 & \ldots & 0 & 0 \\ a_{21} & a_{22} & a_{23} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \ldots & a_{n-2,n-1} & 0 \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \ldots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{n,n-1} & a_{nn} \end{bmatrix} \tag{16a}$$

and

$$A_2 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n-2} & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2,n-2} & a_{2,n-1} & a_{2n} \\ 0 & a_{32} & \dots & a_{3,n-2} & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{n,n-1} & a_{nn} \end{bmatrix} \tag{16b}$$

are cyclic matrices if the matrix (16a) satisfies the condition

$$a_{12},\ a_{23},\ \dots,\ a_{n-2,n-1},\ a_{n-1,n} \neq 0 \tag{17a}$$

and the matrix (16b)

$$a_{21},\ a_{32},\ \dots,\ a_{n-1,n-2},\ a_{n,n-1} \neq 0, \tag{17b}$$

respectively.

Remark 1. Every square matrix with only one nonzero entry in each row and in each column and its inverse are cyclic matrices.

Remark 2. Every nonsingular matrix $A \in \mathbb{R}^{2\times 2}$ ($\det A \neq 0$) is cyclic.

Definition 6. The descriptor system is called normal if its matrix $\bar{A}_G$ is cyclic.

Normal systems have very useful properties and play important role in technical sciences [2–4, 24].

3. POSITIVE DESCRIPTOR ELECTRICAL CIRCUITS WITH CYCLIC STATE MATRICES

Consider the electrical circuit shown in Fig. 1 with given resistances $R_1, R_2, \dots, R_n$, inductances $L_1, L_2, \dots, L_n$ and source voltage $e = e(t)$.

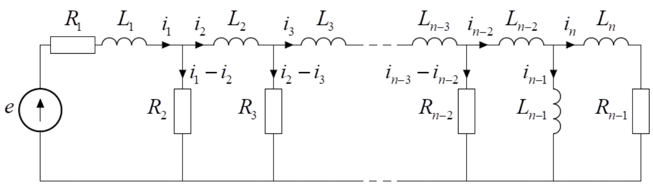

Fig. 1. Electrical circuit with inductances

Using Kirchoff's laws we may write the equations

$$\begin{aligned} &L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + (R_1 + R_2)i_1 - R_2 i_2 = e, \\ &L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} + (R_2 + R_3)i_2 - R_2 i_1 - R_3 i_3 = 0, \\ &\vdots \\ &L_{n-2} \frac{\mathrm{d}i_{n-2}}{\mathrm{d}t} + L_{n-1} \frac{\mathrm{d}i_{n-1}}{\mathrm{d}t} - R_{n-2}\left(i_{n-3} - i_{n-2}\right) = 0, \\ &L_n \frac{\mathrm{d}i_n}{\mathrm{d}t} - L_{n-1} \frac{\mathrm{d}i_{n-1}}{\mathrm{d}t} + R_{n-1} i_n = 0, \\ &i_n + i_{n-1} - i_{n-2} = 0, \end{aligned} \tag{18}$$

which can be written in the form

$$E_L \frac{\mathrm{d}x_L}{\mathrm{d}t} = A_L x_L + B_L e, \tag{19a}$$

where

$$E_L = \begin{bmatrix} L_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & L_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & L_{n-2} & L_{n-1} & 0 \\ 0 & 0 & \dots & 0 & -L_{n-1} & L_n \\ 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}, \quad B_L = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$A_L = \begin{bmatrix} -R_{12} & R_2 & \dots & 0 & 0 & 0 & 0 \\ R_2 & -R_{23} & \dots & 0 & 0 & 0 & 0 \\ 0 & R_3 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & R_{n-2} & -R_{n-2} & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & -R_{n-1} \\ 0 & 0 & \dots & 0 & -1 & 1 & 1 \end{bmatrix}, \tag{19b}$$

$$x_L = \begin{bmatrix} i_1 & i_2 & \dots & i_{n-1} & i_n \end{bmatrix}^T, \quad R_{ij} = R_i + R_j.$$

As the output $y = y(t)$ of the electrical circuit the voltage on the resistor R_{n-1} is chosen

$$y = R_{n-1} i_n = C_L x_L, \quad C_L = \begin{bmatrix} 0 & \dots & 0 & R_{n-1} \end{bmatrix}. \tag{19c}$$

To simplify the notation we assume $n = 4$. Thus, we have

$$E_L = \begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & L_3 & 0 \\ 0 & 0 & -L_3 & L_4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_L = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$A_L = \begin{bmatrix} -(R_1+R_2) & R_2 & 0 & 0 \\ R_2 & -R_2 & 0 & 0 \\ 0 & 0 & 0 & -R_3 \\ 0 & -1 & 1 & 1 \end{bmatrix}, \tag{20}$$

$$C_L = \begin{bmatrix} 0 & 0 & 0 & R_4 \end{bmatrix}.$$

Next, using (20) and (3b) for $c = 0$ we compute

$$\bar{E}_L = \begin{bmatrix} \frac{L_1}{R_1} & \frac{L_2}{R_1} & \frac{L_3}{R_1} & 0 \\ \frac{L_1}{R_1} & \frac{L_2(R_1+R_2)}{R_1R_2} & \frac{L_3(R_1+R_2)}{R_1R_2} & 0 \\ \frac{L_1}{R_1} & \frac{L_2(R_1+R_2)}{R_1R_2} & \frac{L_3}{R_3} + \frac{L_3(R_1+R_2)}{R_1R_2} & -\frac{L_4}{R_3} \\ 0 & 0 & -\frac{L_3}{R_3} & \frac{L_4}{R_3} \end{bmatrix} \tag{21a}$$

$$\bar{A}_L = -\mathbb{I}_4, \quad \bar{B}_L = \begin{bmatrix} \frac{1}{R_1} & \frac{1}{R_1} & \frac{1}{R_1} & 0 \end{bmatrix}^T$$

$$\bar{E}_L^D = \begin{bmatrix} \frac{R_1+R_2}{L_1} & -\frac{L_2R_2(L_3+L_4)}{L_1\Delta_L} & -\frac{L_3L_4R_2}{L_1\Delta_L} & -\frac{L_3L_4R_2}{L_1\Delta_L} \\ -\frac{R_2(L_3+L_4)}{\Delta_L} & \frac{\bar{e}_{22}}{\Delta_L^2} & \frac{\bar{e}_{23}}{\Delta_L^2} & \frac{\bar{e}_{24}}{\Delta_L^2} \\ -\frac{L_4R_2}{\Delta_L} & \frac{\bar{e}_{32}}{\Delta_L^2} & \frac{\bar{e}_{33}}{\Delta_L^2} & \frac{\bar{e}_{34}}{\Delta_L^2} \\ -\frac{L_3R_2}{\Delta_L} & \frac{\bar{e}_{42}}{\Delta_L^2} & \frac{\bar{e}_{43}}{\Delta_L^2} & \frac{\bar{e}_{44}}{\Delta_L^2} \end{bmatrix} \tag{21b}$$

and from (8c), (21a), (21b) we have

$$\bar{A}_{1L} = -\bar{E}_L^D, \quad \bar{B}_{1L} = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{B}_{2L} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\bar{N}_L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{21c}$$

$$\bar{E}_L\bar{E}_L^D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{L_2(L_3+L_4)}{\Delta_L} & \frac{L_3L_4}{\Delta_L} & \frac{L_3L_4}{\Delta_L} \\ 0 & \frac{L_2L_4}{\Delta_L} & \frac{L_3(L_2+L_4)}{\Delta_L} & -\frac{L_2L_4}{\Delta_L} \\ 0 & \frac{L_2L_3}{\Delta_L} & -\frac{L_2L_3}{\Delta_L} & \frac{L_4(L_2+L_3)}{\Delta_L} \end{bmatrix},$$

where

$$\begin{aligned} \Delta_L &= L_2L_3+L_2L_4+L_3L_4, \\ \bar{e}_{22} &= L_2(L_3^2R_2+L_3^2R_3+L_4^2R_2+2L_3L_4R_2), \\ \bar{e}_{23} &= L_3(R_2L_4^2+R_2L_3L_4-L_2L_3R_3), \\ \bar{e}_{24} &= L_3L_4(L_2R_3+L_3R_2+L_3R_3+L_4R_2), \\ \bar{e}_{32} &= L_2(R_2L_4^2+R_2L_3L_4-L_2L_3R_3), \\ \bar{e}_{33} &= L_3(R_3L_2^2+R_2L_4^2), \\ \bar{e}_{34} &= -L_4(R_3L_2^2+R_3L_2L_3-R_2L_3L_4), \\ \bar{e}_{42} &= L_2L_3(L_2R_3+L_3R_2+L_3R_3+L_4R_2), \\ \bar{e}_{43} &= -L_3(R_3L_2^2+L_2L_3R_3-R_2L_3L_4), \\ \bar{e}_{44} &= L_4(L_2^2R_3+L_3^2R_2+L_3^2R_3+2L_2L_3R_3). \end{aligned} \tag{21d}$$

For the matrix

$$G = \begin{bmatrix} 0 & \frac{R_2}{L_1} & \frac{R_2}{L_2} & -\frac{L_4R_2}{L_2L_3} \\ 0 & -\frac{g_{22}}{\Delta_L} & 0 & 0 \\ 0 & 0 & -\frac{g_{33}}{L_4\Delta_L} & 0 \\ 0 & 0 & 0 & -\frac{g_{44}}{\Delta_L} \end{bmatrix}, \tag{22a}$$

$$\begin{aligned} g_{22} &= L_2R_3+L_3R_2+L_3R_3+L_4R_2, \\ g_{33} &= R_2L_4^2+L_3R_2L_4-L_2L_3R_3, \\ g_{44} &= L_2R_3+L_3R_2+L_3R_3+L_4R_2 \end{aligned} \tag{22b}$$

we obtain

$$\bar{A}_{GL} = \bar{E}_L^D\bar{A}_L + G(\mathbb{I}_4 - \bar{E}_L\bar{E}_L^D) = \begin{bmatrix} -\frac{R_1+R_2}{L_1} & \frac{R_2}{L_1} & 0 & 0 \\ \frac{R_2(L_3+L_4)}{\Delta_L} & -\frac{\bar{a}_{22}}{\Delta_L^2} & \frac{L_3R_3}{\Delta_L} & 0 \\ \frac{L_4R_2}{\Delta_L} & 0 & -\frac{L_4R_2}{\Delta_L} & \frac{\bar{a}_{34}}{\Delta_L} \\ \frac{L_3R_2}{\Delta_L} & 0 & -\frac{L_3R_2}{\Delta_L} & -\frac{\bar{a}_{44}}{\Delta_L} \end{bmatrix}, \tag{23a}$$

$$\begin{aligned} \bar{a}_{22} &= L_3R_2+L_3R_3+L_4R_2, \\ \bar{a}_{34} &= L_2R_3-L_4R_2, \\ \bar{a}_{44} &= L_2R_3+L_3R_2+L_3R_3, \end{aligned} \tag{23b}$$

where Δ_L is defined by (21d). From (23) it follows that there does not exist a matrix $G \in \mathbb{R}^{4\times 4}$ such that $\bar{A}_{GL} = \bar{E}_L^D\bar{A}_L + G(\mathbb{I}_4 - \bar{E}_L\bar{E}_L^D)$ is a Metzler matrix (i.e. the negative element in the fourth row and third column of $\bar{A}_{GL}$ can not be eliminated). However, by Theorem 3, the matrix $\bar{A}_{GL}$ is a cyclic matrix.

Conclusion 1. The descriptor electrical circuit of R, L, e type shown in Fig. 1 is a non-positive cyclic one. Similar results can be obtained for any n.

Consider the electrical circuit shown in Fig. 2 with given resistances $R_1, R_2, \ldots, R_n$ capacitances $C_1, C_2, \ldots, C_n$, and source voltages $e_1 = e_1(t)$, $e_2 = e_2(t)$.

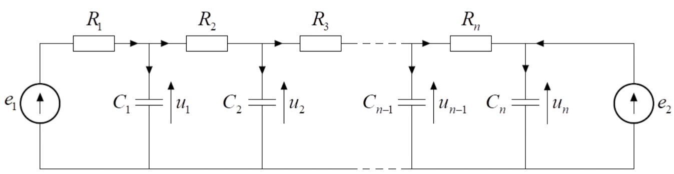

Fig. 2. Electrical circuit with capacitances

Using Kirchoff's laws we may write the equations

$$\begin{aligned} &\frac{e_1-u_1}{R_1} = \frac{u_1-u_2}{R_2} + C_1\frac{\mathrm{d}u_1}{\mathrm{d}t}, \\ &\frac{u_1-u_2}{R_2} = \frac{u_2-u_3}{R_3} + C_2\frac{\mathrm{d}u_2}{\mathrm{d}t}, \\ &\vdots \\ &\frac{u_{n-2}-u_{n-1}}{R_{n-1}} = \frac{u_{n-1}-u_n}{R_n} + C_{n-1}\frac{\mathrm{d}u_{n-1}}{\mathrm{d}t}, \\ &u_n - e_2 = 0, \end{aligned} \tag{24}$$

which can be written in the form

$$E_C\frac{\mathrm{d}x_C}{\mathrm{d}t} = A_Cx_C + B_Ce, \tag{25}$$

where

$$E_C = \begin{bmatrix} C_1 & 0 & \dots & 0 & 0 \\ 0 & C_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & C_{n-1} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad B_C = \begin{bmatrix} \frac{1}{R_1} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -1 \end{bmatrix},$$

$$A_C = \begin{bmatrix} -a_1 & \frac{1}{R_2} & 0 & \dots & 0 \\ \frac{1}{R_2} & -a_2 & \frac{1}{R_3} & \dots & 0 \\ 0 & \frac{1}{R_3} & -a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}, \quad a_n = \left(\frac{1}{R_n} + \frac{1}{R_{n+1}}\right), \tag{26}$$

$$x_C = \begin{bmatrix} u_1 & u_2 & \dots & u_{n-1} & u_n \end{bmatrix}^T.$$

As the output $y = y(t)$ of the electrical circuit we choose the voltage on the capacitor C_1

$$y = u_1 = C_C x_C, \quad C_C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}. \tag{27}$$

To simplify the notation we assume $n = 4$. Thus, we have

$$E_C = \begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_C = \begin{bmatrix} \frac{1}{R_1} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix},$$

$$A_C = \begin{bmatrix} -\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & \frac{1}{R_2} & 0 & 0 \\ \frac{1}{R_2} & -\left(\frac{1}{R_2}+\frac{1}{R_3}\right) & \frac{1}{R_3} & 0 \\ 0 & \frac{1}{R_3} & -\left(\frac{1}{R_3}+\frac{1}{R_4}\right) & \frac{1}{R_4} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{28}$$

$$C_C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

Next, using (28) and (3b) for $c = 0$ we compute

$$\bar{E}_C = \frac{1}{R_{14}} \begin{bmatrix} C_1R_1R_{24} & C_2R_1R_{34} & C_3R_1R_4 & 0 \\ C_1R_1R_{34} & C_2R_{12}R_{34} & C_3R_{12}R_4 & 0 \\ C_1R_1R_4 & C_2R_{12}R_4 & C_3R_{13}R_4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{A}_C = -\mathbb{I}_4, \quad \bar{B}_C = \frac{1}{R_{14}} \begin{bmatrix} R_{24} & R_1 \\ R_{34} & R_{12} \\ R_4 & R_{13} \\ 0 & R_{14} \end{bmatrix}, \quad R_{ij} = \sum_{k=i}^{j} R_k, \tag{29a}$$

$$\bar{E}_C^D = \begin{bmatrix} \frac{R_1+R_2}{C_1R_1R_2} & -\frac{1}{C_1R_2} & 0 & 0 \\ -\frac{1}{C_2R_2} & \frac{R_2+R_3}{C_2R_2R_3} & -\frac{1}{C_2R_3} & 0 \\ 0 & -\frac{1}{C_3R_3} & \frac{R_3+R_4}{C_3R_3R_4} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{29b}$$

and from (8c), (29a), (29b) we have

$$\bar{A}_{1C} = -\bar{E}_C^D,$$

$$\bar{B}_{1C} = \begin{bmatrix} \frac{1}{C_1R_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{C_3C_4} \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_{2C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \tag{29c}$$

$$\bar{N}_C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{E}_C\bar{E}_C^D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For the matrix

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \tag{30}$$

we obtain

$$\bar{A}_{GC} = \bar{E}_C^D\bar{A}_C + G(\mathbb{I}_4 - \bar{E}_C\bar{E}_C^D)$$
$$= \begin{bmatrix} -\frac{R_1+R_2}{C_1R_1R_2} & \frac{1}{C_1R_2} & 0 & 0 \\ \frac{1}{C_2R_2} & -\frac{R_2+R_3}{C_2R_2R_3} & \frac{1}{C_2R_3} & 0 \\ 0 & \frac{1}{C_3R_3} & -\frac{R_3+R_4}{C_3R_3R_4} & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}. \tag{31}$$

From (29)-(31) it follows that all positivity conditions are met and $\bar{A}_{GC}$ is a cyclic Metzler matrix.

Conclusion 2. The descriptor electrical circuit of R, C, e type shown in Fig. 2 is a positive cyclic one. Similar results can be obtained for any n.

Therefore, the following theorem has been proved.

Theorem 4. The following statements are true:

1) descriptor electrical circuits of *RL*-type chain structure (as shown in Fig. 1) are non-positive and cyclic,
2) descriptor electrical circuits of *RC*-type chain structure (as shown in Fig. 2) are positive and cyclic.

4. FRACTIONAL DESCRIPTOR ELECTRICAL CIRCUITS

Consider the fractional linear electrical circuit [5, 7, 20] described by the equations

$$E\frac{\mathrm{d}^{\alpha}x(t)}{\mathrm{d}t^{\alpha}} = Ax(t)+Bu(t), \quad 0<\alpha<1, \tag{32a}$$

$$y(t) = Cx(t), \tag{32b}$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^p$ are the state, input and output vectors and E, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{p\times n}$,

$$\frac{\mathrm{d}^{\alpha}x(t)}{\mathrm{d}t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^{\alpha}}\mathrm{d}\tau,\ \ \dot{x}(\tau)=\frac{\mathrm{d}x(t)}{\mathrm{d}t} \tag{32c}$$

is the Caputo fractional derivative and

$$\Gamma(z)=\int_0^{\infty} t^{z-1}e^{-t}dt,\ \ \mathrm{Re}(z)>0 \tag{32d}$$

is the gamma function [5, 11]. Assuming that for some chosen $c\in\mathbb{C}$ we have $\det[Ec-A]\neq 0$ and premultiplying (32a) by $[Ec-A]^{-1}$ we obtain

$$\bar{E}\frac{\mathrm{d}^{\alpha}x(t)}{\mathrm{d}t^{\alpha}} = \bar{A}x(t)+\bar{B}u(t), \tag{33a}$$

where

$$\bar{E}=[Ec-A]^{-1}E,\ \ \bar{A}=[Ec-A]^{-1}A,\ \ \bar{B}=[Ec-A]^{-1}B. \tag{33b}$$

Definition 7. [20] The fractional descriptor continuous-time linear system (33a), (32b) (or equivalently (32)) is called (internally) positive if $x(t)\in\mathbb{R}^n_+$ and $y(t)\in\mathbb{R}^p_+$, $t\geq 0$ for any consistent initial conditions $x(0)\subset\mathbb{R}^n_+$ and all admissible inputs $u(t)\subset\mathbb{R}^m_+$, $t\geq 0$ such that $\frac{\mathrm{d}^{k\alpha}u(t)}{\mathrm{d}t^{k\alpha}}\in\mathbb{R}^m_+$, $k=1,\dots,q-1$.

Theorem 5. [20] The fractional descriptor continuous-time linear system (33a), (32b) (or equivalently (32)) is positive if and only if there exists a matrix $G\in\mathbb{R}^{n\times n}$ such that

$$\bar{A}_G\in\mathbb{M}_n \tag{34a}$$

and

$$\begin{aligned}&\mathrm{Im}\,\bar{E}\bar{E}^D\subset\mathbb{R}^n_+,\ \ \bar{B}_1\in\mathbb{R}^{n\times m}_+,\ \ C\in\mathbb{R}^{p\times n}_+,\ \ D\in\mathbb{R}^{p\times m}_+\\ &\quad -\bar{N}^k\bar{B}_2\in\mathbb{R}^{n\times m}_+,\ \ k=1,\dots,q-1,\end{aligned} \tag{34b}$$

where the matrix $\bar{A}_G$ is defined by (9) and the matrices $\bar{B}_1$, $\bar{B}_2$, $\bar{N}$ are given by (8c).

Definition 8. The positive fractional descriptor continuous-time linear system (33a), (32b) (or equivalently (32)) is called asymptotically stable if

$$\lim_{t\to\infty}x(t)=0 \tag{35}$$

for all consistent initial conditions $x_0\in\bar{E}\bar{E}^D v$ (where $v\in\mathbb{R}^n$ is an arbitrary vector) and $u(t)=0$.

Theorem 6. [20] The positive fractional descriptor continuous-time linear system (33a), (32b) (or equivalently (32)) for $u(t)=0$ is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. All coefficients of the characteristic equation

$$\begin{aligned}\det[Es^{\alpha}-A] &= a_r s^{r\alpha}+a_{r-1}s^{(r-1)\alpha}+\dots\\ &\quad +a_1 s^{\alpha}+a_0=0\end{aligned} \tag{36}$$

are positive, i.e. $a_i>0$ for $i=0,1,\dots,r$, where $r<n$.

2. There exists strictly positive vector $\lambda^T=[\,\lambda_1\ \ \dots\ \ \lambda_n\,]$, $\lambda_k>0$, $k=1,\dots,n$ such that

$$\bar{A}_G\lambda<0 \quad \text{for an arbitrary matrix } G\in\mathbb{R}^{n\times n}. \tag{37}$$

In a similar way as for integer order circuits we can write using Kirchhoff's laws similar equations as (18) and (24) substituting the first order derivatives by corresponding fractional α-order derivatives of the currents in the coils and voltages on the capacitors. In a similar way as for descriptor electrical circuits we can prove for the fractional descriptor electrical circuits the following theorem.

Theorem 7. The following statements are true:

1) fractional descriptor electrical circuits of *RL*-type chain structure (as shown in Fig. 1) are non-positive and cyclic,
2) fractional descriptor electrical circuits of *RC*-type chain structure (as shown in Fig. 2) are positive and cyclic.

5. CONCLUDING REMARKS

The positivity and cyclicity of descriptor electrical circuits with chain structure have been investigated. Some new results concerning non-positive and positive descriptor electrical circuits with cyclic state matrices have been established (Theorem 4) and next extended to fractional electrical circuits (Theorem 7). An open problem is an extension of these considerations to fractional descriptor linear electrical circuits and systems with different fractional orders.

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