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SPECIAL SECTION

Application of heuristic methods to the identification of the parameters of discrete-continuous models

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Abstract. The article presents the process of identifying discrete-continuous models with the use of heuristic algorithms. A stepped cantilever beam was used as an example of a discrete-continuous model. The theoretical model was developed based on the formalism of Lagrange multipliers and the Timoshenko theory. Based on experimental research, the theoretical model was validated and the optimization problem was formulated. Optimizations were made for two algorithms: genetic (GA) and particle swarm (PSO). The minimization of the relative error of the obtained experimental and numerical results was used as the objective function. The performed process of identifying the theoretical model can be used to determine the eigenfrequencies of models without the need to conduct experimental tests. The presented methodology regarding the parameter identification of the beams with the variable cross-sectional area (according to the Timosheno theory) with additional discrete components allows us to solve similar problems without the need to exit complex patterns.

Key words: discrete-continuous model; experimental modal analysis; optimization; identification; vibrations.

1. INTRODUCTION

Due to economic aspects and time constraints, experimental studies of prototypes of working machines are not profitable [1]. In the systems that are exposed to continual vibrations, it is necessary to protect these systems against the resonance phenomenon [2]. Modeling is often used to determine the behavior of these systems under operating conditions. However, during the design or construction phase, it is necessary to identify and optimize the adopted theoretical models. In order to identify the designated theoretical models, it is necessary to conduct experimental research that will enable the determination of the accuracy and fidelity to the real object.

In the literature, one can find works that most often use heuristic methods to optimize discrete or discrete-continuous models. The determination of the location and depth of the crack in the Euler-Bernoulli cantilever beam was presented in the work [3]. A modified cuckoo optimization algorithm was used to detect the damage. The difference of squares between the experimental and numerical results was adopted as the objective function. The work [4] presents an inverse analysis of the crack location problem based on the obtained theoretical and experimental frequencies. A modified particle swarm optimization (PSO) algorithm was used for identification. The modification of the algorithm was to use the strategy of squeezing the domain of searching space in each iteration. As a result, the process of finding the optimal solution was accelerated and the inherent structure of the algorithm was preserved. The assessment of the cantilever beam damage detection using the PSO algorithm is also presented in the article [5]. Apart from determining the location of the crack, the influence of particular parameters of the algorithm on the structural condition monitoring was also determined. The mean fitness value and success rate were used as the criteria for measuring the convergence and stability of the algorithm. The process of identifying the viscous damping parameter of the cantilever beam has been described in the article [6]. The PSO algorithm was used as an identification tool. Objective functions were determined using incomplete complex eigenvectors that appear in the model in relation to external damping. The proposed method can be used even in the situation when incomplete modal data is available. In the optimization processes of dynamic problems, apart from the PSO algorithm, a genetic algorithm (GA) or its modifications are also often used [7, 8]. In paper [9], an optimized regression model was developed to determine the damping coefficient of a cantilever beam. Xia *et al*. used a GA to identify the optimal variables of the model. Cross-validation and support vector regression were performed and compared with other regression methods. The obtained numerical results were verified by the experimental results. The problem of identifying cracks in a cantilever beam using both the PSO and GA algorithms is presented in tthe manuscript [8]. Based on the eigenfrequencies

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of the beam and the mode shape, the objective function was determined. A reinforced beam made of graphite-epoxy material was analyzed. The finite element method was used for numerical simulations, in which the Timoshenko beam theory and laminated composite theory were applied. The identification of the parameters of a discrete-continuous system using the finite element method and the iterative method was presented among others in the work [10].

This paper proposes the process of identifying the parameters of discrete-continuous models on the basis of known experimental frequencies with the use of heuristic algorithms. The identification included a solution to the optimization problem based on two methods: GA and PSO. The proposed method enables the analysis of the changes impact of individual model parameters on the vibrations of the considered system without the need to perform experimental tests. The methodology presented in the paper for the identification of parameters of beams with a variable cross-sectional area with additional discrete elements can be used for beams of any shape (variable cross-sectional area) and with any method of attachment, as well as to take into account additional discrete elements. Knowing the solution presented in the work, it is possible to automatically formulate and solve similar problems, without the need to derive complex formulas.

2. RESEARCH METHODS

2.1. Genetic algorithm

The genetic algorithm is one of the most frequently used evolutionary algorithms that were initiated by J. Holland in 1975 [11]. One of their most important advantages is the effective mechanism of searching the large solution space. It is used in optimization problems of complex nonlinear models, where finding the optimal location is often a difficult task. Like the other evolutionary algorithms, the main operators are selection, mutation, and crossover [12].

The principle of the classical GA can be presented in the following points (Fig. 1): first, the population of chromosomes that are solution candidates to a problem is randomly generated;

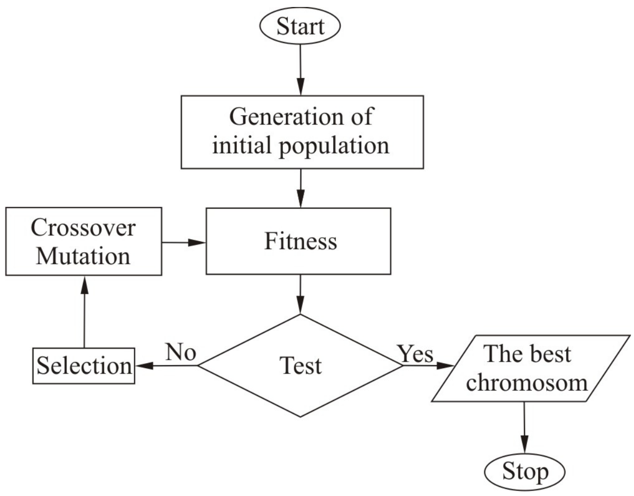

Fig. 1. Genetic algorithm

then the fitness function of each chromosome in the population is calculated; next Selection, Crossover, and Mutation are repeated until a steady number of offspring will be created or the value of the solution will be satisfactory [13].

The Selection operator selects chromosomes in the population for reproduction. The fitter the chromosome, the more times it is likely to be selected to reproduce. Crossover and Mutation are the reproduction operators, the former forms a new chromosome by combining parts of each of the two parental chromosomes, and the latter forms a new chromosome by making alterations to the values of genes in a copy of a single parent chromosome.

2.2. Particle swarm optimization

The PSO algorithm is one of the most modern heuristic global optimization methods and also an optimization algorithm, it was firstly proposed by Kennedy and Eberhart in 1995 in work [14].

The PSO algorithm is based on observation of phenomena occurring in nature, such as foraging of a swarm of insects or shoal of fish. Each particle of the swarm is able to remember and use its experience that is taken from the whole iteration process and, also, is able to communicate with other members. The swarm of particles is able to identify "good" areas of the domain and can search deeply in these areas for an optimum.

The PSO algorithm, used in this work, is presented in a simplified form in Fig. 2.

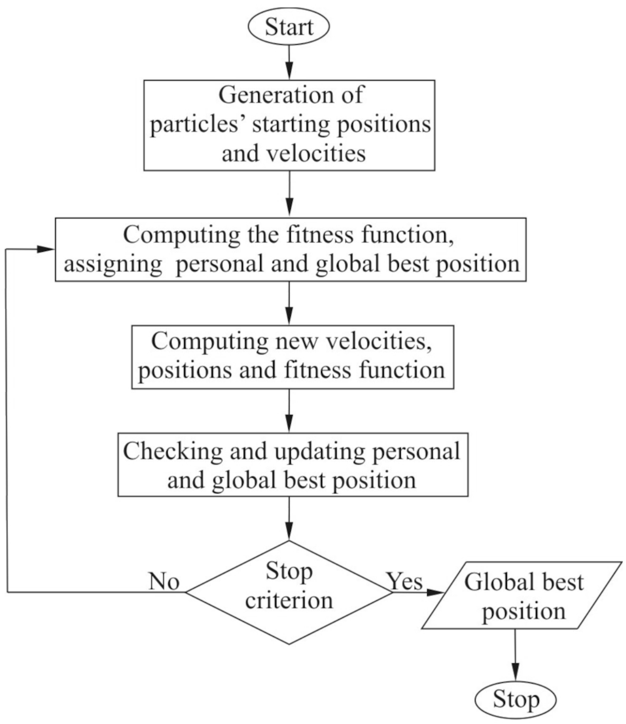

Fig. 2. PSO algorithm

Initial values of sought after variables (the position of particular particle) and starting velocities are random. Then, in an iteration step $n+1$, distance covered by a particle in *m-th* direction (the velocity of the particle in m-th direction) can be described as following [15, 16]

$$V_m^{(n+1)} = \chi\left(wV_m^{(n)} + c_1 r_1\left(p_m - xm^{(n)}\right)\right.$$
$$\left.+ c_2 r_2\left(g_m - x_m^{(n)}\right)\right), \tag{1}$$

where $m = 1, 2$, χ is a constriction factor, w is a weight of coefficient, $V_m^{(n)}$ is a velocity in the previous iteration step, c_1 and c_2 are cognitive and social parameters suitable, r_1 and r_2 are random numbers taken from (0, 1), p_m is a personal best position of the considered particle from the whole iteration process, g_m is a global best position obtained by the entire swarm and $x_m^{(n)}$ is a particle position in the previous iteration step.

A new position for each particle in each considered direction is equal to

$$x_m^{(n+1)} = x_m^{(n)} + V_m^{(n+1)}. \tag{2}$$

3. PROBLEM FORMULATION

A rigid restraint is often used as a type of fixing in computational models. However, in real systems, such fastening is impossible to obtain. It is used because of the simplicity and lack of real parameters values in the fixing system. The theoretical models developed based on such fastening do not fully reflect the real object. Therefore, it is necessary to conduct experimental verification of the mathematical models. When the obtained experimental results differ significantly from the theoretical results, it is necessary to perform the identification of the developed model.

The problem of identifying the parameters of the discrete-continuous model was presented in the example of the stepped cantilever beam (Fig. 3), which was circumscribed according to the Timoshenko theory [17].

Multi-stepped beams are important in the design process of various types of structures (e.g. aircraft, manipulators, buildings). Their variable geometry and/or material properties along the length can be used to increase the strength and stability of the structure, but also to reduce weight and volume [18, 19].

To formulate and solve the problem of free vibrations of the analyzed system, the formalism of Lagrange multipliers was used [20]. The method is especially used for determining the system frequency consisting of many continuous and discrete components. The free vibration problem of the cantilever beam can be reduced to the matrix system of equations, which can be presented in the form:

$$\mathbf{C}\vec{\Lambda} = 0, \tag{3}$$

where:

$$\vec{\Lambda} = [\Lambda_1,\ \Lambda_2,\ \ldots,\ \Lambda_6]^T \tag{4}$$

is the vector of Lagrange multipliers, $\Lambda_1,\ \Lambda_2,\ \ldots,\ \Lambda_6$ are amplitudes of Lagrange multipliers and the square matrix $\mathbf{C}$ has the form

$$\mathbf{C} = \begin{bmatrix} C_{1_{1,1}} & C_{1_{1,2}} & C_{1_{1,3}} & C_{1_{1,4}} & 0 & 0 \\ C_{1_{2,1}} & C_{1_{2,2}} & C_{1_{2,3}} & C_{1_{2,4}} & 0 & 0 \\ C_{1_{3,1}} & C_{1_{3,2}} & c_{3,3} & c_{3,4} & -C_{2_{3,5}} & -C_{2_{3,6}} \\ C_{1_{4,1}} & C_{1_{4,2}} & c_{4,3} & c_{4,4} & -C_{2_{4,5}} & -C_{2_{4,6}} \\ 0 & 0 & -C_{2_{5,3}} & -C_{2_{5,4}} & c_{5,5} & c_{5,6} \\ 0 & 0 & -C_{2_{6,3}} & -C_{2_{6,4}} & c_{6,5} & c_{6,6} \end{bmatrix}, \tag{5}$$

where:

$$\begin{aligned} c_{3,3} &= C_{1_{3,3}} + C_{2_{3,3}}, & c_{3,4} &= C_{1_{3,4}} + C_{2_{3,4}}, \\ c_{4,3} &= C_{1_{4,3}} + C_{2_{4,3}}, & c_{4,4} &= C_{1_{4,4}} + C_{2_{4,4}}, \\ c_{5,5} &= C_{2_{5,5}} + C_{3_{5,5}}, & c_{5,6} &= C_{2_{5,6}} + C_{3_{5,6}}, \\ c_{6,5} &= C_{2_{6,5}} + C_{3_{6,5}}, & c_{6,6} &= C_{2_{6,6}} + C_{3_{6,6}}. \end{aligned} \tag{6}$$

Coefficients $C_{n_{k,r}}$, which can be defined as

$$C_{n_{k,r}} = \sum_{i=0}^{m} \frac{b_{n_{i,k}} b_{n_{i,r}}}{K_{n_i} - f^2 M_{n_i}}, \tag{7}$$

describe the dynamic properties of the individual parts of the beam. The selected denotations $b_{n_{i,r}}$ map the i-th translational and rotational vibrational modes of n-th beam segments without additional elements. The descriptions of the mathematical expressions presented above are discussed in detail in works [20, 21].

Using the existence of non-trivial solutions of the system of equations (equation (3)), the equation describing free vibrations of a beam takes the form:

$$\det \mathbf{C} = 0. \tag{8}$$

On the basis of the dependencies determined above, an algorithm and a script were developed to enable numerical simulations. The analyzed cantilever beam was made of S235 steel

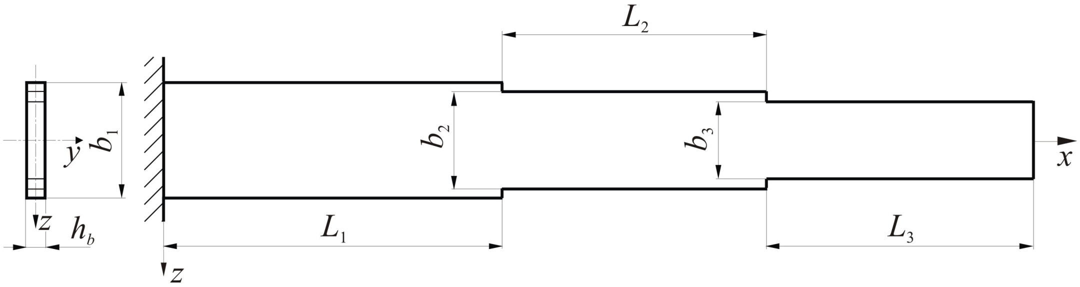

Fig. 3. The cantilever stepped beam

with the following material data: density – 7850 kg/m^3, Young modulus – $2.1 \cdot 10^{10}$ N/m^2. In addition, the following characteristic parameters of the beam were adopted $h_b = 0.005$ m, $L_1 = 0.25$ m, $L_2 = 0.15$ m, $L_3 = 0.10$ m, $b_1 = 0.06$ m, $b_2 = 0.04$ m, $b_3 = 0.02$ m.

Table 1 presents the results of numerical calculations obtained for a different number of terms of coefficients $C_{n_{k,r}}$ (equation (7)).

Table 1
Frequencies from the numerical calculations of the cantilever stepped beam

Number of terms of coefficients $C_{n_{k,r}}$	Frequency [Hz]				
	f_1	f_2	f_3	f_4	f_5
10	24.71	126.45	322.67	635.99	1002.51
20	23.95	121.12	311.72	597.90	963.03
30	23.74	119.72	308.34	588.22	952.29
40	23.63	118.95	306.59	583.84	946.77
60	23.52	118.23	305.06	579.44	941.82

The first stage of the identification process concerned the validation of the adopted theoretical model. The validation was performed by comparing the theoretical and experimental results. Moreover, the relative error was calculated (Table 2).

Table 2
Comparison of numerical and experimental results and the relative error between them

	Frequency [Hz]				
	f_1	f_2	f_3	f_4	f_5
Presented method	23.52	118.23	305.06	579.44	941.82
EMA* [21]	20.8	112.0	290.0	542.0	901.0
Relative error [%]	13.08	5.56	5.19	6.90	4.50

*Experimental Modal Analysis

Based on the values of the obtained relative errors of the natural frequencies (Table 2), one can state that the theoretical model does not correctly reflect the real object.

The differences in the results, obtained through experimental tests and numerical simulations, are caused by the inaccuracy of the beam material data and/or the method of fixing during the experimental tests (ideal restraint). The solution to the first issue is to conduct the strength tests, while the representation of an ideal restraint is impossible to achieve in experimental tests.

4. PROBLEM SOLUTION

Assuming that the beam material data used for calculation are correct, then the theoretical model should be modified by replacing the rigid fixing by rotational C_b and translational K_b spring (Fig. 4). Spring constants values are unknown and must be identified.

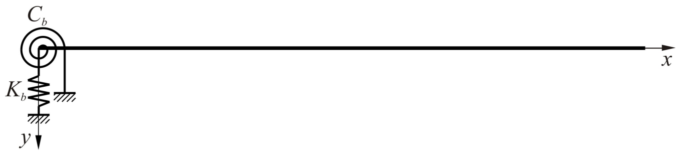

Fig. 4. Elastic fixed beam

Taking into account the elastic fixing of the beam, the matrix **C** can be written in the form

$$\mathbf{C} = \begin{bmatrix} c_{1,1} & C_{1_{1,2}} & C_{1_{1,3}} & C_{1_{1,4}} & 0 & 0 \\ C_{1_{2,1}} & c_{2,2} & C_{1_{2,3}} & C_{1_{2,4}} & 0 & 0 \\ C_{1_{3,1}} & C_{1_{3,2}} & c_{3,3} & c_{3,4} & -C_{2_{3,5}} & -C_{2_{3,6}} \\ C_{1_{4,1}} & C_{1_{4,2}} & c_{4,3} & c_{4,4} & -C_{2_{4,5}} & -C_{2_{4,6}} \\ 0 & 0 & -C_{2_{5,3}} & -C_{2_{5,4}} & c_{5,5} & c_{5,6} \\ 0 & 0 & -C_{2_{6,3}} & -C_{2_{6,4}} & c_{6,5} & c_{6,6} \end{bmatrix}, \quad (9)$$

where:

$$c_{1,1} = C_{1_{1,1}} + \varepsilon_1, \qquad c_{2,2} = C_{1_{2,2}} + \varepsilon_2 \quad (10)$$

and

$$\varepsilon_1 = \frac{1}{K_b}, \qquad \varepsilon_2 = \frac{1}{C_b}. \quad (11)$$

In order to formulate the identification problem the GA [22–25] and PSO method [26–28] have been used. The following objective function was adopted in optimization process [29]:

$$f(K_b, C_b) = \frac{\sum_{i=1}^{n} \left| \frac{f_t^{(n)} - f_e^{(n)}}{f_e^{(n)}} \right| 100\%}{n}, \quad \text{for } n = 5. \quad (12)$$

In this function, the average relative error between the first five frequencies values of free vibration from the experimental tests (f_e) and the numerical simulations (f_t) has been minimized.

4.1. Genetic algorithm

Using the GA, the worked-out algorithm and the numerical program made it possible to obtain optimum values of spring constants K_b and C_b. The values of the springs constants have been searched in the range $[1000,\ 1 \cdot 10^{10}]$, and the rank selection has been used in the genetic algorithm. A two-point crossover was selected as the crossover function in which two points are randomly selected from the parents' chromosomes. The mutation process was done by using bit string mutation which consists in changing the value of the randomly selected bit to the opposite. The first population was obtained by using the random initialization process.

The numerical calculations have been conducted for two different values of genetic operators. However, in the case, where crossover probability has been equal to 80% and mutation probability has been equal to 2%, calculations were performed twice, to verify the correctness of the computations. Due to the long computation time in the matrix **C** 10 terms of coefficients $C_{n_{k,r}}$ have been taken. The population has consisted of 20 chromosomes. The computations have been repeated until the number of offspring reached 400. The obtained results are shown in Table 3. The target function values for the best individuals are presented in Fig. 5.

Table 3
Objective function and values of spring constants representing the elastic support system obtained on the basis of numerical calculations (GA)

No.	c_r	m_r	n_s	Objective function	K_b [N/m]	C_b [Nm]
1	0.50	0.10	400	1.974	$1.97 \cdot 10^7$	3080.2
2	0.65	0.15	400	2.078	$1.47 \cdot 10^7$	3065.1
3	0.80	0.20	400	2.222	$1.00 \cdot 10^7$	3117.5

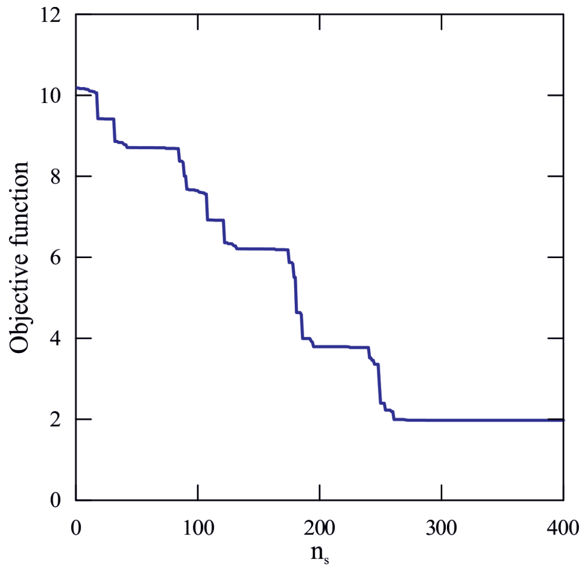

Fig. 5. The GA objective function values for best individuals

Taking into account the above values (spring constants) in the theoretical discrete-continuous model, numerical research has been conducted for each case, and the obtained results are presented in Table 4.

Table 4
Free vibration frequencies obtained after the identification of the theoretical model with the help of GA

	Frequency [Hz]				
	f_1	f_2	f_3	f_4	f_5
1	20.81	110.14	282.17	555.13	873.57
2	20.79	110.00	281.54	552.76	868.24
3	20.84	110.00	280.84	549.02	858.91

Based on the results from Tables 3 and 4, one can state that although the obtained values of springs constants are different from each other (but the order of magnitude is the same), the free vibration frequencies are very similar.

4.2. Particle swarm optimization

Based on the presented PSO algorithm, the algorithm and numerical program have been worked out, and the next values of spring constants K_b and C_b have been determined. In the same way, as in the case of GA, the values of the springs constants have been searched in the range $[1000,\ 1 \cdot 10^{10}]$, and the numerical calculations have been conducted for two different values of coefficients of the velocity of the particle. The individual parameters of the PSO algorithm were adopted as [30,31]: $w = 1$, $c_1 = 1.49445$, $c_2 = 1.49445$. As was the case of the genetic algorithm, in the matrix $\mathbf{C}$ 10 terms of coefficients $C_{n_{k,r}}$ and 400 iterations have been taken to calculation. The particle swarm consisted of 20 individuals. The obtained results are shown in Table 5. In Table 5 the free vibration frequencies for calculated spring constants are also presented.

Table 5
Objective function and values of spring constants representing the elastic support system obtained on the basis of numerical calculations and free vibration frequencies obtained after the identification of the theoretical model using PSO method

χ	w	C_1	C_2	Objective function	K_b [N/m]	C_b [Nm]
1	$\frac{it_{\max} - it}{it_{\max}}$	1.49445	1.49445	2.252	$9.8182 \cdot 10^6$	3241.14
0.8	0.6	2.1	2.0	2.282	$9.04976 \cdot 10^6$	3135.37
Frequency [Hz]		f_1	f_2	f_3	f_4	f_5
		20.96	110.36	281.45	549.69	859.04
		20.86	110.00	280.59	547.72	855.68

where: $it_{\max}$ – maximum iteration, it – actual number of iteration.

Assuming different input parameters of the PSO algorithm, a similar value of the objective function is obtained. Also the stiffness values of the translational spring K_b and the rotational spring C_b are similar.

4.3. Comparison of obtained results

In Table 6, the obtained natural frequencies for identified theoretical models with the help of a GA and PSO as well as the relative errors are summarized.

Comparing the free vibration frequencies of the identified theoretical models and the real object, the high correlation of the results is visible, which demonstrates the correct matching of the mathematical models.

Comparing the results shown in Table 6, it is noted that in the case of the first free vibration frequency the relative error is minimal (0.05%) for the theoretical model identified using GA and is more than 0.7% for use in the identification process of the PSO method. For the second free vibration frequency, the relative error for both analyzed methods is similar. From the third free vibration frequency values of the relative errors are already different, however, from an engineering point of view, the obtained values are similar and both identified theoretical models can be considered for further studies.

Table 6
Free vibration frequencies obtained after the identification of the theoretical model, experimental free vibration frequencies and the relative errors

Frequency [Hz]	f_1	f_2	f_3	f_4	f_5
EMA[1]	20.8	112	290	542	901
GA[2]	20.81	110.14	282.17	555.13	873.57
PSO[3]	20.96	110.36	281.45	549.69	859.04
Relative error [%]	f_1	f_2	f_3	f_4	f_5
GA[2]	0.05	1.66	2.70	2.42	3.04
PSO[3]	0.77	1.46	2.95	1.42	4.66

where:
[1] – Experimental modal anaylsis [21],
[2] – Identified theoretical model using GA,
[3] – Identified theoretical model using PSO algorithm

Comparing the obtained results for the identification of theoretical models with the use of the GA and PSO algorithm, it can be concluded that the obtained objective function value is better in the case of the genetic algorithm (Table 3) and is equal 1.974 (in the case of PSO the best result is 2.282 – Table 5). As a result, the relative error (between the theoretical and experimental values) for natural frequencies is different in both cases. The relative error is minimal for the first free vibration frequencies for the theoretical model identified using genetic algorithms and is equal to 0.05%. For the remaining eigenfrequencies, this error oscillates in the range from about 1.4% to about 4.7%. However, from an engineering point of view, the obtained values for both identified theoretical models can be considered for further studies.

5. CONCLUSIONS

This paper presents the process of identifying the parameters of discrete-continuous models with the use of heuristic methods. Two types of algorithms were used in this process: GA and PSO. Validation was performed using experimental free vibration frequencies of the modeled object.

Using the proposed method, the discrete-continuous model represented by the stepped cantilever beam was identified. The identification process consisted of determining the translational and rotational spring constants, which allowed us to model the system elastic restraint. Due to the significant differences between experimental research and numerical calculations, this process was necessary.

For the case of genetic algorithms, the best value of objective function has been equal to 1.974%, whereas for PSO 2.252%. In both cases, the same number of iterations has been performed. The calculated values of the rotational and translational springs constant in both cases (using genetic algorithm and PSO) are similar and the free vibration frequencies for identified theoretical models do not differ greatly from each other. This indicates that both proposed mathematical models at the determined values of springs constants represent the real system correctly.

The described heuristic algorithms (PSO and GA) used for identification can be used to determine any parameters of a discrete-continuous system for which the natural frequencies were obtained during experimental tests. In addition, the identified model allows analyzing the impact of parameter changes (length, width and thickness of the beam and its segments) on the vibrations of the considered system without the need to conduct further experimental tests, assuming that the item is made of the same material. In the case of shape change of the beam and/or the material would be required to obtain the experimental vibration frequency of the "new" system.

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