

# New stability tests for positive descriptor linear systems

Tadeusz KACZOREK  \*

Białystok University of Technology, Faculty of Electrical Engineering, Wiejska 45D, 15-351 Białystok, Poland

**Abstract.** The asymptotic stability of positive descriptor continuous-time and discrete-time linear systems is considered. New sufficient conditions for stability of positive descriptor linear systems are established. The efficiency of the new stability conditions are demonstrated on numerical examples of continuous-time and discrete-time linear systems.

**Key words:** descriptor; positive; continuous-time; discrete-time; system; sufficient condition; stability.

## 1. INTRODUCTION

In positive systems inputs, state variables and outputs take only nonnegative values for any nonnegative inputs and nonnegative initial conditions [1–4]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollutions models. A variety of models having positive behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems with different fractional orders have been addressed in [3–5]. Descriptor positive systems have been analyzed in [6–8] and descriptor fractional linear systems in [8–11]. Linear positive electrical circuits with state feedbacks have been addressed in [2,4,6]. The stability of fractional linear discrete-time systems has been investigated in [12–16].

In this paper the asymptotic stability of positive descriptor continuous-time and discrete-time linear systems will be addressed.

The paper is organized as follows. In Section 2 the new stability tests for positive descriptor linear continuous-time systems are presented. The corresponding stability tests for positive descriptor discrete-time linear systems are given in Section 3. Concluding remarks are given in Section 4.

The following notation will be used:  $\mathfrak{R}$  – the set of real numbers,  $\mathfrak{R}^{n \times m}$  – the set of  $n \times m$  real matrices,  $\mathfrak{R}_+^{n \times m}$  – the set of  $n \times m$  real matrices with nonnegative entries and  $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$ ,  $M_n$  – the set of  $n \times n$  Metzler matrices (real matrices with nonnegative off-diagonal entries),  $I_n$  – the  $n \times n$  identity matrix.

## 2. POSITIVE DESCRIPTOR CONTINUOUS-TIME LINEAR SYSTEMS

Consider the autonomous continuous-time linear system

$$E\dot{x}(t) = Ax(t), \quad (1)$$

where  $x(t) \in \mathfrak{R}^n$  is the state vector and  $E \in \mathfrak{R}^{n \times n}$ ,  $A \in \mathfrak{R}^{n \times n}$ .

The system (1) satisfies the conditions  $\det E = 0$  and

$$\det[Es - A] \neq 0. \quad (2)$$

The following elementary operations on real matrices will be used [2,3]:

1. Multiplication of any  $i$ -th row (column) by the number  $a$ . This operation will be denoted by  $L[i \times a]$  for row operation and by  $R[i \times a]$  for column operation.
2. Addition to any  $i$ -th row (column) of the  $j$ -th row (column) multiplied by any number  $b$ . This operation will be denoted by  $L[i + j \times b]$  for row operation and by  $R[i + j \times b]$  for column operation.
3. The interchange of rows  $i$  and  $j$  will be denoted by  $L[i, j]$  and for columns by  $R[i, j]$ .

The elementary operations do not change the rank of the matrices [2,3].

Performing elementary row operations on the array

$$E, A \quad (3)$$

or equivalently on (1) by elimination the linearly dependent rows in the matrix  $E$  we obtain

$$\begin{bmatrix} E_1 & A_1 \\ 0 & A_2 \end{bmatrix} \quad (4)$$

and

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad (5)$$

\*e-mail: t.kaczorek@pb.edu.pl

Manuscript submitted 2021-11-09, revised 2022-01-14, initially accepted for publication 2022-02-06, published in June 2022.

where

$$\begin{aligned} x_1(t) &\in \mathfrak{R}^{n_1}, & x_2(t) &\in \mathfrak{R}^{n_2}, & A_1 &= \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}, \\ A_2 &= \begin{bmatrix} A_{21} & A_{22} \end{bmatrix}, & A_{11} &\in \mathfrak{R}^{n_1 \times n_1}, & A_{12} &\in \mathfrak{R}^{n_1 \times n_2}, \\ A_{21} &\in \mathfrak{R}^{n_2 \times n_1}, & A_{22} &\in \mathfrak{R}^{n_2 \times n_2}, & n_1 + n_2 &= n. \end{aligned} \quad (6)$$

The matrix  $A \in \mathfrak{R}_+^{n_2 \times n_2}$  is called monomial if each its column (row) contains only one positive entry and its remaining entries are zero.

**Lemma 1.** The system (1) is (internally) positive if the matrix  $E_1 \in \mathfrak{R}_+^{n_1 \times n}$  contains  $n_1$  monomial columns and

$$\begin{aligned} A_{11} &\in M_{n_1}, & A_{12} &\in \mathfrak{R}_+^{n_1 \times n_2}, \\ A_{21} &\in \mathfrak{R}_+^{n_2 \times n_1}, & A_{22} &\in M_{n_2}, & n_1 + n_2 &= n. \end{aligned} \quad (7)$$

**Proof.** If the matrix  $E_1$  contains  $n_1$  monomial columns than by elementary column operations the remaining its columns may be eliminated and we obtain  $E_1 = \begin{bmatrix} E_{11} & 0 \end{bmatrix}$ , where  $E_{11} \in M_{n_1}$ . In this case from (5) we obtain  $\dot{x}_1(t) = E_{11}^{-1}A_{11}x_1(t) + E_{11}^{-1}A_{12}x_2(t)$ , where  $E_{11}^{-1}A_{11} \in M_{n_1}$ ,  $E_{11}^{-1}A_{12} \in \mathfrak{R}_+^{n_1 \times n_2}$ . This completes the proof.  $\square$

A vector is called strictly positive if all its components are positive.

**Theorem 1.** The positive descriptor system (1) is asymptotically stable if there exists strictly positive vectors  $\lambda_1 \in \mathfrak{R}_+^{n_1}$ ,  $\lambda_2 \in \mathfrak{R}_+^{n_2}$  such that

$$A_{11}\lambda_1 + A_{12}\lambda_2 < 0 \quad \text{and} \quad A_{21}\lambda_1 + A_{22}\lambda_2 = 0. \quad (8)$$

**Proof.** Integrating (5) we obtain

$$E_1 \int_0^\infty \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} dt = A_{11} \int_0^\infty x_1(t) dt + A_{12} \int_0^\infty x_2(t) dt \quad (9)$$

$$0 = A_{21} \int_0^\infty x_1(t) dt + A_{22} \int_0^\infty x_2(t) dt. \quad (10)$$

Taking into account that

$$E_1 \int_0^\infty \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} dt = E_1 \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix} - E_1 \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \quad (11)$$

and for asymptotically stable system  $x_1(\infty) = 0, x_2(\infty) = 0$  from (9) and (10) we obtain

$$-E_1 \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} < 0 \quad (12)$$

and

$$\begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 0, \quad (13)$$

where  $\lambda_1 = \int_0^\infty x_1(t) dt, \lambda_2 = \int_0^\infty x_2(t) dt$  which is equivalent to (8).

If  $A_{22} \in M_{n_2}$  is asymptotically stable then  $-A_{22}^{-1} \in \mathfrak{R}_+^{n_2 \times n_2}$  and from (10) we obtain

$$\lambda_2 = -A_{22}^{-1}A_{21}\lambda_1 \in \mathfrak{R}_+^{n_2}. \quad (14)$$

Substituting (14) into  $A_{11}\lambda_1 + A_{12}\lambda_2 < 0$  we obtain

$$\bar{A} = A_{11} - A_{12}A_{22}^{-1}A_{21} \in M_{n_1} \quad (15)$$

which is asymptotically stable. This completes the proof.  $\square$

In particular case when the system is standard and  $\det E \neq 0$  then from (1) we have  $\dot{x} = \bar{A}x, \bar{A} = E^{-1}A$  and the relations (8) take the well-known form [3, 4]

$$\bar{A}\lambda < 0, \quad (16)$$

where  $\lambda \in \mathfrak{R}_+^n$  is strictly positive vector.

**Example** Consider the autonomous descriptor system (1) with the matrices

$$E = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad (17)$$

$$A = \begin{bmatrix} -3 & 1 & 0 & 1 \\ 0 & 4 & -1 & -2 \\ -2 & 1 & -1 & 2 \\ 1 & -2 & 1 & 0 \end{bmatrix}.$$

The system satisfies the assumption (2) since

$$\begin{aligned} \det[Es - A] &= \begin{vmatrix} 3 & s-1 & 0 & 2s-1 \\ -s & -4 & 1 & -s+2 \\ 2 & s-1 & 1 & 2s-2 \\ s-1 & 2 & -1 & s \end{vmatrix} \\ &= 7s^2 - 28s + 5. \end{aligned} \quad (18)$$

Performing on the array

$$E, A = \left[ \begin{array}{cccc|cccc} 0 & 1 & 0 & 2 & -3 & 1 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 & 4 & -1 & -2 \\ 0 & 1 & 0 & 2 & -2 & 1 & -1 & 2 \\ 1 & 0 & 0 & 1 & 1 & -4 & 1 & 0 \end{array} \right], \quad (19)$$

the following elementary operations  $L[2, 4], L[4+2]$  and  $L[3+1(-1)]$  we obtain

$$\begin{matrix} E_1 & A_1 \\ 0 & A_2 \end{matrix} = \left[ \begin{array}{cccc|cccc} 0 & 1 & \vdots & 0 & 2 & -3 & 1 & \vdots & 0 & 1 \\ 1 & 0 & \vdots & 0 & 1 & 1 & -4 & \vdots & 1 & 0 \\ \dots & \dots & \vdots & \dots & \dots & \dots & \dots & \vdots & \dots & \dots \\ 0 & 0 & \vdots & 0 & 0 & 1 & 0 & \vdots & -1 & 1 \\ 0 & 0 & \vdots & 0 & 0 & 1 & 0 & \vdots & 0 & -2 \end{array} \right]. \quad (20)$$

Note that the first two columns of  $E_1$  are the columns of monomial matrix and we may eliminate the fourth column of  $E_1$ . In this case we have

$$A_{11}\lambda_1 + A_{12}\lambda_2 = \begin{bmatrix} -3 & 1 \\ 1 & -4 \end{bmatrix} \lambda_1 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \lambda_2 < \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (21)$$

and

$$A_{21}\lambda_1 + A_{22}\lambda_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \lambda_1 + \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \lambda_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (22)$$

From (22) and (21) we obtain

$$\begin{aligned} \lambda_2 &= -A_{22}^{-1}A_{21}\lambda_1 \\ &= \frac{1}{2} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \lambda_1 = \frac{1}{2} \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix} \lambda_1 \end{aligned} \quad (23)$$

and

$$\begin{aligned} \bar{A} &= A_{11} - A_{12}A_{22}^{-1}A_{21} \\ &= \begin{bmatrix} -3 & 1 \\ 1 & -4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -2.5 & 1 \\ 2.5 & -4 \end{bmatrix}. \end{aligned} \quad (24)$$

The descriptor system with (17) satisfies the condition (8) for  $\lambda_1 = [1 \ 1]^T$ ,  $\lambda_2 = [1.5 \ 0.5]^T$  and by Theorem 1 it is asymptotically stable.

### 3. POSITIVE DISCRETE-TIME DESCRIPTOR LINEAR SYSTEMS

Consider the autonomous descriptor discrete-time linear system

$$\bar{E}x_{i+1} = \bar{A}x_i, \quad (25)$$

where  $x_i \in \mathfrak{R}^n$  is the state vector and  $\bar{E} \in \mathfrak{R}^{n \times n}$ ,  $\bar{A} \in \mathfrak{R}^{n \times n}$ .

It is assumed that the system (25) satisfies the conditions  $\det \bar{E} = 0$  and

$$\det [\bar{E}z - \bar{A}] \neq 0. \quad (26)$$

Performing elementary row operations on the array

$$\bar{E}, \bar{A} \quad (27)$$

or equivalently on the system (25) by elimination the linearly dependent rows in the matrix  $\bar{E}$  we obtain

$$\begin{matrix} \bar{E}_1 & \bar{A}_1 \\ 0 & \bar{A}_2 \end{matrix} \quad (28)$$

and

$$\begin{bmatrix} \bar{E}_1 \\ 0 \end{bmatrix} \begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}, \quad (29)$$

where

$$\begin{aligned} x_{1,i} &\in \mathfrak{R}^{n_1}, & x_{2,i} &\in \mathfrak{R}^{n_2}, \\ \bar{A}_1 &= [\bar{A}_{11} \ \bar{A}_{12}], & \bar{A}_2 &= [\bar{A}_{21} \ \bar{A}_{22}], \\ \bar{A}_{11} &\in \mathfrak{R}^{n_1 \times n_1}, & \bar{A}_{12} &\in \mathfrak{R}^{n_1 \times n_2}, \\ \bar{A}_{21} &\in \mathfrak{R}^{n_2 \times n_1}, & \bar{A}_{22} &\in \mathfrak{R}^{n_2 \times n_2}, \\ \bar{E}_1 &= [\bar{E}_{11} \ \bar{E}_{12}], & \bar{E}_{11} &\in \mathfrak{R}_+^{n_1 \times n_1}, \\ \bar{E}_{12} &\in \mathfrak{R}_+^{n_1 \times n_2}, & n_1 + n_2 &= n. \end{aligned} \quad (30)$$

The discrete-time system (25) is positive if

$$\begin{aligned} \bar{A}_{11} - \bar{E}_{11} &\in \mathfrak{R}_+^{n_1 \times n_1}, & \bar{A}_{12} - \bar{E}_{12} &\in \mathfrak{R}_+^{n_1 \times n_2} \\ \text{and } \bar{A}_{22} &\in M_{n_2} \text{ is asymptotically stable.} \end{aligned} \quad (31)$$

Note that  $-\bar{A}_{22} \in \mathfrak{R}^{n_2 \times n_2}$  if and only if  $\bar{A}_{22}$  is asymptotically stable Metzler matrix [3].

**Theorem 2.** The positive descriptor system (25) is asymptotically stable if there exists strictly positive vectors  $\lambda_1 \in \mathfrak{R}_+^{n_1}$ ,  $\lambda_2 \in \mathfrak{R}_+^{n_2}$ , such that

$$\begin{aligned} (\bar{A}_{11} - \bar{E}_{11}) \lambda_1 + (\bar{A}_{12} - \bar{E}_{12}) \lambda_2 &< 0, \\ \bar{A}_{21} \lambda_1 + \bar{A}_{22} \lambda_2 &= 0. \end{aligned} \quad (32)$$

**Proof.** Taking into account (29) we obtain

$$\bar{E}_{11} \sum_{i=0}^{\infty} x_{1,i+1} + \bar{E}_{12} \sum_{i=0}^{\infty} x_{2,i+1} = \bar{A}_{11} \sum_{i=0}^{\infty} x_{1,i} + \bar{A}_{12} \sum_{i=0}^{\infty} x_{2,i}, \quad (33)$$

$$0 = \bar{A}_{21} \sum_{i=0}^{\infty} x_{1,i} + \bar{A}_{22} \sum_{i=0}^{\infty} x_{2,i}. \quad (34)$$

Note that

$$\bar{E}_{11} \sum_{i=0}^{\infty} x_{1,i+1} = \bar{E}_{11} \left( \sum_{i=0}^{\infty} x_{1,i} - x_{1,0} \right), \quad (35)$$

$$\bar{E}_{12} \sum_{i=0}^{\infty} x_{2,i+1} = \bar{E}_{12} \left( \sum_{i=0}^{\infty} x_{2,i} - x_{2,0} \right)$$

and using (33) we obtain

$$-\bar{E}_{11}x_{1,0} - \bar{E}_{12}x_{2,0} = [\bar{A}_{11} - \bar{E}_{11}] \lambda_1 + [\bar{A}_{12} - \bar{E}_{12}] \lambda_2 < 0, \quad (36)$$

$$\bar{A}_{21} \lambda_1 + \bar{A}_{22} \lambda_2 = 0, \quad (37)$$

where

$$\lambda_1 = \sum_{i=0}^{\infty} x_{1,i}, \quad \lambda_2 = \sum_{i=0}^{\infty} x_{2,i}. \quad (38)$$

This completes the proof.  $\square$

**Remark 1.** The condition (36) is not satisfied if at least one row of the matrix  $\bar{A}_{11} + \bar{A}_{12} - \bar{E}_{11} - \bar{E}_{12}$  is not negative.

**Remark 2.** Assuming  $\lambda_1 \in \mathfrak{R}_+^{n_1}$  is strictly positive vector we may find the desired vector  $\lambda_2 \in \mathfrak{R}_+^{n_2}$  from the equation

$$\lambda_2 = -\bar{A}_{22}^{-1} \bar{A}_{21} \lambda_1. \quad (39)$$

**Example 3.1.** Consider the positive descriptor system (25) with the matrices

$$E = \begin{bmatrix} 0.8 & 0.1 & \vdots & 0.6 & 0.2 \\ 0.1 & 0.7 & \vdots & 0.3 & 0.8 \\ \dots & \dots & \vdots & \dots & \dots \\ 0.9 & 0.8 & \vdots & 0.9 & 1 \\ -0.1 & -0.7 & \vdots & -0.3 & -0.8 \end{bmatrix}, \quad (40)$$

$$A = \begin{bmatrix} 0.3 & 0.3 & \vdots & 0.8 & 0.3 \\ 0.2 & 0.4 & \vdots & 0.4 & 0.5 \\ \dots & \dots & \vdots & \dots & \dots \\ 1.3 & 1 & \vdots & -0.2 & 1.1 \\ -0.1 & -0.2 & \vdots & 0 & -1.2 \end{bmatrix}.$$

The system satisfied the assumption (26) since

$$\det[Ez - A] = -0.0414z^3 + 1.0981z^2 - 1.1839z + 2.174. \quad (41)$$

Performing on the array

$$E, A = [E \mid A]. \quad (42)$$

the following elementary operations  $L[3+1(-1)], L[3+2(-1)], L[4+2(+1)]$  we obtain

$$\left[ \begin{array}{cc|cc} \bar{E}_{11} & \bar{E}_{12} & \bar{A}_{11} & \bar{A}_{12} \\ 0 & 0 & \bar{A}_{21} & \bar{A}_{22} \end{array} \right]$$

$$= \left[ \begin{array}{cc|cc|cc} 0.8 & 0.1 & \vdots & 0.6 & 0.2 & 0.3 & 0.3 & \vdots & 0.2 & 0.3 \\ 0.1 & 0.7 & \vdots & 0.3 & 0.8 & 0.2 & 0.4 & \vdots & 0.4 & 0.5 \\ \dots & \dots & \vdots & \dots & \dots & \dots & \dots & \vdots & \dots & \dots \\ 0 & 0 & \vdots & 0 & 0 & 0.8 & 0.3 & \vdots & -1.4 & 0.3 \\ 0 & 0 & \vdots & 0 & 0 & 0.1 & 0.2 & \vdots & 0.4 & -0.7 \end{array} \right]. \quad (43)$$

In this case from (39) for  $\lambda_1 = [1 \ 1]^T$  we have

$$\lambda_2 = -\bar{A}_{22}^{-1} \bar{A}_{21} \lambda_1$$

$$= \begin{bmatrix} -1.4 & 0.3 \\ 0.4 & -0.7 \end{bmatrix}^{-1} \begin{bmatrix} 0.8 & 0.3 \\ 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (44)$$

and

$$(\bar{A}_{11} - \bar{E}_{11}) \lambda_1 + (\bar{A}_{12} - \bar{E}_{12}) \lambda_2$$

$$= \begin{bmatrix} -0.5 & 0.2 \\ 0.1 & -0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.4 & 0.1 \\ 0.1 & -0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= - \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (45)$$

Therefore, the positive descriptor system with (40) is asymptotically stable.

#### 4. CONCLUDING REMARKS

The asymptotic stability of positive descriptor continuous-time and discrete-time linear systems has been investigated. New sufficient conditions for stability of positive descriptor continuous-time linear systems have been given in Section 2 and for discrete-time systems in Section 3. The efficiency of the new stability conditions are demonstrated on numerical examples of continuous-time and discrete-time linear systems. The considerations can be extended to positive descriptor fractional orders continuous-time and discrete-time systems.

#### ACKNOWLEDGEMENTS

This work was carried out in the framework of work No. WZ/WE – IA/6/2020.

#### REFERENCES

- [1] A. Berman and R. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. SIAM, 1994.
- [2] T. Kaczorek, *Positive 1D and 2D Systems*. Springer-Verlag, 2002.
- [3] T. Kaczorek, *Selected Problems of Fractional Systems Theory*. Springer, 2011.
- [4] T. Kaczorek and K. Borawski, *Descriptor Systems of Integer and Fractional Orders*. Springer, 2021.
- [5] T. Kaczorek, "Positive linear systems consisting of n subsystems with different fractional orders," *IEEE Trans. Circuits Syst.*, vol. 58, no. 7, pp. 1203–1210, 2011.
- [6] L. Farina and S. Rinaldi, *Positive Linear Systems; Theory and Applications*. J. Wiley, 2000.
- [7] T. Kaczorek, "Descriptor positive discrete-time and continuous-time nonlinear systems," in *Proc. SPIE: Photonics Applications in Astronomy, Communications, Industry, and High-Energy Physics Experiments 2014*, vol. 9290, 2014, pp. 805–814.
- [8] T. Kaczorek, "Positive singular discrete-time linear systems," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 45, no. 4, pp. 619–631, 1997.

- [9] T. Kaczorek, "Positive fractional continuous-time linear systems with singular pencils," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 60, no. 1, pp. 9–12, 2012.
- [10] L. Sajewski, "Descriptor fractional discrete-time linear system and its solution – comparison of three different methods," in *Challenges in Automation, Robotics and Measurement Techniques, Advances in Intelligent Systems and Computing*, 2016, vol. 440, pp. 37–50.
- [11] L. Sajewski, "Descriptor fractional discrete-time linear system with two different fractional orders and its solution," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 64, no. 1, pp. 15–20, 2016.
- [12] P. Ostalczyk, *Discrete Fractional Calculus: Application in Control and Image Processing*. Word Scientific, 2016.
- [13] K. Rogowski, "General response formula for cfd pseudo-fractional 2d continuous linear systems described by the roesser model," *Symmetry-Basel*, vol. 12, no. 12, pp. 1–12, 2020.
- [14] A. Ruszewski, "Practical and asymptotic stabilities for a class of delayed fractional discrete-time linear systems," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 67, no. 3, pp. 509–515, 2019.
- [15] A. Ruszewski, "Stability of discrete-time fractional linear systems with delays," *Arch. Control Sci.*, vol. 29, no. 3, pp. 549–567, 2019.
- [16] H.W.J. Zhang, Z. Han, and J. Hung, "Robust stabilization of discrete-time positive switched systems with uncertainties and average dwell time switching," *Circuits Syst. Signal Process.*, vol. 33, pp. 71–95, 2014.