

# Another Proof of Ambiguity of the Formula Describing Aliasing and Folding Effects in Spectrum of Sampled Signal

Andrzej Borys

**Abstract**—In this paper, a new proof of ambiguity of the formula describing the aliasing and folding effects in spectra of sampled signals is presented. It uses the model of non-ideal sampling operation published by Vetterli et al. Here, their model is modified and its black-box equivalent form is achieved. It is shown that this modified model delivers the same output sequences but of different spectral properties. Finally, a remark on two possible understandings of the operation of non-ideal sampling is enclosed as well as fundamental errors that are made in perception and description of sampled signals are considered.

**Keywords**—signal sampling; occurrence of spectrum aliasing and folding; modelling of non-ideal signal sampling operation; Vetterli's model

## I. INTRODUCTION

MANY researchers working in the areas of computer science, telecommunications, electrical and electronics engineering, and signal processing still believe that this highly celebrated and commonly used [1]–[3] expression

$$X_s(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(f - k/T) \quad (1)$$

for describing the spectrum aliasing and folding effects in the signal sampling is fully correct – despite receiving from the author of this paper a strong evidence that just the opposite might be valid [4]. Probably, this is due to the fact that they do not read journals such as the Intl Journal of Electronics and Telecommunications. It appears that they simply consider it to be a low-ranking journal, which is not worth reading. Unduly.

However, the purpose of this paper is not to present the results achieved in [4] from another perspective. Here, we aim in presenting a new proof of what has been shown in [4] with the use of quite different tools.

In (1),  $X(f)$  means the spectrum of an energy, bandlimited signal  $x(t)$ ;  $X(f - k/T)$  is this spectrum shifted by  $k/T$  (to the left or to the right, depending upon a sign of the integer  $k$ ) on the frequency  $f$  axis. Further,  $f_s$  stands for the sampling frequency (rate) used in sampling the signal  $x(t)$ , where  $t$  is a

continuous time variable. Moreover,  $T = 1/f_s$ , where  $T$  is a sampling period.

Furthermore, let us denote by  $x_s(t)$  the signal  $x(t)$  sampled ideally or non-ideally (the difference between these two cases will be explained later). So, with this,  $X_s(f)$  in (1) means the Fourier transform of  $x_s(t)$ . That is it is a spectrum of the sampled signal.

As already mentioned, this paper contains a quite different approach to the problem mentioned above to show that (1) is not a relevant formula for a correct description of the spectrum of a sampled signal (independently of whether the sampling operation is carried out ideally or not). Here, the model of signal sampling published in [5], together with the principles accompanying its construction, are exploited. Whereby this model is slightly modified in this paper. But, it is shown that its modification is fully legitimate.

As well known, see, for example, [1]–[3], the form of (1) follows exclusively from modelling the sampled signal  $x_s(t)$  as a series of Dirac deltas (impulses). That is as

$$x_s(t) = \delta_T(t) \cdot x(t) = x_{D,T}(t), \quad (2)$$

where the Dirac comb  $\delta_T(t)$  is defined as

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \quad (3)$$

with  $\delta(t - kT)$ ,  $k = \dots, -1, 0, 1, \dots$ , meaning the time-shifted Dirac deltas. Furthermore, note the use of an equivalent notation  $x_{D,T}(t)$  in (2) for  $x_s(t)$ . In it, the first index,  $D$ , stands for the name of Dirac. The latter notation (not  $x_s(t)$ ) will be used in what follows to distinguish one another possible manner of modelling of the sampled signal from the one given by (2).

The remainder of the paper is organized as follows. In the next section, we present a description of the model of the signal sampling demonstrated by Vetterli et al. in [5]. Their model is modified in Section III and used afterwards to develop a new

The author is with the Department of Marine Telecommunications, Faculty of Electrical Engineering, Gdynia Maritime University, Gdynia, Poland; e-mail: a.borys@we.umg.edu.pl.



formula for the spectrum of sampled signals. The formula derived differs from the one which is used in the current theory. Moreover, it does not predict any aliasing and folding effects in spectra of sampled signals; its correctness follows from the fact that any real analog/digital converters that produce sequences of Dirac deltas do not exist. The next section contains a remark on two possible understandings of the operation of non-ideal sampling. One of them is used in this paper; it corresponds to the model of Vetterli et al. In Section V, some fundamental errors that are made in perception and description of sampled signals are analyzed and explained in detail. The paper ends with a final conclusion.

## II. DESCRIPTION OF THE VETTERLI'S MODEL OF SIGNAL SAMPLING

A model of signal sampling that takes into account non-idealities occurring in carrying out this operation by real sampling devices (real A/D converters) has been demonstrated in an article [5] written by Vetterli et al. This model is shown here on Fig. 1, after [5, see Fig. 1].

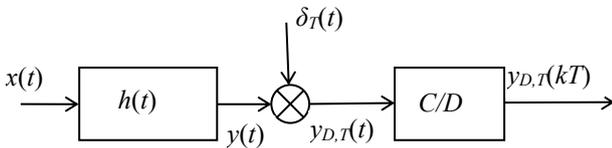


Fig. 1. Graphical representation of the Vetterli's model of signal sampling operation that reflects formula (1) complemented with the preceding and following operations (averaging of sampled signal and picking up samples, respectively).

In Fig. 1, the whole non-ideal behavior of a real A/D converter is "put into" the first block named " $h(t)$ ". And, it is modelled by a linear filter possessing an impulse response  $h(t)$ , which represents a local signal averaging process or any other appropriate one [6]. Hence, we can express the signal  $y(t)$  in Fig. 1 as a result of a convolution of the continuous time signal  $x(t)$ , which is applied to the input of the A/D converter, with  $h(t)$ . The resulting signal is then sampled ideally as foreseen by formula (2), where obviously  $\delta_T(t)$  in Fig. 1 (as in (2)) means the Dirac comb. (The symbol  $\otimes$  in Fig. 1 means a multiplication.) In effect, we get the signal  $y_{D,T}(t)$  (replacing now the signal  $x_s(t)$  standing on the left-hand side of (2)).

Finally,  $y_{D,T}(kT)$ 's on the right-hand side of Fig. 1 stand for the samples of the signal  $y(t)$ . They are picked up from the signal  $y_{D,T}(t)$  in the block named „C/D” (in fact, picking up the samples from  $y_{D,T}(t)$  is the only task of this processing unit).

Note now that in the current theory of signal sampling the spectra of the signals  $y_{D,T}(t)$  and  $y_{D,T}(kT)$  occurring at the input and output of the C/D block, respectively, are the same.

This is so because we have

$$\begin{aligned} F(y_{D,T}(t)) &= F(\delta_T(t)) * F(y(t)) = \\ &= \left( \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(2\pi f - k \frac{2\pi}{T}\right) \right) * Y(f), \end{aligned} \quad (4)$$

where the symbol  $*$  stands for the convolution operation and  $F(\cdot)$  means performing the Fourier transform of a signal indicated (for example,  $F(y(t)) = Y(f)$ ). Or, alternatively,  $F(y_{D,T}(t))$  can be calculated as

$$\begin{aligned} F(y_{D,T}(t)) &= F\left(\sum_{k=-\infty}^{\infty} y_{D,T}(kT)\delta(t-kT)\right) = \\ &= \sum_{k=-\infty}^{\infty} y_{D,T}(kT)\exp(-j2\pi f kT). \end{aligned} \quad (5)$$

Moreover, observe that the right-hand side of (5) is in fact nothing else than (per definition; see, for example, [3]) the Discrete Time Fourier Transform (DTFT) of the sequence of  $y_{D,T}(kT)$ 's. Hence, we can write

$$\begin{aligned} F(y_{D,T}(t)) &= \sum_{k=-\infty}^{\infty} y_{D,T}(kT)\exp(-j2\pi f kT) = \\ &= \text{DTFT}(y_{D,T}(kT)). \end{aligned} \quad (6)$$

(Note that all the derivations presented in (4), (5), and (6) are standard. They use standard properties of Fourier transforms and Dirac deltas referenced to in textbooks; see, for instance, [1]–[3].)

Furthermore, the spectrum of the sequence of discrete values, as these ones  $y_{D,T}(kT)$ 's at the output of the C/D processing unit in Fig. 1, is well defined in the literature. It is just the  $\text{DTFT}(y_{D,T}(kT))$  in this case.

## III. MODIFICATION OF THE VETTERLI'S MODEL AND PROOF OF ANOTHER FORMULA FOR THE SAMPLED SIGNAL SPECTRUM

As noted in the previous section, the only task of the processing unit „C/D” of Fig. 1 is picking up the samples  $y_{D,T}(kT)$ 's associated with the signal  $y_{D,T}(t)$ .

In what follows, we explain this in more detail; and, to this end, we start with rewriting (2) – with  $x(t)$  and  $x_{D,T}(t)$  replaced now by  $y(t)$  and  $y_{D,T}(t)$ , respectively – in the following form:

$$\begin{aligned} y_{D,T}(t) &= \left( \sum_{k=-\infty}^{\infty} \delta(t-kT) \right) \cdot y(t) = \\ &= \sum_{k=-\infty}^{\infty} y(kT)\delta(t-kT). \end{aligned} \quad (7)$$

Now, we see in (7) that the samples  $y(kT) = y_{D,T}(kT)$ 's are the coefficients which multiply the successive time-shifted Dirac deltas  $\delta(t-kT)$ ,  $k = \dots, -1, 0, 1, \dots$ . Thus, the processing unit C/D simply identifies them and transports to its output.

Next, note that for performing the action described above we do not necessarily need to use a sequence of the weighted Dirac deltas (with the coefficients being the samples  $y_{D,T}(kT)$ 's) as in (7). In fact, we can use any other appropriate formula, for example, the following one:

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} y_{D,T}(kT) \operatorname{sinc}(t/T - k), \quad (8)$$

where the function  $\operatorname{sinc}(t)$  is defined as

$$\operatorname{sinc}(t) = \sin(\pi t)/\pi t \text{ for } t \neq 0 \text{ and } 1 \text{ for } t = 0 \quad (9)$$

and  $\hat{y}(t)$  means a function being an approximation of  $y(t)$  shown in Fig. 1 that exploits the aforementioned set of  $y_{D,T}(kT)$ 's.

Note that (8) can be always written and makes sense, as shown in [7]. In more detail, if the sampling period  $T$  in (8) is so chosen that the following:

$$T \leq 1/(2f_m) \quad (10)$$

holds, then (8) represents the formula for a perfect reconstruction [1]–[3] of the signal  $y(t)$  from its samples  $y(kT) = y_{D,T}(kT)$ ,  $k = \dots -1, 0, 1, \dots$ . Whereby, in (10),  $f_m$  denotes the maximal frequency present in the spectrum of the signal  $y(t)$ . Further, the perfect reconstruction in this case means that  $\hat{y}(t)$  is exactly equal to  $y(t)$  (i.e.  $\hat{y}(t) = y(t)$ ) [1]–[3].

When the condition (10) is not satisfied, then (8) plays a role of an approximating function of the signal  $y(t)$ , as shown in [8]. (By the way, note that any physical signal used in electronics, telecommunications, and signal processing can be treated as a bandlimited one (for arguments, see, for example, [9]). Hence, the maximal frequency  $f_m$  in its spectrum can be always determined or, at least, approximated. Thereby, we can always check (precisely or approximately) whether (10) is fulfilled or not.)

Furthermore, as shown in [8] and [10], (8) plays a role of the reconstruction formula for an interpolating function of the signal  $y(t)$  mentioned above. Let us denote this interpolating function as  $y_a(t)$  (similarly as in [10]). Further, the latter signal is a bandlimited one and its maximal frequency which is present in its spectrum – denote it here as  $f_{ma}$  – is equal to  $1/(2T)$  (that is  $f_{ma} = 1/(2T)$ ).

So, in summary of the latter case, we can rewrite (8) for  $y_a(t)$  as

$$y_a(t) = \sum_{k=-\infty}^{\infty} (y_a(kT) = y(kT)) \operatorname{sinc}(t/T - k), \quad (11)$$

having in mind that  $y_a(kT) = y(kT) = y_{D,T}(kT)$  also holds.

What are the disadvantages and dangers of choosing the formula (7) in modelling of a sampled signal? The most relevant here is the fact that there are no physical analog/digital (A/D) signal converters that produce series of weighted Dirac impulses (as in (7)).

The author of this paper consulted practitioners, designers of A/D converters on this issue and heard from them the following: signals inside or at outputs of real A/D converters have either a form of sequences of impulses of a finite duration or sequences of numbers of a finite value. Whereby the first form is rather very rarely exploited in signal processing. (By the way, calculation of its spectrum does not pose any problems – as shown in [11].) Primary form in the digital signal processing applying microprocessors is the second one mentioned above. (And here, unlike in the first case, calculation of its spectrum is problematic; see, for example, [4].) Further, their opinion is that neither of these two types of signals mentioned above can be perceived as sequences of weighted Dirac impulses.

In opinion of the author of this paper, the above excludes (7) from the candidates, which can describe the sampled signal correctly. As not existing in real A/D signal converters, the signals expressed by (7) can obscure a true picture of what happens in these devices.

Contrary to the above, a natural and a reasonable choice offers the description (8). Simply because signals of the type given by (8), that is  $\hat{y}(t)$ , are present in real A/D signal converters. Obviously, this choice still remains an arbitrary one. But, there is no danger that it will introduce any undesirable side effects into the description.

Having the above in mind, let us modify the Vetterli's model of Fig. 1 accordingly. It is shown in Fig. 2.

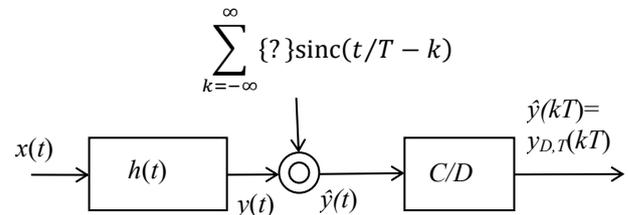


Fig. 2. Modified Vetterli's model of Fig. 1.

Let us now note the changes in Fig. 2 when compared with Fig. 1. We see two graphical changes. Instead of the signal  $\delta_T(t)$  (a sequence of Dirac impulses), we have now the signal

$\sum_{k=-\infty}^{\infty} \{?\} \operatorname{sinc}(t/T - k)$  with unknown coefficients. These

coefficients depend upon the index  $k$  and are “worked out” in the element marked as  $\odot$  (which replaces here the operation of multiplication  $\otimes$  of Fig. 1). Its task is to calculate the values of the signal  $y(t)$  at the successive instants for which the function  $\operatorname{sinc}(t/T - k)$  achieves its maxima (equal to 1), when the index  $k$  changes. And afterwards, to substitute them in places of  $\{?\}$ 's

in  $\sum_{k=-\infty}^{\infty} \{?\} \operatorname{sinc}(t/T - k)$ .

The third difference between Figs. 1 and 2 (not visualized in Fig. 2) regards the description of the processing unit C/D. Now, in Fig. 2, this unit picks up, successively, as the index  $k$  changes, the coefficients of the expanded signal

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} y(kT) \operatorname{sinc}(t/T - k)$$

at the instants  $kT$  for which the function  $\operatorname{sinc}(t/T - k)$  achieves its maximum. Moreover, we recall here that  $\hat{y}(kT) = y(kT) = y_{D,T}(kT)$  holds. This means that the output signal of the device in Fig. 2 is exactly the same as the one of the device of Fig. 1.

In [5], in constructing the model visualized in Fig. 1, it has been assumed that the input and output signals of the processing unit C/D have the same spectrum. This principle is taken over in our model presented in Fig. 2. That is the spectra of the signals  $\hat{y}(t)$  and  $\operatorname{seq}(\hat{y}(kT) = y(kT) = y_{D,T}(kT))$ , where “seq” stands for a sequence of the indexed elements ( $k$  plays here a role of an index), are the same.

To proceed further, let us now calculate the spectrum of the signal  $\hat{y}(t)$ . And, to this end, note that this spectrum depends upon whether the signal  $y(t)$  was sampled in a way enabling its perfect reconstruction or not (that is it was under-sampled); for more details regarding this point, see [8]. In other words, it depends upon whether the condition (10) is satisfied or not; using the results presented in [8], we can write

$$F(\hat{y}(t)) = \begin{cases} Y(f) & \text{when } T \leq 1/(2f_m) \\ Y_a(f) & \text{when } T > 1/(2f_m) \end{cases}, \quad (12)$$

where  $Y(f) = F(y(t))$  and  $Y_a(f) = F(y_a(t))$  mean the spectra of the signals  $y(t)$  and  $y_a(t)$ , respectively.

In the next step, we apply the principle regarding the spectra of the input and output signals of the processing unit C/D mentioned above (which says that they are the same). Hence, we write

$$\begin{aligned} \operatorname{SPECT}(\hat{y}(kT) = y(kT) = y_{D,T}(kT)) &= \\ = F(\hat{y}(t)) &= \begin{cases} Y(f) & \text{when } T \leq 1/(2f_m) \\ Y_a(f) & \text{when } T > 1/(2f_m) \end{cases}, \end{aligned} \quad (13)$$

where a new symbol  $\operatorname{SPECT}(\cdot)$  for denoting the spectrum of a sequence of indexed values indicated in the brackets is used because it obviously differs from  $\operatorname{DTFT}(y_{D,T}(kT))$ .

Finally, note that the result (13) does not predict any aliasing and folding effects in the spectra of sampled signals, in contrast to the theory in force.

#### IV. REMARK ON SAMPLES OF IDEALLY AND NON-IDEALLY SAMPLED SIGNALS

What does the notion of an ideal or a non-ideal sampling of a continuous time signal mean? And, what is the difference between them? Here, we remark shortly on our understanding of this issue.

A continuous time signal is sampled ideally, when the samples emerging in operation of its sampling are exactly equal

to its values at the corresponding instants  $kT$ ,  $k = \dots, -1, 0, 1, \dots$ . So, interpreting this with the use of the model of Fig. 1 or of Fig. 2, we assume that then the block named “ $h(t)$ ” does not occur

and the output sequence at the processing unit C/D is  $x(kT)$ 's. (The above understanding of ideal sampling was exploited in the analysis presented in [4].)

Now, note that the non-ideal case of sampling can be understood in two different ways. The first of them is as visualized in Figs. 1 and 2. In these two models, the non-ideality of sampling is taken into account by introducing the block “ $h(t)$ ”. This causes that the values of samples at the output of the processing unit C/D,  $y_{D,T}(kT)$ 's, differ from those following from the ideal case,  $x(kT)$ 's.

Next, observe that both the cases of ideal and of non-ideal sampling defined above are identical with respect to the spectrum considerations presented in this paper because they lead to the same form of the formula: (1) or (13), depending upon the model used. Only thing which must be then done is an appropriate interpretation of the quantities used in the aforementioned formulas – what follows from the fact that we are dealing then with either the sequence of  $x(kT)$ 's or the sequence of  $y_{D,T}(kT)$ 's. So, in this context, see that the  $y_{D,T}(kT)$ 's can be treated as the corrupted  $x(kT)$ 's when we write  $y_{D,T}(kT) = x(kT) + e(kT)$ ,  $k = \dots, -1, 0, 1, \dots$ , where  $e(kT)$ ,  $k = \dots, -1, 0, 1, \dots$  mean the successive differences between the ideal and non-ideal values of samples. (Note that a similar scheme is used in acoustic signal processing to describe the quantization noise.)

A quite different is the second possible way of modelling the non-ideal sampling operation. This approach uses a sampled signal description in form of a sequence of short impulses; see, for example, [11]. However, it is rather very rarely exploited in signal processing.

Additionally, we would like to underline here that the calculations of the sampled signal spectrum, which have been presented in [11], apply exclusively to the case of a description of the sampled signal in form of a sequence of short impulses. And that the model developed in [11] is principally different from those given in Figs. 1 and 2.

#### V. FUNDAMENTAL ERRORS IN PERCEPTION AND DESCRIPTION OF SAMPLED SIGNALS

The first version of this article was intended to consider only the modified Vetterli's model of Fig. 2 and to show consequences of such modelling on the form of the sampled signal spectrum. It has been read by many experts in the area, who gave the author a lot of feedback. Some of their opinions were, however, surprising. Moreover, their opinions were in many cases contradictory and, obviously, still remain so. Or, they reject a priori other points of view.

In opinion of the author of this paper, the controversies mentioned above should not be ignored, under no circumstances. On the contrary, something has to be done about this. These controversies should be given full attention because they touch on fundamental aspects of signal sampling process. And, for this reason, they should be thoroughly examined (once again); a task of performing this (even briefly) has been undertaken in the present section. By the way, it is also worth noting that this attempt of clarification of ambiguities associated with the perception and description of the signal sampling process was supported by the Editors of the Intl Journal of Electronics and Telecommunications.

A standard, and it can be even said, a flagship argument for the use of the weighted Dirac comb in description of the sampled signal is a belief that it is analogous to describing the so-called point masses by physicists.

Let us take a closer look at the latter and, to this end, let us consider a single point mass for simplicity, which corresponds (in its mathematical description) to a single element in the weighted Dirac comb (that is to a Dirac impulse multiplied by a number which in our case of a mass point has a meaning of the mass located at this point). But, before starting with this situation, let us consider first a simpler environment that is a mass distributed in some volume. Further, for simplicity, consider a one-dimensional case. This environment is illustrated in Fig. 3.

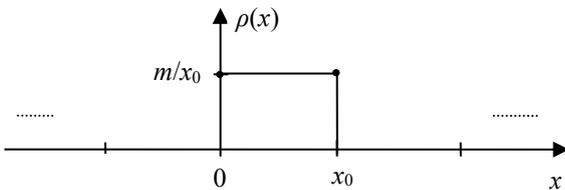


Fig. 3. Illustration to an example of a mass distributed in some “one-dimensional volume” extending from  $x = 0$  to  $x = x_0$  and described by a mass density function  $\rho(x)$  shown in this figure.

The one-dimensional object represented by the mass density function  $\rho(x)$  of Fig. 3 has a mass equal to  $m$ , which is a result of integration of the function  $\rho(x)$  over the “volume”  $\{0 \leq x \leq x_0\}$ . That is

$$m = \int_0^{x_0} \rho(x) dx = \int_0^{x_0} m/x_0 dx \quad (14)$$

Observe further that when we weigh a material object with the use of a scale, it gives us the weight  $m$  of this object, not its mass density function  $\rho(x)$ . Moreover, note that the physicists use a concept called the center of mass, which can be, obviously, also used to characterize the “mass properties” of an object. It is a concrete point of the space occupied by this object. So, knowing this point and the value of  $m$ , we can express the “mass properties” of an object in another way. In our example, it will look as shown in Fig. 4.

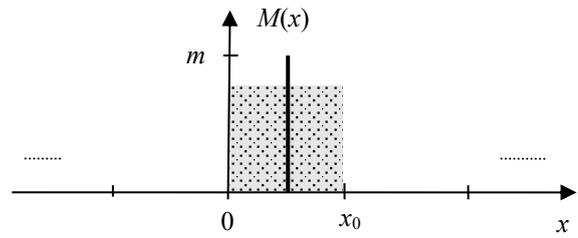


Fig. 4. Another way of expressing the location of the distributed mass in a “one-dimensional volume” occupied by our example object possessing the mass density function  $\rho(x)$  shown in Fig. 3. This illustration is at the same time a descriptive definition of a quasi-function  $M(x)$ .

Note that  $M(x)$  defined in Fig. 4 is not a function; it would be a function if it were not for the shaded and dotted area. Any values of  $M(x)$  for the values of  $x$  ranging from 0 (inclusive) to  $x_0$  (also inclusive), but except the value of  $x_0/2$ , are forbidden (and this is just marked in Fig. 4 by that shaded and dotted area). Further, we emphasize that we are talking here about the forbidden values, not about unknown ones or zeros. Therefore,  $M(x)$  is called here a quasi-function because it is well-determined except the two intervals on the  $Ox$  axis mentioned above.

Now, we will draw some conclusions from a comparison of the two possibilities of description of the “mass properties” of our example object. First of all, note that these descriptions are not compatible with each other; they are completely different because the mass density function  $\rho(x)$  is a well-defined function unlike the  $M(x)$ , which is not a function at all. However, this does not mean at the same time that there is no relationship between them. The relationship is through the functional (14); obviously, this relationship is an indirect one (a direct one is not possible because of the nature of  $\rho(x)$  and  $M(x)$ ). Second, the latter is clearly reflected in the frequency domain. That is the Fourier transform of  $\rho(x)$  is well-defined unlike the Fourier transform of  $M(x)$ , which does not exist at all.

Let us now turn to the special case in the above descriptions, that is to the point mass. Then, as it is taught and explained in the textbooks, the case illustrated in Fig. 3 goes into the one presented in Fig. 5, i.e. when  $x_0$  goes to zero.

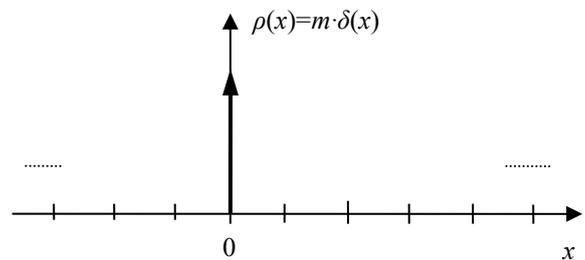


Fig. 5. Version of Fig. 3 for illustration of an example of a mass  $m$  concentrated in one point only ( $x = 0$ ). Here, the mass density function  $\rho(x)$  equals the Dirac impulse  $\delta(x)$  multiplied by  $m$ .

Further, note that in this case, as the mathematical literature on distributions (see, for example, [12]) shows, an equivalent of the functional given by (14) exists and it can be expressed in the following form:

$$m = \int_{-\infty}^{\infty} m \cdot \delta(x) dx = \langle \delta | m \rangle, \quad (15)$$

where the usual engineers' notation for the operation of integration of the Dirac delta is used. Moreover, on the extreme right-hand side of (15), the symbol  $\langle \cdot | \cdot \rangle$  denoting a scalar product of two functions is used. It indicates that  $m$  can be also viewed, at least formally, as a scalar product of  $\delta(x)$  and  $m$ .

Finally, in this case, the object  $M(x)$  defined in a descriptive form in Fig. 4 assumes the form which is illustrated in Fig. 6.

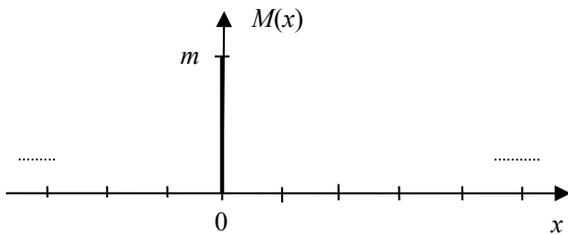


Fig. 6. Version of Fig. 4 for illustration of an example of a mass  $m$  concentrated in one point only – according to the second method discussed in this paper. Here, the “distributed” mass occupies only one point ( $x = 0$ ) and has its mass density function  $\rho(x)$  shown in Fig. 5. Furthermore, this illustration is at the same time a descriptive definition of a quasi-function  $M(x)$  for the case of the point mass (remark: here,  $M(x)$  becomes a function).

Observe now that the descriptions for the point mass shown in Figs. 5 and 6 are not identical, similarly as the previous ones. They differ from each other, but in another way. In this context, note that  $M(x)$  represents a function (before it was an object not being a function). Unlike this, the mass density function  $\rho(x)$  is not now a function (previously it was). However, as before, it has a Fourier transform; its spectrum equals  $m$  for all frequencies. Further,  $M(x)$  possesses now the Fourier transform, which is however identically equal to zero (remark: the definition of the Lebesgue integral must be applied in calculation of the Fourier transform to get this result [4]). Moreover, the “measured” quantity here, as before, is the mass  $M(0)$  (well defined) – not the mass density at the point  $x = 0$  (besides, as  $\rho(0) = m \cdot \delta(0)$ , it is not well defined). In other words, the scale provides the information about  $M(0)$ , not about  $\rho(0)$ .

After discussion of the case of single masses (occupying some volume as well as those called point ones), let us now continue our example with the infinite sequences of masses. To this end, assume that the masses occur in our one-dimensional space uniformly in the distance  $d = x_0$  from each other. And,

consider separately continuations of the single mass versions discussed before; that is of the one represented by Figs. 3 and 4, and of the second visualized in Figs. 5 and 6.

Moreover, assume in the first one that our one-dimensional space is fully (strictly) filled with masses. That is the masses in the segments:  $\dots, \{-x_0 \leq x < 0\}, \{0 \leq x < x_0\}, \dots$  touch each other. And, the mass density functions may be different in the successive segments. However, assume that they are fixed (constant) in each of these segments. So, taking all this into account, we can sketch the corresponding graphs as shown in Fig. 7.

Next, consider the second variant which replicates what we have in Figs. 5 and 6, for the case of an infinite sequence of point masses. It is summarized in Fig. 8.

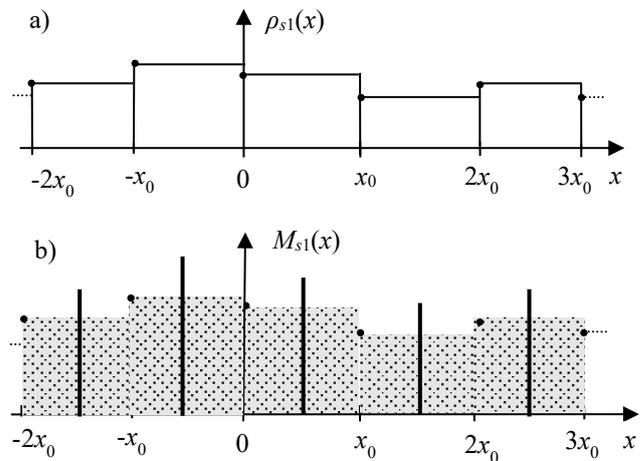


Fig. 7. a) Illustration to the mass density function  $\rho_{s1}(x)$  for an example of an infinite sequence of masses having the properties described in the text; b) An alternative description through the object  $M_{s1}(x)$ , where the latter means extension of  $M(x)$  (which is shown in Fig. 4) to the case of a sequence of masses.

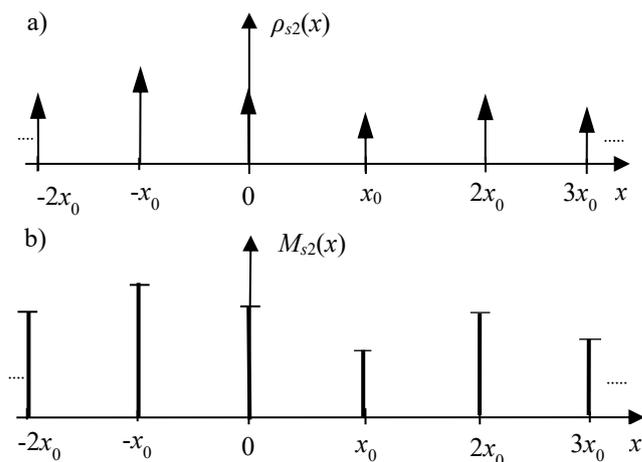


Fig. 8. a) Illustration to the mass density function  $\rho_{s2}(x)$  for an infinite sequence of point masses; b) An alternative description through the object  $M_{s2}(x)$ , where the latter means extension of  $M(x)$  (which is shown in Fig. 6) to the case of a sequence of masses.

What is common in these two different descriptions presented in Figs. 7 and 8? Looking at the objects  $M_{s1}(x)$  of Fig. 7b and  $M_{s2}(x)$  of Fig. 8b, we see that the values provided by the scales are the same in both the cases (here, the “phase shift” of  $x_0/2$  between them on the Ox axis has no significance; it does not matter). More precisely, we have:  $\dots, M_{s1}(-(3/2)x_0) = M_{s2}(-2x_0), M_{s1}(-(1/2)x_0) = M_{s2}(-x_0), M_{s1}(x_0/2) = M_{s2}(0), M_{s1}((3/2)x_0) = M_{s2}(x_0), M_{s1}((5/2)x_0) = M_{s2}(2x_0), \dots$ . This means obviously that “from the point of view of the measuring equipment” or, in other words, from the point of view of the series of weights provided by the scales, these descriptions are identical. However, the entire objects  $M_{s1}(x)$  and  $M_{s2}(x)$  are not identical. This is so because in the case of the former all the points on the Ox axis not belonging to the following set:  $\{\dots, -(3/2)x_0, -(1/2)x_0, x_0/2, (3/2)x_0, \dots\}$  are forbidden (see Fig. 7b). But, unlike this, in the case of the latter all the points on the Ox axis which do not belong to the set:  $\{\dots, -2x_0, -x_0, 0, x_0, 2x_0, \dots\}$  are also allowed. Furthermore, all the values of the function  $M_{s2}(x)$  at the points outside the set  $\{\dots, -2x_0, -x_0, 0, x_0, 2x_0, \dots\}$  are identically equal to zero (see Fig. 8b).

Comparison of Figs. 7a and 8a shows that the function  $\rho_{s1}(x)$  and the weighted Dirac comb  $\rho_{s2}(x)$  differ substantially from each other. And, not only graphically. Their analytical descriptions are also significantly different. Because, as already mentioned, the first of them is a function but the second a distribution.

Now, we go to application of the descriptions derived to modelling of the sampled signal. And, in this context, let us recall briefly what is available to us: Our first description refers to a series of point masses which occupy zero volumes but the second to a series of distributed mass centers treated as distributed (mass) points in a space. In the next step, we replace the spatial variable  $x$  in Figs. 7 and 8 with the continuous time variable  $t$  and  $x_0 = d$  with the sampling period  $T$ . In this way, we get the following functions and objects:  $\rho_{s1}(t)$ ,  $\rho_{s2}(t)$ ,  $M_{s1}(t)$ , and  $M_{s2}(t)$ , with the corresponding illustrations given by Figs. 7 and 8 (which are referred to our example, but now interpreted as a sampled signal). We will refer to them in what follows, when we will discuss on whether the first or the second description is more realistic.

Note that the description given by  $\rho_{s2}(t)$  and  $M_{s2}(t)$ , and illustrated in Fig. 8 is promoted in all papers and textbooks on the signals theory as well as on the signal processing. This is the only representation for sampled signals recognized by the researchers.

Looking at Fig. 8, we see immediately that by reasoning in this way, these researchers are making two basic errors. In an indefinite way, they identify the sampled signal  $M_{s2}(t)$  that

appears at the output of an analog/digital (A/D) converter with the object denoted here by  $\rho_{s2}(t)$ . (Probably, they do this because of the fact that the function  $M_{s2}(t)$  does not possess a spectrum or it is identically equal to zero (at least in the conventional sense of this notion, i.e. as a Fourier transform; for more details, see, for example, [4]. But, unlike this,  $\rho_{s2}(t)$  has a well-defined Fourier transform.) Secondly,  $\rho_{s2}(t)$  plays here a role of a “density” of some object. And, what does it mean? If interpreted as the “density” of the sampled signal  $M_{s2}(t)$ , it does not, obviously, equal this signal (Fig. 8 demonstrates clearly that  $\rho_{s2}(t) \neq M_{s2}(t)$ ). So, we have inconsistencies in this reasoning.

However, one can get rid of the inconsistencies mentioned above. For example, an approach we want to propose now is quite reasonable. Namely, as the sampled signal  $M_{s2}(t)$  possesses no spectrum in the conventional sense or it is identically equal to zero (as already mentioned above), one resorts to defining a substitute spectrum for  $M_{s2}(t)$  by assuming that it is a spectrum of a function or of an object very closely related with  $M_{s2}(t)$ . Here,  $\rho_{s2}(t)$  is obviously such an object. And, this is precisely what in fact do the advocates of the sampled signal description presented in Fig. 8. However, this is not all we would like to say about it.

One very important thing has been overlooked by its advocates. Namely, all the values of  $M_{s2}(t)$  equal identically to zero that occur at the points of the Ox axis which are not contained in the set:  $\{\dots, -2x_0, -x_0, 0, x_0, 2x_0, \dots\}$  are false. Why? Because the A/D converter produces nothing at these points, not zeros. So, in other words, we should rather interpret this fact as follows: these points are forbidden. And, for this reason, we should also ask what implications this has on the form of the associated (with  $M_{s2}(t)$ ) density object  $\rho_{s2}(t)$ .

Note now however that we have already found here a description for the latter situation (described above). Namely, the sampled signal as it is really outputted by the A/D converter (with no values for the forbidden points mentioned above) is well described by  $M_{s1}(t)$ , and its associated density object  $\rho_{s1}(t)$ . This is illustrated in Fig. 7.

Further, observe that the density object  $\rho_{s1}(t)$  (see Fig. 7a) is in this case an ordinary function and therefore possesses a well-defined Fourier transform. Moreover, it can be well approximated by

$$\hat{\rho}_{s1}(t) = \sum_{k=-\infty}^{\infty} M_{s1}(kT + (1/2)T) \text{sinc}(t/T - k). \quad (16)$$

We observe also that the functions  $\rho_{s1}(t)$  and  $\hat{\rho}_{s1}(t)$  are not periodic ones. Therefore, their spectra are not periodic, too. So, in the case of this description, we cannot talk about occurrence of any aliasing and folding effects.

## VI. FINAL CONCLUSION

Once again, the main conclusion following from the results presented in this paper is that any aliasing and folding effects in spectra of sampled signals cannot occur. The primary cause of this lies in that there exist no real A/D signal converters which produce sequences of Dirac impulses. And, in our opinion, this fact gives an incentive to introduce some revisions into the existing theory.

## REFERENCES

- [1] A. V. Oppenheim, R. W. Schaffer, and J. R. Buck, *Discrete-Time Signal Processing*. New Jersey: Prentice Hall, 1998.
- [2] R. J. Marks II, *Introduction to Shannon Sampling and Interpolation Theory*, New York: Springer-Verlag, 1991.
- [3] R. N. Bracewell, *The Fourier Transform and Its Applications*, New York: McGraw-Hill, 2000.
- [4] A. Borys, "Spectrum aliasing does not occur in case of ideal signal sampling," *Intl Journal of Electronics and Telecommunications*, vol. 67, no. 1, pp. 71-77, 2021. <https://doi.org/10.24425/ijet.2021.135946>
- [5] M. Vetterli, P. Marziliano, T. Blu, "Sampling signals with finite rate of innovation", *IEEE Transactions on Signal Processing*, vol. 50, no. 6, pp. 1417-1428, 2002. <https://doi.org/10.1109/TSP.2002.1003065>
- [6] M. Vetterli, J. Kovacevic, V. K. Goyal, *Foundations of Signal Processing*. Cambridge, England: Cambridge University Press, 2014.
- [7] A. Borys, "Some useful results related with sampling theorem and reconstruction formula," *Intl Journal of Electronics and Telecommunications*, vol. 65, no. 3, pp. 471-475, 2019. <https://doi.org/10.24425/ijet.2019.129801>
- [8] A. Borys, "Filtering property of signal sampling in general and under-sampling as a specific operation of filtering connected with signal shaping at the same time," *Intl Journal of Electronics and Telecommunications*, vol. 66, no. 3, pp. 589-594, 2020. <https://doi.org/10.24425/ijet.2020.134016>
- [9] A. Borys, "Some topological aspects of sampling theorem and reconstruction formula," *Intl Journal of Electronics and Telecommunications*, vol. 66, no. 2, pp. 301-307, 2020. <https://doi.org/10.24425/ijet.2020.131878>
- [10] A. Borys, "On derivation of discrete time Fourier transform from its continuous counterpart," *Intl Journal of Electronics and Telecommunications*, vol. 66, no. 2, pp. 355-368, 2020. <https://doi.org/10.24425/ijet.2020.131885>
- [11] A. Borys, "Spectrum aliasing does occur only in case of non-ideal signal sampling," *Intl Journal of Electronics and Telecommunications*, vol. 67, no. 1, pp. 79-85, 2021. <https://doi.org/10.24425/ijet.2021.135947>
- [12] R. F. Hoskins, *Delta Functions: Introduction to Generalised Functions*. Oxford: Woodhead Publishing, 2010.