

# Nonlinear magnetic circuit – self-inductance definitions, passivity and waveforms distortion

 Dariusz SPAŁEK<sup>✉\*</sup>

Silesian University of Technology, Electrical Engineering Faculty, Akademicka 10, 44-100 Gliwice, Poland

**Abstract.** The generalized magnetizing curve series for the nonlinear magnetic circuit is proposed. Subsequently, three definitions of self-inductance for the nonlinear magnetic circuit are compared. The passivity of the magnetic circuit is reconsidered. Three theorems that describe features of Fourier harmonics of distorted waveforms have been proved.

**Key words:** generalized magnetizing curve; nonlinear magnetic circuits; self-inductance definitions; passivity; Fourier harmonics of distorted waveforms.

## LIST OF MAIN SYMBOLS

$c_{k+1}$	–	coefficients of current Maclaurin series,
$d_{k+1}$	–	coefficients of flux Maclaurin series,
$B$	–	magnetic flux density (magnitude $B_m > 0$ ),
$E_\mu$	–	magnetic field energy stored in the magnetic circuit,
$f$	–	frequency,
$H$	–	magnetic field strength,
$\text{sgn}()$	–	sign function,
$\mu()$	–	magnetic permeability (isotropic parameter or one of the principal axes values),
$\nu()$	–	reluctivity $\nu() = 1/\mu()$ ,
$\nu_k$	–	coefficients of reluctivity Maclaurin series,
$\omega = 2\pi f$	–	angular speed.

## 1. INTRODUCTION

The paper deals with a few theoretical fundamental problems of the nonlinear magnetic circuit. The main attention is paid to

- three definitions of self-inductance coefficient,
- passivity of the nonlinear circuit,
- Fourier series and waveform distortion for nonlinear magnetic circuit supplied by sine source.

The investigations aim to reach some criteria for the verification of models of nonlinear magnetic circuits. There are formulated criteria for generalized nonlinear magnetizing curves by means of Maclaurin series coefficients.

The novelty of the approaches [1–3] presented in this paper lies in taking into account

- terms of even powers  $2p$  (not only of odd powers  $2p + 1$ ) for current–flux (flux–current) and  $H$ – $B$  magnetizing curves,

- the passivity of the nonlinear magnetic circuit,
- three new theorems (and proofs) describing harmonics of distorted waveforms,
- examples of distorted waveform harmonics and their signatures for different nonlinear magnetic circuits.

## 2. THREE DEFINITIONS OF SELF INDUCTANCE COEFFICIENT

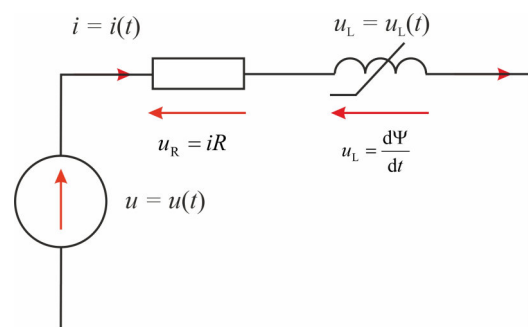
Let us consider three very fundamental definitions of the self-inductance coefficient. From the mathematical point of view and for engineering purposes, it is necessary to consider the comparison between all definitions. The wide bibliography reveals three inductance definitions. Firstly, the so-called flux (static) definition of self-inductance [4, 5]

$$L \stackrel{\text{df}}{=} \Psi/i, \quad (1)$$

where  $\Psi$  denotes magnetic flux of magnetic circuit,  $i$  is current (Fig. 1).

Secondly, the energy (integral) definition of self-inductance is based on magnetic field energy  $E_\mu$  as follows

$$L_E \stackrel{\text{df}}{=} 2E_\mu/i^2, \quad (2)$$



**Fig. 1.** Nonlinear inductance (magnetic circuit) as a part of electric circuit

\*e-mail: [Dariusz.Spalek@polsl.pl](mailto:Dariusz.Spalek@polsl.pl)

Manuscript submitted 2021-10-03, revised 2022-02-16, initially accepted for publication 2022-02-28, published in June 2022.

where magnetic field energy  $E_\mu$  equals to

$$E_\mu = \int_0^\Psi i d\Psi. \quad (3)$$

Thirdly, the dynamic (differential) self-inductance is defined by the following derivative

$$L_D \stackrel{\text{df}}{=} d\Psi/di, \quad (4)$$

A unique relation between magnetic flux and current for the stationary and non-hysteresis magnetic circuit is the most important curve. The relation must be a homeomorphic (bicontinuous) function (Table 1)

$$\Psi = \Psi(i) \Leftrightarrow i = i(\Psi). \quad (5)$$

These relations are valid for isotropic, stationary, and non-hysteresis magnetic regions where it can be written

$$H = v(B)B. \quad (6)$$

The functions (5) should be of the class  $C^\infty$  ( $C$  – infinity function) while quantum phenomena are not taken into account [1, 6, 7]. The functions (5) have to be also odd, thus one of the more extended formulas is proposed in the form of a series as follows

$$\Psi = \Psi(i) = \sum_{k=0}^{\infty} d_{k+1} |i|^k i, \quad (7)$$

where powers (exponents) are not only odd but also even (!). Nevertheless, the function (7) is odd (because of the modulus). The function is called the generalized magnetizing curve (series).

Table 1 presents some generalized magnetizing curves of saturation features different, with odd and even powers (e.g. curves 4<sup>th</sup>, 6<sup>th</sup>, 8<sup>th</sup>).

The inverse function for current is also odd as given below

$$i = i(\Psi) = \sum_{k=0}^{\infty} c_{k+1} |\Psi|^k \Psi. \quad (8)$$

The derivative of each term (8) is set to zero at central point  $\Psi = 0$  (excluding the first term  $k = 0$ ). Therefore the derivatives on both sides of the central point are equal to zero, thus the function (8) is of class  $C^\infty$ .

The magnetic field energy stored equals to

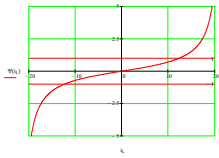
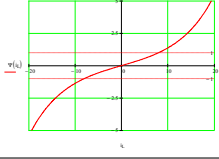

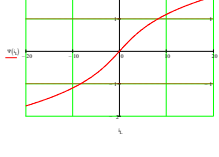
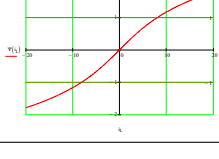
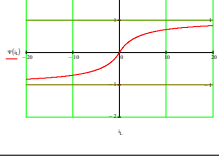
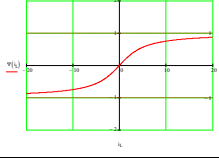
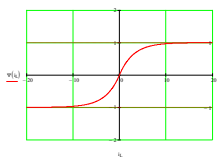
$$E_\mu = \int_0^\Psi i d\Psi = \sum_{k=0}^{\infty} \frac{c_{k+1}}{k+2} |\Psi|^{k+2}, \quad (9)$$

hence

$$\frac{L_E}{L^2} = \sum_{k=0}^{\infty} \frac{c_{k+1}}{k/2+1} |\Psi|^k \Psi / \Psi. \quad (10)$$

**Table 1**

Nonlinear magnetic circuits – exemplary magnetizing curves

	Curve $\Psi = \Psi(i)$	Chart $\Psi = \Psi(i)$
1	$\Psi = \Psi_T \tan(i/I_T)$	
	Odd powers, nontypical Convex curve ( $i > 0$ ), nontypical Saturation (Asymptote) of $i$	
2	$\Psi = \Psi_{SH} \sinh(i/I_{SH})$	
	Odd powers, nontypical Convex curve ( $i > 0$ ), Infinite $i$ , $\Psi$	
3	$i = \frac{\Psi}{L_a} + \frac{\Psi}{a}$	
	First power, Linear, Infinite $i$ , $\Psi$	
4	$i = \frac{\Psi}{L_a} + \frac{\Psi \Psi ^{\kappa-1}}{a^\kappa}$	
	Real power, Concave curve ( $i > 0$ ), Infinite $i$ , $\Psi$ ; For $a < 0$ possible activity	
5	$\Psi = \Psi_{AH} \text{asinh}(i/I_{AH})$	
	Odd powers, Concave curve ( $i > 0$ ), Infinite $i$ , $\Psi$	
6	$\Psi = \Psi_R \frac{i}{I_R (1 +  i/I_R )}$	
	Odd and Even powers, Concave curve ( $i > 0$ ), Saturation (Asymptote) of $\Psi$	
7	$\Psi = \Psi_{AT} \text{atan}(i/I_{AT})$	
	Odd powers, Concave curve ( $i > 0$ ), Saturation (Asymptote) of $\Psi$	
8	$\Psi = \pm \Psi_{EX} \left(1 - e^{-\frac{i}{I_{EX}}}\right)$	
	Odd and Even powers, Concave curve ( $i > 0$ ), Saturation (Asymptote) of $\Psi$	

If  $c_{k+1} \geq 0$  (considering  $k/2 + 1 \geq 1$ ), then for  $\Psi > 0$  is

$$\frac{L_E}{L^2} \leq \frac{i}{\Psi} = \frac{1}{L}, \quad (11)$$

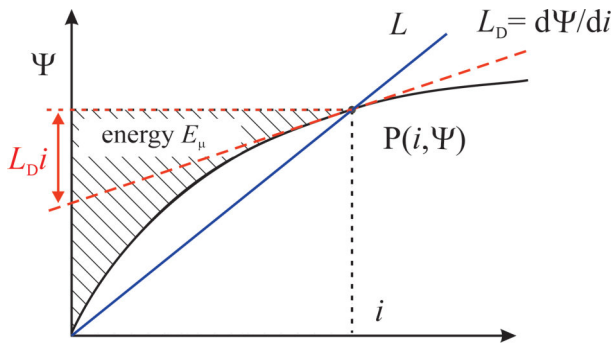
thus

$$L_E \leq L, \quad (12)$$

and the same inequality is valid for  $\Psi < 0$ , due to the odd symmetry of functions (7). Geometrical interpretation of inequality (12) presents Fig. 2, where the area crossed (magnetic energy) for concave functions satisfies the inequality

$$E_\mu \leq E_\Delta = \frac{1}{2}\Psi i, \quad (13)$$

which is equivalent to (12).



**Fig. 2.** Nonlinear concave magnetization curve of magnetic circuit ( $i > 0$ )

Further, the reciprocal of dynamic inductance for  $\Psi > 0$  satisfies the following equality

$$\frac{1}{L_D} = \frac{di}{d\Psi} = \sum_{k=0}^{\infty} (k+1)c_{k+1}|\Psi|^k, \quad (14)$$

which represents the monotonically increasing function of  $\Psi$  (as well as  $i$  and  $i^2$ ). Because the energy inductance is the mean value of dynamic inductance in  $i^2$  space as follows

$$L_E = \frac{2}{i^2} \int_0^i L_D i di = \frac{1}{i^2} \int_0^i L_D d(i^2), \quad (15)$$

thus the mean value satisfies the relation for monotonically decreasing function  $L_D$

$$L_D(i^2 = 0) \geq L_E \geq L_D(i^2). \quad (16)$$

The geometrical interpretation of (16) is shown in Fig. 2. The triangle area of laterals  $L_D i$  and  $i$  satisfies the relation

$$\frac{1}{2}L_D i^2 \leq E_\mu = \frac{1}{2}L_E i^2, \quad (17)$$

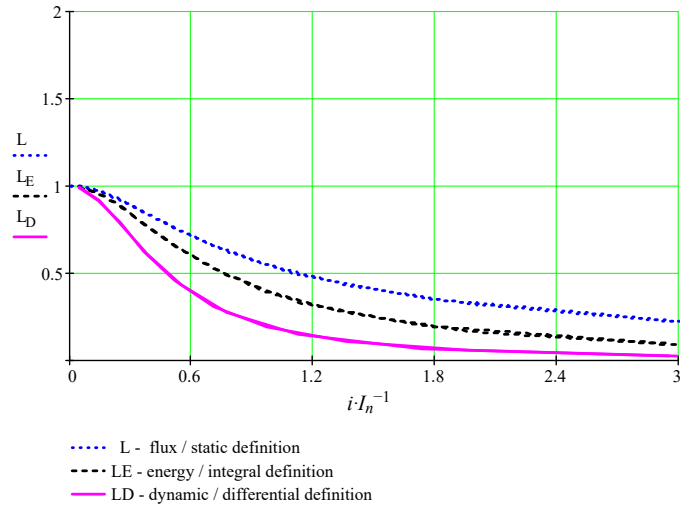
which confirms (16).

The relations (12) and (16) lead to the main concluding relations for self inductances

$$L_D \leq L_E \leq L, \quad (18)$$

which are obtained for convex ( $\Psi > 0$ ) generalized magnetizing curves. The equalities in (18) appear for linear magnetic circuits.

The examples 4–8 (Table 1) satisfy the proved inequalities (18). Figure 3 confirms the proved inequalities (18).

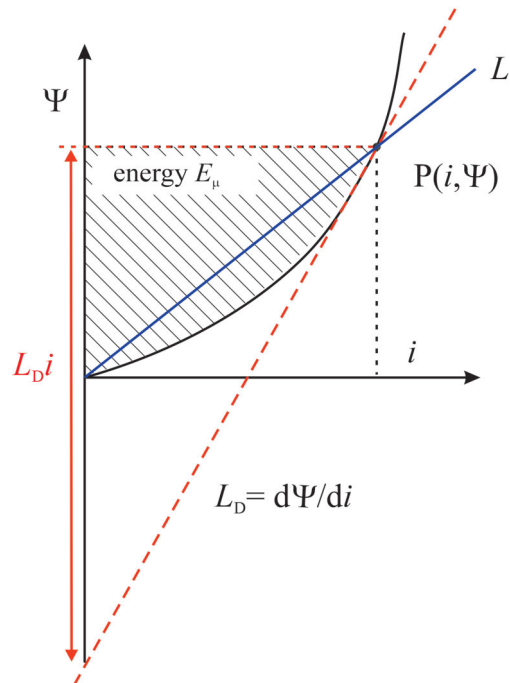


**Fig. 3.** Inductances vs. current for concave functions 4<sup>th</sup> up to 8<sup>th</sup> (Table 1)

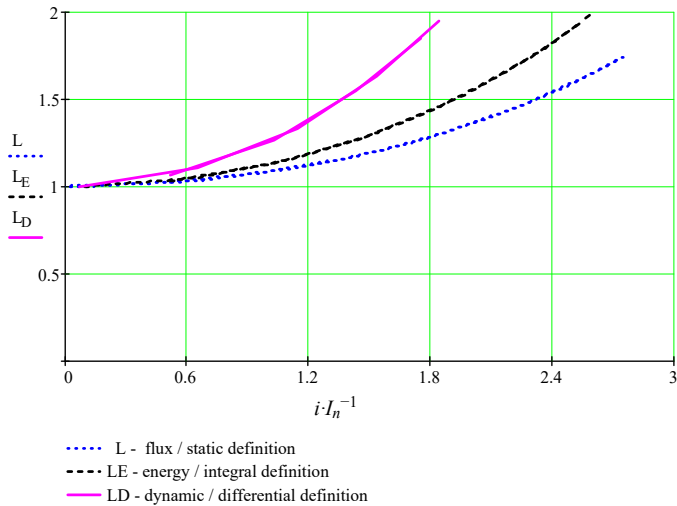
For the curve characteristics which are given by the convex magnetizing curve ( $i > 0$ ), the relations are inversely set then (18), i.e.

$$L_D \geq L_E \geq L. \quad (19)$$

Figures 4 and 5 confirm the inequalities (19) graphically for the convex magnetizing curve ( $i > 0$ ).



**Fig. 4.** Nonlinear convex magnetization curve of magnetic circuit ( $i > 0$ )



**Fig. 5.** Inductances vs. current for convex curves 1<sup>st</sup> and 2<sup>nd</sup> (Table 1)

### 3. PASSIVITY OF NONLINEAR INDUCTANCE

An element of any circuit is passive if two conditions are satisfied. Firstly, the following integral is nonnegative

$$W(t) = \int_{-\infty}^t u(t)i(t) dt \geq 0, \quad (20)$$

secondly, the following equivalence is satisfied

$$\bigcap_{t \in (-\infty, t_0)} (u(t) = 0 \Leftrightarrow i(t) = 0). \quad (21)$$

On the contrary, if the element is not passive, then it is called active. Hence, each element of the magnetic circuit is either passive or active (exclusive or). Passivity is the feature of the circuit that classifies an element of the circuit as a receiver of energy.

Regarding (21) for  $t_0 = 0$  is satisfied

$$W(t) = \int_0^t ui dt = \int_0^\Psi id\Psi = \sum_{k=0}^{\infty} \frac{c_{k+1}}{k+2} \Psi^{k+2}. \quad (22)$$

If all coefficients  $c_{k+1}$  are nonnegative, thus the passivity is guaranteed

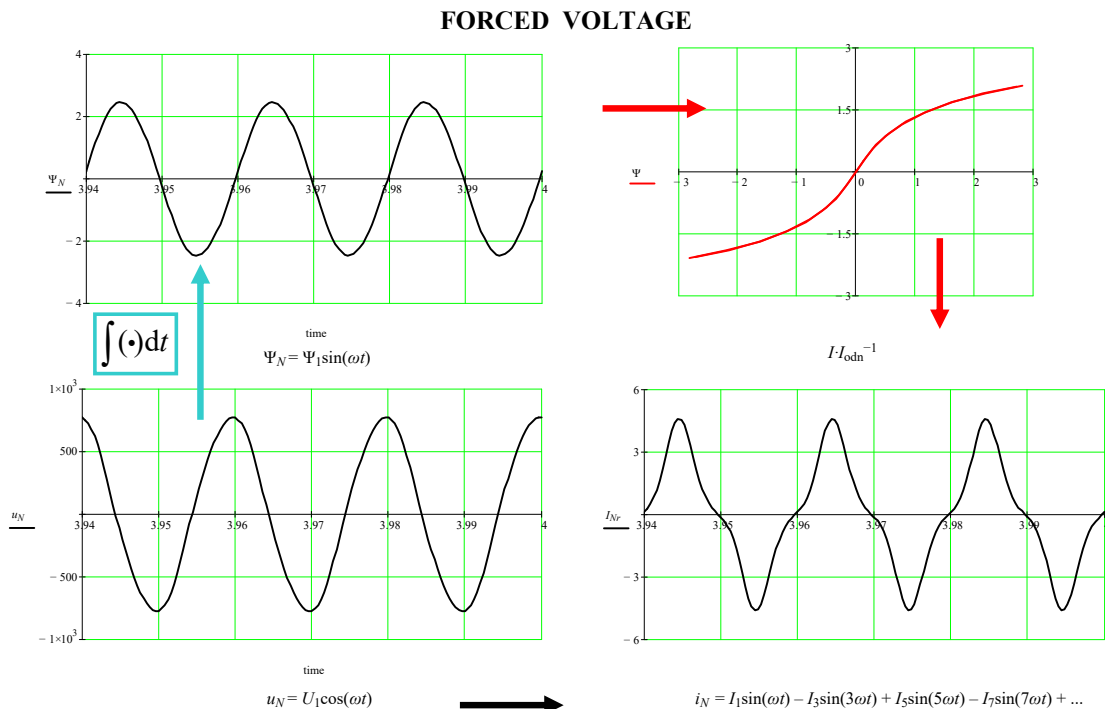
$$W(t) \geq 0. \quad (23)$$

If some coefficients  $c_{k+1}$  are negative the passivity for a particular nonlinear magnetic circuit must be thoroughly checked. In some cases, the nonlinear magnetic circuit may be active. This effect could be technically caused by magnets or electromagnets set into a magnetic circuit.

For example, for the magnetizing curve number 4 (Table 1) at parameters as follows  $U_1 = 650\sqrt{2}$  V,  $R = 0.09 \Omega$ ,  $L_a = 0.159$  H,  $\kappa = 3$ ,  $a = -0.8$  Wb·A<sup>-1/κ</sup>, the condition (23) is not satisfied. The magnetic circuit is active (is not passive).

### 4. FOURIER HARMONICS FOR NONLINEAR MAGNETIC CIRCUIT – MAIN REMARKS

The nonlinear magnetic circuit working under sine source of either voltage or magnetic flux generates waveform distortion in either current or magnetic field strength, respectively – Fig. 6.



**Fig. 6.** Graphical method of analysis of nonlinear magnetic circuit while forcing magnetic flux density (flux, voltage) – waveform distortion (Theorem 3)

Let us assume that magnetic flux density is forced along with one of the reluctivity principal axes as follows

$$B(t) = B_m \sin(\omega t), \quad (24)$$

thus appears waveform distortion [4, 8–10] of magnetic field strength

$$H(t) = v(B(t))B(t), \quad (25)$$

which is described by the Fourier series as follows

$$H(t) = \sum_{h=0}^{\infty} a_h \cos(h\omega t) + b_h \sin(h\omega t). \quad (26)$$

The coefficients of series (26) are given by relations

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} v(B_m \sin(\omega t)) B_m \sin(\omega t) dt, \quad (27)$$

$$a_h = \frac{2}{T} \int_{-T/2}^{T/2} v(B_m \sin(\omega t)) B_m \sin(\omega t) \cdot \cos(h\omega t) dt, \quad (28)$$

$$b_h = \frac{2}{T} \int_{-T/2}^{T/2} v(B_m \sin(\omega t)) B_m \sin(\omega t) \cdot \sin(h\omega t) dt. \quad (29)$$

Subsequently, it is assumed that reluctivity is even a function of magnetic flux density

$$v(-B) = v(B). \quad (30)$$

As a consequence, the following conclusions are derived

- 1)  $a_0 = 0$  – no constant term,
  - 2)  $a_h = 0$  –  $H(t)$  is odd function of time,
  - 3)  $b_{2k} = 0$  – harmonics of orders  $h = 2k$  vanish,
- hence  $H(t)$  is odd  $H(-t) = -H(t)$  and periodically asymmetric  $H(t + T/2) = -H(t)$  function of time [4].

## 5. FOURIER HARMONICS FOR NONLINEAR MAGNETIC CIRCUIT – THREE THEOREMS

In this section, the features of the distorted waveform are considered by means of rigorous harmonics analysis.

Firstly, the third harmonic is investigated. According to (29) it can be written

$$b_{2k+1} = (-1)^k \frac{4B_m}{\pi} \int_0^{\pi} v(B_m \sin \varphi) \cdot \sin \varphi \cdot \sin((2k+1)\varphi) d\varphi, \quad (31)$$

for  $h = 2k + 1$ ,  $\varphi = \omega t$ .

**Theorem 1.** If the reluctivity function  $v(\cdot)$  is positive, odd, and increasing (or decreasing) constantly, then for magnetic flux density (29) it follows

$$H(t) = H_1 \sin(\omega t) \pm H_3 \sin(3\omega t) \dots \quad (32)$$

**Proof.** According to (31) it follows.

$$b_3 = -\frac{8B_m}{\pi} \int_0^{\pi/2} v(B_m \sin \varphi) \sin \varphi \sin(3\varphi) d\varphi. \quad (33)$$

Applying theorem about mean value ([11] p. 186) for certain  $c \in \left(0, \frac{\pi}{2}\right)$ , one obtains

$$b_3 = -\frac{8B_m}{\pi} v(B_m \sin 0) \int_0^c \sin \varphi \sin(3\varphi) d\varphi - \frac{8B_m}{\pi} v\left(B_m \sin\left(\frac{\pi}{2}\right)\right) \int_c^{\pi/2} \sin \varphi \sin(3\varphi) d\varphi, \quad (34)$$

hence, finally

$$b_3 = -\frac{2B_m}{\pi} \{v(B_m) - v(0)\} (1 - \cos 2c) \sin 2c, \quad (35)$$

which leads to conclusions

if  $v(\cdot)$  monotonically increases then

$$b_3 = -|b_3| = -H_3 < 0,$$

if  $v(\cdot)$  monotonically decreases then

$$b_3 = +|b_3| = +H_3 > 0. \quad \square$$

**Theorem 2.** If reluctivity  $v(\cdot)$  takes the form of uniformly convergent even series as follows

$$v(B) = \sum_{n=0}^{\infty} v_n |B|^n, \quad (36)$$

thus for (24), the magnetic field strength takes the form of

$$H(t) = \sum_{i=0}^{\infty} b_{2i+1} \sin((2i+1)\omega t), \quad (37)$$

where harmonic coefficients are equal to

$$b_{2k+1} = \frac{4B_m}{\pi} \sum_{n=0}^{\infty} \alpha_{k,n} v_n B_m^n = \text{sgn}(b_{2k+1}) H_{2k+1}, \quad (38)$$

where  $H_{2+1} = |b_{2k+1}|$ .

For even  $n = 2p$  it is satisfied

$$\alpha_{k,n} = \begin{cases} \frac{(-1)^k \pi}{2^{2p+1}} \binom{2p+1}{p-k} & \text{if } p \geq k, \\ 0 & \text{if } p < k, \end{cases} \quad (39)$$

and for odd  $n = 2p - 1$

$$\alpha_{k,n} = \begin{cases} \frac{(-1)^k 2^{p+1} p! (2p-1)!!}{(2p+2k+1)!! (2p-2k-1)!!} & \text{if } p \geq k, \\ \frac{(-1)^p 2^{2p+1} p! (2p-1)!! (2k-2p-1)!!}{(2p+2k+1)!!} & \text{if } k \geq p. \end{cases} \quad (40)$$

**Remark 1.** Double factorial can be presented for even and odd numbers as follows

$$(2p)!! = 2^p p! = 2^p \Gamma(p+1), \quad (2p-1)!! = \frac{2^p}{\sqrt{\pi}} \Gamma\left(p + \frac{1}{2}\right),$$

respectively.

**Proof.** According to (31) and (36), it follows

$$b_{2k+1} = \frac{4B_m}{\pi} \sum_{n=0}^{\infty} v_n B_m^n \int_0^{\pi} |\sin(\varphi)|^n \sin \varphi \sin((2k+1)\varphi) d\varphi. \quad (41)$$

The integral (denoted by  $I_{k,n}$ ) is determined by means of direct integrations. Finally, Authors of [12] (p. 397, equation 3.631.6) give the results for even  $n = 2p$

$$I_{k,n} = \left\{ \begin{array}{ll} \frac{(-1)^k \pi}{2^{2p+1}} \binom{2p+1}{p-k} & \text{if } p \geq k \\ 0 & \text{if } p < k \end{array} \right\} = \alpha_{k,2p} \quad (42)$$

and for odd  $n = 2p - 1$  ([12] p. 397, equation 3.631.5)

$$I_{k,n} = \left\{ \begin{array}{ll} \frac{(-1)^k 2^{2p+1} p! (2p-1)!!}{(2p+2k+1)!! (2p-2k-1)!!} & \text{if } p \geq k \\ \frac{(-1)^p 2^{2p+1} p! (2p-1)!! (2k-2p-1)!!}{(2p+2k+1)!!} & \text{if } k \geq p \end{array} \right\} = \alpha_{k,2p-1}. \quad (43)$$

□

Often, when the constitutive relation for the magnetic circuit (36) is given only by even coefficients, i.e.  $v_{2p-1} = 0$  and odd coefficients  $v_{2p}$  are nonnegative, hence Theorem 2. leads to Theorem 3.

**Theorem 3.** Under the assumptions of Theorem 2,  $v_{2p-1} = 0$  and  $v_{2p} \geq 0$  is satisfied

$$H(t) = H_1 \sin(\omega t) - H_3 \sin(3\omega t) + H_5 \sin(5\omega t) - H_7 \sin(7\omega t) + \dots, \quad (44)$$

where the coefficients of harmonics are changing sign alternatively, i.e. signature is  $(+, -, +, -, +, -, \dots)$ .

**Proof.** Equations (38), (39) and  $v_{2p-1} = 0$  enable us to put down the following relation for harmonics coefficients

$$b_{2k+1} = (-1)^k 4 \sum_{p=k}^{\infty} v_{2p} \binom{2p+1}{p-k} \left(\frac{B_m}{2}\right)^{2p+1} = (-1)^k H_{2k+1}. \quad (45)$$

Relation (45) implies the sufficient and necessary condition for alternative changes of signs of coefficients  $b_{2k+1}$ . Namely, the equivalence is as given below

$$\sum_{p=k}^{\infty} v_{2p} \binom{2p+1}{p-k} \left(\frac{B_m}{2}\right)^{2p+1} \geq 0 \Leftrightarrow H_{2k+1} \geq 0. \quad (46)$$

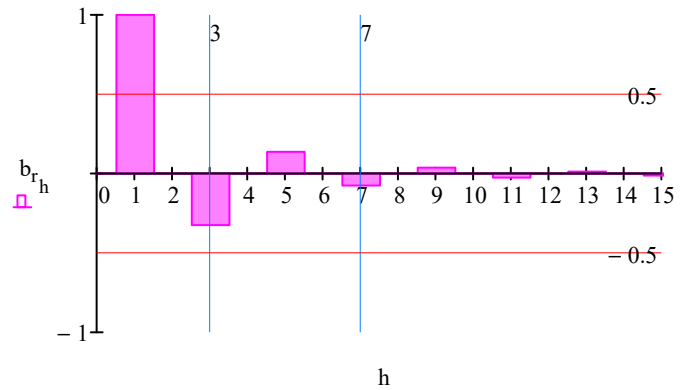
In particular, the sum is positive for  $v_{2p} \geq 0$ , which means the signs of consecutive coefficients of series (44) change alternatively. □

**Remark 2.** If for  $k > N$  the coefficients  $v_{2k} = 0$  (i.e. Maclaurian series reduces to polynomial of  $N$ -th degree), then  $b_{2k+1} = 0$ .

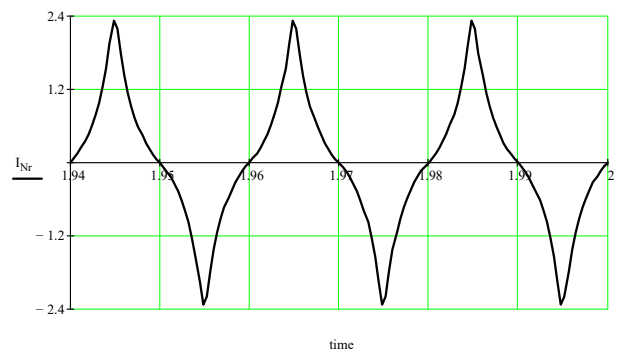
## 6. EXAMPLES

The presented Theorem 2 gives us insight into all harmonic values, generally. Particularly, Theorem 3 determines signs and moduli of harmonics of distorted waveforms by means of necessary and sufficient conditions (46).

For example, for concave magnetizing curves (Table 1, 4<sup>th</sup>–8<sup>th</sup> rows), according to Theorem 2 and (38), the signs of harmonics are changing alternatively as shown in Fig. 7. The appropriate prolate distorted waveform is presented in Fig. 8 (and in Fig. 6).



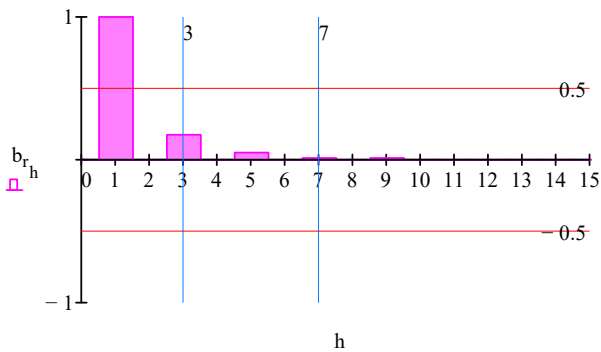
**Fig. 7.** Current harmonics vs. harmonic numbers  $h$  of the concave generalized magnetizing curve  $\text{sgn}(x)[1 - \exp(-|x|)]$  – the distorted waveform signature is alternate  $(+, -, +, -, +, -, \dots)$



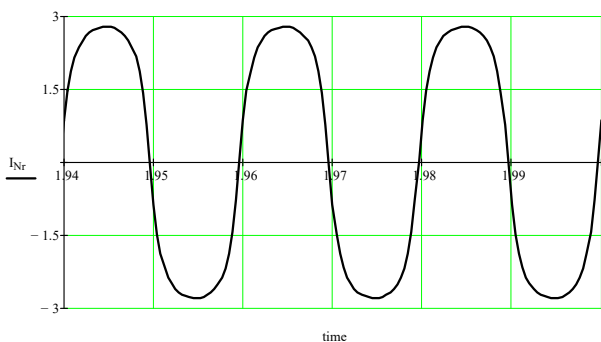
**Fig. 8.** Current vs. time of the concave generalized magnetizing curve  $\text{sgn}(x)[1 - \exp(-|x|)]$  – the distorted waveform is prolate

On the contrary, for the convex magnetizing curve (Table 1, 1<sup>st</sup> and 2<sup>nd</sup> rows), according to Theorem 2 and (38), it results that the signs remain unaltered – Fig. 9. The appropriate oblate distorted waveform is presented in Fig. 10.





**Fig. 9.** Current harmonics vs. harmonic numbers  $h$  of the convex magnetizing curve  $\tan(x)$  – the distorted waveform signature is unaltered  $(+, +, +, +, +, +, \dots)$



**Fig. 10.** Current vs. time of the convex magnetizing curve  $\tan(x)$  – the distorted waveform is oblate

## 7. CONCLUSIONS

There are considered nonlinear magnetic circuits described by the generalized magnetizing curve (series) given by (7) and (36). These series contain not only odd and also even powers. The even terms are incorporated with the help of the modulus function. The generalized magnetizing curve allows us to describe more precisely the nonlinear magnetic circuit. Exemplary magnetizing curves are presented in Table 1.

For the generalized magnetizing curves the three fundamental self-inductance definitions are revised and are ordered by means of inequality relations.

The passivity of the nonlinear magnetic circuit is reconsidered with respect to the generalized magnetizing curve formula. Examples of passive and active nonlinear magnetic circuits are presented.

Fourier series of current and distorted magnetic field strength are focused on nonlinear magnetic circuits. Three theorems are proved that describe values and signatures of harmonics (mag-

netic field strength, current). The first theorem is about the third harmonic sign (32). The second theorem describes generally the values (sign and modulus) of harmonics for distorted waveforms (38). The second and third theorems describe harmonics signatures of distorted waveforms, e.g.  $(+, +, +, +, +, +, \dots)$  or  $(+, -, +, -, +, -, \dots)$ . Moreover, the third theorem formulates the necessary and sufficient condition (46) for harmonics signature of distorted waveform.

The presented generalized magnetizing curve and three theorems enable us to verify models of the nonlinear magnetic circuits, e.g. the accuracy of approximation applied, the passivity of the circuit, and the harmonics signature of distorted waveforms.

The presented theoretical approaches can be used for comparison proposes and verification of other results.

## REFERENCES

- [1] D. Spalek, "Theorems about higher harmonics of distorted waveform," *5-th Slovak and Tschech International Conference Sensor and Sensors System*, Presov, Slovakia, 1994, pp. 58–63, [in Polish].
- [2] D. Spalek, "Fourier series of nonlinear magnetic circuit magnetizing currents," *Elektryka, Silesian University of Technology*, vol. 154, pp. 101–108, 1996, [in Polish].
- [3] D. Spalek, "Two relations for generalized discrete Fourier transform coefficients," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 66, no. 3, pp. 275–281, 2018, doi: [10.24425/123433](https://doi.org/10.24425/123433).
- [4] S. Bolkowski, *Electric circuit theory*, WNT, Warsaw, 1995, [in Polish].
- [5] B.S. Guru and H.R. Hiziroglu, *Electromagnetic field theory fundamentals*, University Press, Cambridge, 2004.
- [6] R.S. Ingarden and A. Jamiołkowski, *Classical electrodynamics*, Elsevier-PWN, Warsaw, 1985.
- [7] I. Białynicki-Birula and Z. Białynicka-Birula, *Quantum Electrodynamics*, Pergamon Press, 1975.
- [8] S. Hui and S.H. Žak, "Discrete Fourier transform and permutations," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 67, no. 6, pp. 995–1005, 2019, doi: [10.24425/bpasts.2019.130874](https://doi.org/10.24425/bpasts.2019.130874).
- [9] P. Kapler, "An application of continuous wavelet transform and wavelet coherence for residential power consumer load profiles analysis," *Bull. Bull. Polish Acad. Sci. Tech. Sci.*, vol. 69, no. 1, e136216, 2021, doi: [10.24425/bpasts.2020.136216](https://doi.org/10.24425/bpasts.2020.136216).
- [10] H. Yi, Y.-L. Ru, and Y.-Y. Dai, "Time invariant property of weighted circular convolution and its application to continuous wavelet transform," *Bull. Polish Acad. Sci. Tech. Sci.*, vol. 69, no. 4, e137726, 2021, doi: [10.24425/bpasts.2021.137726](https://doi.org/10.24425/bpasts.2021.137726).
- [11] K. Kuratowski, *Differential and integral calculus*, PWN, Warsaw, 1978, [in Polish].
- [12] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Seventh Edition, Elsevier, 2007.