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Controllability of time varying semilinear non-instantaneous impulsive systems with delay, and nonlocal conditions

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In this paper we prove the exact controllability of a time varying semilinear system considering non-instantaneous impulses, delay, and nonlocal conditions occurring simultaneously. It is done by using the Rothe's fixed point theorem together with some sub-linear conditions on the nonlinear term, the impulsive functions, and the function describing the nonlocal conditions. Furthermore, a control steering the semilinear system from an initial state to a final state is exhibited.

Key words: exact controllability, semilinear time varying control systems, non-instantaneous impulses, delay, nonlocal conditions, Rothe's fixed point theorem

1. Introduction

Control systems appear naturally in the technological development of humanity, but almost no one had bothered to give the mathematical formulation of each one of the improvements made. In real life, it seems that all control systems are controllable, which means transferring the system from an initial state to a final desired state conveniently. But it was until the Kalman's algebraic rank condition (1930–2016) to verify the controllability of linear autonomous systems that mathematicians realized that many systems are not controllable. So, control problems have been studied for a long time, but not from the mathematical point of view;

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one of the first applications dates back to regulatory mechanisms with float in Greece. Motivated by the need of measuring the time, Greeks built the ktesibius clock, a water clock, around 250 BC [1], which is considered the first control system of history. However, the first significant work in control was James Watt's centrifugal regulator, which was developed in 1788 and worked with automatic feedback [2]. Control problems that arose at that time until approximately the end of the 19th century were generally characterized by being eminently intuitive. This was no longer the case as the desire to improve the transient responses and the precision of the control systems allowed control theory to be developed.

By the mid-twentieth century, mainly because of the world wars, control theory played an important role in some fields such as navigation and communication. For instance, it was crucial for controlling the directions of the ships, in problems related to radars, and in the study of the precise positioning of weapons on military ships and aircraft; as well as in the aerospace field. The 20th century, in particular after the second world war, was characterized mainly by the appearance of several works and books. To mention some of them, we have the works done by Xu, Solodovniko, Liu Bao, Izawa, Aizerman, Krasovski, Cheon et al., Kim, La Salle and Bellman [3].

Between 1960 and 1980 optimal control was thoroughly investigated: deterministic and stochastic systems, as well as adaptive control and learning of complex systems. The most notable works during this period of time were three studies presented by Kalman et al. These works were characterized by the introduction of linear algebra, so that systems with multiple inputs and outputs could be treated [4]. From the 1980s to the 1990s, the advances in modern control theory were focused on robust control and related issues where some perturbations of the control systems are considered as intrinsic phenomena of it [5]. Currently, this constitutes a promising branch of research, since the more perturbations a control system has, the better is the representation of real-life problems.

Nowadays, the study of controllability for this type of system (with perturbations) is being increasingly investigated. To mention some of them, we have the work done by Leiva about the controllability of semilinear impulsive nonautonomous systems by the use of Rothe's fixed point theorem [6]. The work done by Balachandran and Arthi about the controllability of nonlinear system with impulses and nonlocal conditions using Banach fixed point theorem [7], as well as the work done by Selvi and Malik about the controllability of impulsive differential systems with finite delay by using measures of noncompactness and Monch fixed point theorem [8]. And recently, the work done by Malik et al. about the controllability of a non-autonomous nonlinear differential system with non-instantaneous impulses [9]. Also, the work done by Chen et al. about the approximate controllability of non-autonomous evolution systems with nonlocal conditions [10]. Some of the literature that was used initially to study the aforementioned and other related works were the works done by Lee and Markus, Curtain and Pritchard, Curtain and Zwart, Sontag among others (see [11–13], and [14] respectively). Here, the authors give the first base concepts and definitions to study both finite and infinite dimensional control linear systems via the controllability and the Gramian operators.

Recently, Leiva et al. reported results on controllability considering simultaneously instantaneous impulses, delay and nonlocal conditions using Rothe's fixed point theorem [15]. Motivated by this work and by the work done by Malik, on controllability of nonlocal non-autonomous neutral differential systems including non-instantaneous impulsive effects [16], in this paper, we study the controllability of non-autonomous systems considering non-instantaneous impulses, delay, and nonlocal conditions occurring simultaneously. We are interested in study the controllability of non-instantaneous impulsive systems encouraged by the several applications that it has in science and engineering. This kind of system describes the dynamics of processes in which a change happens abruptly at a fixed time and this change remains on a finite time interval. It is observed in lasers, and in the intravenous introduction of drugs in the bloodstream [17]. The starters of this new class of problems were Hernandez and O'Regan with their study about the existence of mild and classical solutions [18].

In order to read another works on non-instantaneous impulses the reader can find them in [19–22]. On the other side, we consider delay and nonlocal conditions in our system since these types of problems occur naturally when modeling real-life problems. For instance, one may consider problems with feedback controls, such as the steady-states of a thermostat where a controller at one of its ends adds or removes heat, depending upon the temperature registered in another point, or phenomena with dependence in the equation and in the boundary conditions, with delays or advances. To end this introduction, we present the outline of the paper: Section 2 describes the analyzed system, notation, preliminary concepts, the hypotheses and the results used throughout this work. Section 3 is devoted to show the controllability for system (3) in the light of Rothe's fixed point theorem. Finally, Section 4 presents a concluding remark.

2. Preliminares

For the purpose of this paper, consider the time varying (non-autonomous) semilinear system given by:

$$\begin{cases} z'(t) = \mathsf{A}(t)z(t) + \mathsf{B}(t)u(t) + f(t, z_t, u(t)) & t \in \bigcup\limits_{i=0}^{N} (s_i, t_{i+1}], \\ z(t) = \mathsf{G}_i(t, z(t)) & t \in \bigcup\limits_{i=1}^{N} (t_i, s_i], \\ z(s) = \varphi(s) - g(z_{\tau_1}, z_{\tau_2} \ldots, z_{\tau_q})(s) & s \in [-r, 0], \end{cases} \tag{1}$$

where s_i, t_i, $\tau_j \in (0, T)$ with $i = 1, 2, \cdots, N$, $j = 1, 2, \cdots, q$ and $0 = s_0 = t_0$; $t_i \leqslant s_i < t_{i+1} < T$ and $\mathsf{A}(t)$, $\mathsf{B}(t)$ are continuous $n \times n$ and $n \times m$ matrices respectively. The control $u(t) \in \mathbb{R}^m$ and state $z(t) \in \mathbb{R}^n$ where the translation function z_t is given by $z_t(s) = z(t+s)$ and $s \in [-r, 0]$. The nonlinear function f depends on the state and the control; the function φ is the initial conditions, and g represent the nonlocal conditions, this function acts as a feedback operator which adjusts a part of the past when the initial function is present, or even, the whole past when the function φ is absent according to some precise future requirements. More specifically, there exists a fixed number $\zeta > 0$ such that $\tau_q \leqslant \min\{\zeta, T\}$, where $[0, T)$ is the maximal interval of local existence for the solutions of problem (1); and $0 \leqslant \tau_1 < \tau_2 < \cdots < \tau_q$, selected under certain rules marked by the real life problem that the mathematical model could represent, such as: $\tau_j = \dfrac{j\tau_q}{q}$, $j = 1, \ldots, q$. The advantage of using nonlocal conditions is that measurements at more places can be incorporated to get better models. For more details and physical interpretations see [23–27] and references therein. On the other hand, the functions G_i represent the non-instantaneous impulses, and they will be discuss next.

In order to consider a general class of impulsive evolution equations Fečkan et. al. in [28] have the following very important remark on the condition:

$$z(t) = \mathsf{G}_i(t, z(t)), \qquad t \in (t_i, s_i], \quad i = 1, 2, \cdots, N, \tag{2}$$

where $\mathsf{G}_i \in C\left([t_i, s_i]; \mathbb{R}^n\right)$ and there are positive constants d_i, $i = 1, 2, \ldots, N$ such that

$$\|\mathsf{G}_i(t, z^1) - \mathsf{G}_i(t, z^2)\| \leqslant d_i \|z^1 - z^2\|, \quad \forall\, z^1, z^2 \in \mathbb{R}^n, \ t \in [t_i, s_i],$$

and $\max\{d_i : i = 1, 2, \ldots, N\} < 1$ is a necessary condition. Then the Banach fixed point theorem gives a unique $y_i \in C([t_i, s_i], \mathbb{R}^n)$ such that $z = \mathsf{G}_i(t, z)$ iff $z = y_i(t)$. So (2) is equivalent to

$$z(t) = y_i(t), \quad t \in (t_i, s_i], \quad i = 1, 2, 3, \ldots, N,$$

which does not depend on the state variable $z(\cdot)$. Thus, it is necessary to modify the condition (2) and we consider the modify condition as

$$z(t) = \mathsf{G}_i(t, z(t_i^-)), \ t \in (t_i, s_i], \ i = 1, 2, 3, \ldots, N.$$

However, we can prove the exact controllability of system (1) by assuming the following condition:

For any bounded set $\mathcal{D}$ in $\widetilde{\mathcal{PC}}_{(t_1, \cdots, t_N)}$ there exists a continuous function $\rho : [0, \tau] \longrightarrow \mathbb{R}_+$ depending on $\mathcal{D}$ such that $\rho(0) = 0$, and for all $z \in \mathcal{D}$ and $t_1, t_2 \in (t_i, s_i]$, $i = 1, 2, 3, \ldots, N$, we have that

$$\|\mathsf{G}_i(t_2, z(t_2)) - \mathsf{G}_i(t_1, z(t_1))\| \leqslant \rho\left(|t_2 - t_1|\right) \|z\|_{\widetilde{\mathcal{PC}}_{(t_1, \cdots, t_N)}}.$$

This condition allows us to prove that a set of functions is equicontinuous when we apply Rothe's fixed point theorem to prove the exact controllability of the system. However, we are going to modify our original system for a better understanding of the main result proof. So, we shall work with the following system instead:

$$\begin{cases} z'(t) = \mathsf{A}(t)z(t) + \mathsf{B}(t)u(t) + f(t, z_t, u(t)) & t \in \bigcup_{i=0}^{N} (s_i, t_{i+1}] \,, \\ z(t) = \mathsf{G}_i(t, z(t_i^-)) & t \in \bigcup_{i=1}^{N} (t_i, s_i] \,, \\ z(s) = \varphi(s) - g(z_{\tau_1}, z_{\tau_2} \ldots, z_{\tau_q})(s) & s \in [-r, 0]. \end{cases} \tag{3}$$

Here, $C(U;V)$ denotes the set of continuous functions z from U to V, similarly the set $C(U\backslash\{p_1, p_2, \cdots, p_l\}; V)$ are the continuous functions except on a finite number l of points p_i; moreover if the side limits exist at each of the discontinuities and $z(p_i) = z(p_i^-)$, it will denoted by $\widetilde{C}(U\backslash\{p_1, \cdots, p_l\}; V)$ where $z(p_i^+) = \lim_{p\to p_i^+} z(p)$ and $z(p_i^-) = \lim_{p\to p_i^-} z(p)$.

The system (3) is set within the following Banach spaces

$$\mathcal{PW} := \mathcal{PW}([0,T];\mathbb{R}^m) = \big\{u\colon [0,T] \to \mathbb{R}^m : u \text{ is bounded and } u \in C(I';\mathbb{R}^m)\big\},$$

where $I' = \bigcup_{i=0}^{N} [s_i, t_{i+1}]$, endowed with the norm

$$\|u\| = \|u\|_0 = \sup_{t\in[0,T]} \|u(t)\|_{\mathbb{R}^m} .$$

$$\widetilde{\mathcal{PC}}_r := \widetilde{\mathcal{PC}}_r([-r,0];\mathbb{R}^n) = \left\{\varphi : [-r,0] \to \mathbb{R}^n : \varphi \in \widetilde{C}\,(I;\mathbb{R}^n)\right\},$$

where $I = [-r, 0]\backslash\{p_1, p_2, \cdots, p_l\}$ and $l \leqslant N$.

Also, the natural Banach spaces for impulsive differential equations given by

$$\begin{aligned} \widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} &:= \widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)}([-r,T];\mathbb{R}^n) \\ &= \left\{z : [-r,T] \to \mathbb{R}^n : z\Big|_{[-r,0]} \in \widetilde{\mathcal{PC}}_r,\ z\Big|_{[0,T]} \in \widetilde{C}(J;\mathbb{R}^n)\right\}, \end{aligned}$$

where $J = [0, T]\backslash\{t_1, t_2, \cdots, t_N\}$, equipped with the norm

$$\|z\| = \|z\|_0 = \sup_{t\in[-r,T]} \|z(t)\|_{\mathbb{R}^n} .$$

Further, $(\mathbb{R}^n)^q$ denotes the Cartesian product of q copies of $\mathbb{R}^n$, and

$$\|z\|_{(\mathbb{R}^n)^q} = \sum_{i=1}^{q} \|z_i\|_{\mathbb{R}^n} ,$$

its induced norm. Thus $\widetilde{\mathcal{PC}}_r^q := \widetilde{\mathcal{PC}}_r^q([-r,0];(\mathbb{R}^n)^q)$ induces the norm

$$\|z\|_{\widetilde{\mathcal{PC}}_r^q} = \sup_{t\in[-r,T]} \|z(t)\|_{(\mathbb{R}^n)^q} .$$

The space $\widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} \times \widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)}$ is endowed with the norm

$$\|(z,u)\| = \|z\|_0 + \|u\|_0 = \|z\| + \|u\| .$$

For $(z,u) \in \widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)}$, we define the following number

$$\left\|f(\cdot, z_{(\cdot)}, u)\right\|_0 := \sup_{t\in[0,T]} \|f(t, z_t, u(t))\|_{\mathbb{R}^n} ,$$

and used the notation

$$\|B\| = \sup_{0\leqslant t\leqslant \tau} \{\|B(t)\|\}.$$

Additionally, the following assumptions are made:

(a) The nonlinear function $f \in C([0,T] \times \widetilde{\mathcal{PC}}_r \times \mathbb{R}^m; \mathbb{R}^n)$ satisfies

$$\|f(t,v,u)\|_{\mathbb{R}^n} \leqslant a_0 \|v(-r)\|_{\mathbb{R}^n}^{\alpha_0} + b_0 \|u\|_{\mathbb{R}^m}^{\beta_0} + c_0, \quad t \in [0,T,], v \in \widetilde{\mathcal{PC}}_r, u \in \mathbb{R}^m.$$

(b) The non-instantaneous impulses function $\mathsf{G}_i \in C([t_i, s_i] \times \mathbb{R}^n; \mathbb{R}^n)$ for all $i = 1,2,3,\ldots,N$ and satisfies

$$\|\mathsf{G}_i(t,z)\|_{\mathbb{R}^n} \leqslant a_i \|z\|_{\mathbb{R}^n}^{\alpha_i} + c_i,$$

and

$$\| \mathsf{G}_i(s,z) - \mathsf{G}_i(t,w)\| \leqslant d_i (|s-t| + \|z-w\|) .$$

(c) The function for the nonlocal condition $g : \widetilde{\mathcal{PC}}_r^q :\longrightarrow \widetilde{\mathcal{PC}}_r$ satisfies

$$\|g(z)\| \leqslant c \|z\|^\eta ,$$

and

$$\|g(z) - g(w)\| \leqslant K \|z-w\| ,$$

where $\eta, \alpha_i, \beta_0 \in [0,1)$ and a_i, b_i, c_i, d_i, c, K are positive constants with $i = 0,1,2,3,\cdots,N$.

Following, the technique applied in [29, 30] proves that for all $\varphi \in \widetilde{\mathcal{PC}}_r$ and $u \in \mathcal{PW}$ the initial value problem (3) has one and only one solution of the form

$$z(t) = \begin{cases} \varphi(t) - g(z_{\tau_1}, z_{\tau_2} \dots, z_{\tau_q})(t), & t \in [-r, 0], \\ \mathcal{U}(t,0)[\varphi(0) - g(z_{\tau_1}, z_{\tau_2} \dots, z_{\tau_q})(0)] & \\ \quad + \int\limits_0^t \mathcal{U}(t,s)[\mathsf{B}(s)u(s) + f(s, z_s, u(s))]\,\mathrm{d}s, & t \in [0, t_1], \\ \mathsf{G}_i(t, z(t_i^-)), & t \in (t_i, s_i], \\ \mathcal{U}(t, s_i)\mathsf{G}_i(s_i, z(t_i^-)) + \int\limits_{s_i}^t \mathcal{U}(t,s)\mathsf{B}(s)u(s)\,\mathrm{d}s & \\ \quad + \int\limits_{s_i}^t \mathcal{U}(t,s) f(s, z_s, u(s))\,\mathrm{d}s, & t \in (s_i, t_{i+1}], \end{cases} \tag{4}$$

for $i = 1, 2, 3, \dots, N$. The evolution operator $\mathcal{U}$ or transition matrix is defined as follows:

$\mathcal{U}(t, s) = \Phi(t)\Phi^{-1}(s)$ with $\Phi(t)$ being the fundamental matrix of the uncontrolled linear system

$$z'(t) = \mathsf{A}(t)z(t), \tag{5}$$

which satisfies

$$\|\mathcal{U}(t,s)\| \leqslant Me^{\omega(t-s)}, \quad 0 \leqslant s \leqslant t \leqslant T, \quad M \geqslant 1, \quad \text{and} \quad \omega \geqslant 0.$$

Note that $\mathcal{U}(t, t) = I$, where I is the identity operator. In addition, if we apply the definition of $\mathcal{U}$ twice, we can obtain clearly that $\mathcal{U}(t_0, t_2) = \mathcal{U}(t_0, t_1)\mathcal{U}(t_1, t_2)$.

The influence of the non-instantaneous impulses, delay and nonlocal conditions in the system can be roughly visualized in the following figure:

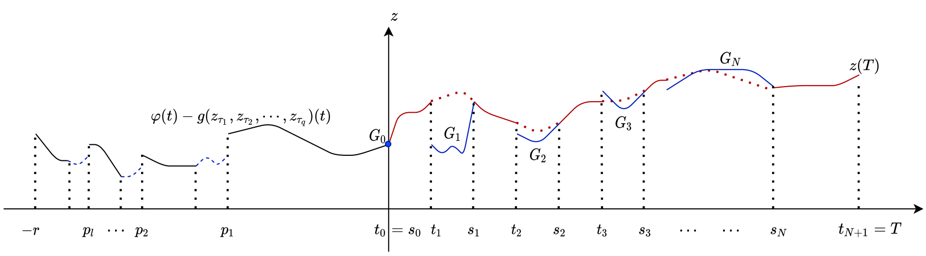

Figure 1: Scheme of the solution

Here, we have the solution on the interval $[-r, 0]$ given by $\varphi(t) - g(z_{\tau_1}, z_{\tau_2} \dots, z_{\tau_q})(t)$ such that the solution is right continuous at the points p_i with $i = 1, 2, 3, \dots N$. In $t_0 = s_0$ we have the function G_0 which is, in this case, a point given by $\mathsf{G}_0 = \varphi(0) - g(z_{\tau_1}, z_{\tau_2} \dots, z_{\tau_q})(0)$. Also, we have the blue lines representing the G_i function which symbolizes the non-instantaneous impulses occurring on $(t_i, s_i]$ and the red lines representing the solution given by the evolution operator on the intervals $(s_i, t_{i+1}]$ for $i = 1, 2, 3, \dots, N$.

The main goal of our work is to prove the controllability of (3), that is, roughly speaking, the ability to steer the system from one state to another state in finite time conveniently. In other words, the system (3) is said to be controllable on $[0, T]$ if for every $\varphi \in \widetilde{\mathcal{PC}_r}$, and $z^1 \in \mathbb{R}^n$, there exists a control $u \in \mathcal{PW}$ such that the corresponding solution $z(t)$ satisfies:

$$z(0) = \varphi(0) - g(z_{\tau_1}, z_{\tau_2} \dots, z_{\tau_q})(0) \quad \text{and} \quad z(T) = z^1.$$

In order to achieve this, the linear control system

$$z'(t) = \mathsf{A}(t)z(t) + \mathsf{B}(t)u(t), \tag{6}$$

associated to (3) it is assumed to be controllable in any interval $[\alpha, \beta] \subseteq [0, T]$.

In light of existing studies of the linear system controllability, (see [12, 13, 31–33]), it is known that the controllability of (6) on $[0, T]$ is obtained from the subjectivity of the operator $\mathcal{G} : L^2([0, T]; \mathbb{R}^m) \longrightarrow \mathbb{R}^n$ defined by

$$\mathcal{G}u = \int_0^T \mathcal{U}(T, s)\mathsf{B}(s)u(s)\,\mathrm{d}s, \tag{7}$$

with corresponding adjoint operator $\mathcal{G}^* : \mathbb{R}^n \longrightarrow L^2([0, T]; \mathbb{R}^m)$ is given by

$$(\mathcal{G}^* z)(s) = \mathsf{B}^*(s)\mathcal{U}^*(T, s)z. \tag{8}$$

And a control $u \in L^2([0, T]; \mathbb{R}^m)$ which directs the system (6) from initial state z^0 to a final state z^1 on $[0, T]$ is given as follows:

$$u(t) = \mathsf{B}^*(t)\mathcal{U}^*(T, t)(\mathcal{W}_{[0,T]})^{-1}(z^1 - \mathcal{U}(T, 0)z^0), \quad t \in [0, T], \tag{9}$$

where $\mathcal{W}_{[0,T]} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the Controllability Gramian Operator in the interval $[0, T]$, defined as

$$\mathcal{W}_{[0,T]}z = \mathcal{G}\mathcal{G}^* z = \int_0^T \mathcal{U}(T, s)\mathsf{B}(s)\mathsf{B}^*(s)\mathcal{U}^*(T, s)z\,\mathrm{d}s. \tag{10}$$

In the same way, the linear system (6) is controllable on $[\alpha, \beta] \subseteq [0, T]$ if, and only if, the controllability operator given by

$$\mathcal{G}_\beta u = \int_\alpha^\beta \mathcal{U}(\beta, s)\mathsf{B}(s)u(s)\,\mathrm{d}s, \tag{11}$$

is surjective. i.e., the Gramian Operator $\mathcal{W}_{[\alpha,\beta]}$ given by

$$\mathcal{G}_\beta\mathcal{G}_\beta^* z = \mathcal{W}_{[\alpha,\beta]} z = \int_\alpha^\beta \mathcal{U}(\beta, s)\mathsf{B}(s)\mathsf{B}^*(s)\mathcal{U}^*(\beta, s)z\,\mathrm{d}s, \tag{12}$$

is invertible. For the aforementioned matrix, there exist δ_α positive value such that $\left\|\mathcal{W}_{[\alpha,\beta]}^{-1}\right\| < \dfrac{1}{\delta_\alpha}$, and a control u steering the linear system (6) from z^α to z^β on $[\alpha, \beta]$ is given by

$$u(t) = \mathsf{B}^*(t)\mathcal{U}^*(\beta, t)(\mathcal{W}_{[\alpha,\beta]})^{-1}(z^\beta - \mathcal{U}(\beta, \alpha)z^\alpha), \quad t \in [\alpha, \beta]. \tag{13}$$

Remark 1 *When we study the exact controllability of finite-dimensional linear systems with controls in L^2-spaces, we must bear in mind that the system is controllable if, and only if, it is controllable with controls in any dense subspace of L^2. Therefore, the system (6) is controllable with controls on L^2 if, and only if, it is controllable with controls on $\mathcal{PW}$(see [6]). Given our main problem, and in order to study the exact controllability of it, we will use the following theorems:*

Theorem 1 ***Rothe's fixed theorem*** *Let E be a Banach space. Let $B \subset E$ be a closed convex subset such that the zero of E is contained in the interior of B. Let $\Psi : B \to E$ be a continuous mapping with $\Psi(B)$ relatively compact in E and $\Psi(\partial B) \subset B$. Then there is a point $x^* \in B$ such that $\Psi(x^*) = x^*$.*

3. Main results

In this section, it is proved the exact controllability of the nonlinear system with non-instantaneous impulses, delay, and nonlocal conditions given by (3). With that purpose, for $t \in [0, T]$ and $i = 0, 1, 2, \cdots, N$, we define the operators $\mathcal{S}_1$ and $\mathcal{S}_2$ as follows

$$\begin{aligned} \mathcal{S}_1 : \widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} \times \mathcal{PW} &\longrightarrow \widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} \\ (z, u)(t) &\longmapsto y(t) = \mathcal{S}_1(z, u), \end{aligned}$$

$$\begin{aligned} \mathcal{S}_2\colon \widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} \times \mathcal{PW} &\longrightarrow \mathcal{PW} \\ (z,u)(t) &\longmapsto v(t) := \mathcal{S}_2(z,u)(t) \end{aligned}$$

where

$$y(t) = \begin{cases} \varphi(t) - g(z_{\tau_1}, z_{\tau_2} \dots, z_{\tau_q})(t), & t \in [-r,0], \\ \mathcal{U}(t,0)\{\varphi(0) - g(z_{\tau_1}, z_{\tau_2} \dots, z_{\tau_q})(0)\} & \\ \quad + \int\limits_0^{t_1} \mathcal{U}(t,s)[\mathsf{B}(s)(\Upsilon_0 \mathfrak{L}_0(z,u))(s) + f(s,z_s,u(s))]\,\mathrm{d}s, & t \in [0,t_1], \\ \mathsf{G}_i(t, z(t_i^-)), & t \in (t_i, s_i], \\ \mathcal{U}(t,s_i)\mathsf{G}_i(s_i, z(t_i^-)) + \int\limits_{s_i}^{t} \mathcal{U}(t,s)\mathsf{B}(\Upsilon_i \mathfrak{L}_i(z,u))(s)\,\mathrm{d}s, & \\ \quad + \int\limits_{s_i}^{t} \mathcal{U}(t,s) f(s,z_s,u(s))\,\mathrm{d}s, & t \in (s_i, t_{i+1}]\,, \end{cases}$$

and

$$v(t) = \begin{cases} \Upsilon_i \mathfrak{L}_i(z,u) := & \\ \mathsf{B}^*(t)\mathcal{U}^*(t_{i+1},t)(\mathcal{W}_{[s_i,t_{i+1}]})^{-1}\mathfrak{L}_i(z,u), & t \in (s_i,t_{i+1}]\,, \\ 0, & t \in (t_i,s_i]\,, \end{cases}$$

here

$$\begin{aligned} \mathfrak{L}_i(z,u) &= z^{t_{i+1}} - \mathcal{U}(t_{i+1},s_i)\mathsf{G}_i(s_i,z(t_i^-)) \\ &\quad - \int\limits_{s_i}^{t_{i+1}} \mathcal{U}(t_{i+1},s) f(s,z_s,u(s))\,\mathrm{d}s, \\ \mathcal{W}_{[s_i,t_{i+1}]} z &= \int\limits_{s_i}^{t_{i+1}} \mathcal{U}(t_{i+1},s)\mathsf{B}(s)\mathsf{B}^*(s)\mathcal{U}^*(t_{i+1},s)z(s)\,\mathrm{d}s. \end{aligned} \tag{14}$$

and there exists $\delta_i > 0$ such that $\left\| \mathcal{W}^{-1}_{[s_i,t_{i+1}]} \right\| < \frac{1}{\delta_i}$.

One the operators $\mathcal{S}_1$, $\mathcal{S}_2$ are defined, we define the operator $\mathcal{S}$ as follows

$$\begin{aligned} \mathcal{S}\colon \widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} \times \mathcal{PW} &\longrightarrow \widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} \times \mathcal{PW} \\ (z(t),u(t)) &\longmapsto \mathcal{S}(z,u) = (\mathcal{S}_1(z,u)(t), \mathcal{S}_2(z,u)(t)) \end{aligned}$$

where $z^{t_{i+1}}$, $i = 0,1,2,\dots,N$ are arbitrary fixed states.

The following remark describes the properties of $\mathcal{S}$ and it can be trivially show from the definition of $\mathcal{S}$.

Remark 2 *The semilinear system with non-instantaneous impulses, delay, and nonlocal conditions* (3) *is controllable on* $[0, T]$*, if and only if, for all initial state* $\varphi \in \widetilde{\mathcal{PC}}_r$ *and a final state* z^1 *the operator* $\mathcal{S}$ *has a fixed point. i.e., there exist* (z, u) *in the domain of* $\mathcal{S}$ *satisfying* $\mathcal{S}(z, u) = (z, u)$.

Before proving of main Theorem, we shall consider the following technical Lemma.

Lemma 1 *Under the assumptions (a)-(c) of the semilinear system* (3)*, the operator* $\mathcal{S}$ *satisfies*

i. $\mathcal{S}$ *is continuous*

ii. $\mathcal{S}$ *maps* $\widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} \times \mathcal{PW}$ *bounded sets into* $\widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} \times \mathcal{PW}$ *equicontinuous sets.*

iii. *The set* $\mathcal{S}(D)$ *is relatively compact for any closed and bounded subset* D *of* $\widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} \times \mathcal{PW}$.

iv. *There exists a closed ball* $B(0, r)$ *of center zero and radius* $r > 0$ *belonging to* $\widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} \times \mathcal{PW}$ *such that* $\mathcal{S}(\partial B(0, r)) \subset B(0, r)$.

Proof. *i.* Considering the hypotheses (a)–(c), $\mathcal{S}_1$ satisfies:

- if $t \in [0, t_1]$

$$\begin{aligned}\|\mathcal{S}_1(z, u)(t) - \mathcal{S}_1(w, v)(t)\| \leqslant \widehat{K}_0 \|z - w\| \\ + M_0 \sup_{s\in[0,t_1]} \|f(s, z_s, u(s)) - f(s, w_s, v(s))\|;\end{aligned}$$

- if $t \in (t_i, s_i]$

$$\|\mathcal{S}_1(z, u)(t) - \mathcal{S}_1(w, v)(t)\| \leqslant d_i \|z - w\|,$$

- if $t \in (s_i, t_{i+1}]$ and $i = 1, 2, 3, \ldots, N$

$$\begin{aligned}\|\mathcal{S}_1(z, u(t) - \mathcal{S}_1(w, v)(t)\| \leqslant \widehat{C}_i \|z - w\| \\ + M_i \sup_{s\in(s_i,t_{i+1}]} \|f(s, z_s, u(s)) - f(s, w_s, v(s))\|,\end{aligned}$$

such that $\widehat{K_i} = C_i E_i K$, $\widehat{C_i} = d_i C_i E_i$, $M_i = D_i E_i$, with $C_i = Me^{\omega(t_{i+1}-s_i)}$, $D_i = \dfrac{M}{\omega}[e^{\omega(t_{i+1}-s_i)} - 1]$ and $E_i = 1 + \dfrac{M^2 \|B\|^2}{\omega\delta_i}[e^{2\omega(t_{i+1}-s_i)} - 1]$.

Then, since f, G_i, g are continuous, then $\mathcal{S}_1$ is continuous. In addition, $\mathcal{S}_2$ is continuous since B and $\mathcal{U}$, $\mathfrak{L}_i$, and $\mathcal{W}_{[s_i,t_{i+1}]}$ are also continuous. As consequence, the operator $\mathcal{S}$ is continuous. Note that in the interval $[-r, 0]$ we get right bound of $\mathcal{S}_1$ from the hypothesis (c), and the operator $\mathcal{S}_2$ is zero there.

ii. Let $D \subset \widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} \times \mathcal{PW}$ be a bounded set. Observe the following estimates:

- if $0 \leqslant \eta_1 \leqslant \eta_2 \leqslant t_1$ then

$$\begin{aligned}\|\mathcal{S}_1(z,u)(\eta_2) - \mathcal{S}_1(z,u)(\eta_1)\| \leqslant{}& \|\mathcal{U}(\eta_2,0) - \mathcal{U}(\eta_1,0)\| \times \\ & \left\{\|\varphi(0)\| + \left\|g(z_{\tau_1}, z_{\tau_2}\dots, z_{\tau_q})(0)\right\|\right\} \\ & + \int\limits_0^{\eta_1} \|\mathcal{U}(\eta_2,s) - \mathcal{U}(\eta_1,s)\| \times \\ & \Big[\|\mathsf{B}(s)\|\,\|\Upsilon_0\mathfrak{L}_0(z,u)(s)\| + \\ & \|f(s,z_s,u(s))\|\Big]\mathrm{d}s \\ & + \int\limits_{\eta_1}^{\eta_2} \|\mathcal{U}(\eta_2,s)\| \times \\ & \Big[\|\mathsf{B}(s)\|\,\|\Upsilon_0\mathfrak{L}_0(z,u)(s)\| + \\ & \|f(s,z_s,u(s))\|\Big]\mathrm{d}s,\end{aligned}$$

- if $t_i < \eta_1 \leqslant \eta_2 \leqslant s_i$

$$\begin{aligned}\|\mathcal{S}_1(z,u)(\eta_2) - \mathcal{S}_1(z,u)(\eta_1)\| &= \left\|\mathsf{G}_i(\eta_2, z(t_i^-)) - \mathsf{G}_i(\eta_1, z(t_i^-))\right\| \\ &\leqslant d_i|\eta_2 - \eta_1|,\end{aligned}$$

- if $s_i < \eta_1 \leqslant \eta_2 \leqslant t_{i+1}$

$$\begin{aligned}\|\mathcal{S}_1(z,u)(\eta_2) - \mathcal{S}_1(z,u)(\eta_1)\| \leqslant{}& \|\mathcal{U}(\eta_2,s_i) - \mathcal{U}(\eta_1,s_i)\|\,\left\|\mathsf{G}_i(s_i, z(t_i^-))\right\| \\ & + \int\limits_{s_i}^{\eta_1} \|\mathcal{U}(\eta_2,s) - \mathcal{U}(\eta_1,s)\| \times\end{aligned}$$

$$
\begin{aligned}
&\Big[\|\mathsf{B}(s)\| \, \|\Upsilon_i \mathfrak{L}_i(z,u)(s)\| + \\
&\|f(s, z_s, u(s))\| \Big] \mathrm{d}s \\
&+ \int_{\eta_1}^{\eta_2} \|\mathcal{U}(\eta_2, s)\| \times \\
&\Big[\|\mathsf{B}(s)\| \, \|\Upsilon_i \mathfrak{L}_i(z,u)(s)\| + \\
&\|f(s, z_s, u(s))\| \Big] \mathrm{d}s,
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathcal{S}_2(z,u)(\eta_2) - \mathcal{S}_2(z,u)(\eta_1)\| \leqslant \|\mathsf{B}^*(\eta_2)\mathcal{U}^*(t_{i+1},\eta_2) - \mathsf{B}^*(\eta_1)\mathcal{U}^*(t_{i+1},\eta_1)\| \\
\times \left\| (\mathcal{W}_{[s_i,t_{i+1}]})^{-1} \mathfrak{L}_i(z,u) \right\|.
\end{aligned}
$$

Since $\mathcal{U}(t,s)$ is continuous and together with the above estimates then $\mathcal{S}_1(D)$ and $\mathcal{S}_2(D)$ are equicontinuous. Moreover, the equicontinuity of $\mathcal{S}$ is attained since

$$
\begin{aligned}
\|\mathcal{S}(z,u)(\eta_2) - \mathcal{S}(z,u)(\eta_1)\|_1 = \|\mathcal{S}_1(z,u)(\eta_2) - \mathcal{S}_1(z,u)(\eta_1)\| \\
+ \|\mathcal{S}_2(z,u)(\eta_2) - \mathcal{S}_2(z,u)(\eta_1)\|.
\end{aligned}
$$

iii. Let $D \subset \widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} \times \mathcal{PW}$ be a closed and bounded subset. Since f, $\mathfrak{L}_i$, and G_i are continuous, for all $(z,u) \in D$ and $i = 0,1,\ldots,N$ there are $T_1, T_2, \ldots, T_{2N+3} \in \mathbb{R}^+$ such that

$$
\|f(\cdot,z,u)\|_0 \leqslant T_1, \quad \|(\mathcal{W}_{[s_i,t_{i+1}]})^{-1}\mathfrak{L}_i(z,u)\| \leqslant T_{i+2}, \quad \|\mathsf{G}_i(\cdot,z)\|_{\mathbb{R}^n} \leqslant T_{N+i+3}.
$$

Since D is bounded there exists $T_D \in \mathbb{R}^+$, such that,

$$
\|(z,u)\| \leqslant T_D, \quad \forall (z,u) \in D.
$$

Note that:

$$
\mathcal{S}(D) = \left\{ (y,v) \in \widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} \times \mathcal{PW} \;\middle|\; \exists (z,u) \in D \text{ and } (y,v) = \mathcal{S}(z,u) \right\}
$$

Then, if we define

$$
\begin{aligned}
D_1 &= \left\{ y \in \widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} \;\middle|\; \exists (z,u) \in D \text{ such that } y = \mathcal{S}_1(z,u) \right\} \\
D_2 &= \left\{ v \in \mathcal{PW} \;\middle|\; \exists (z,u) \in D \text{ such that } v = \mathcal{S}_2(z,u) \right\}
\end{aligned}
$$

hence, $S(D) = D_1 \times D_2$.

Consider an arbitrary countable family of elements $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathcal{S}(D)$, where ϕ_j is given by $\phi_j = (y_j, v_j) \in D_1 \times D_2$ for each $j \in \mathbb{N}$. Next, consider the sequence $\{y_j\}_{j\in\mathbb{N}}$ in D_1, thus for each $j \in \mathbb{N}$, there exists $(z,u) \in D$ such that

$$y_i(t) = \begin{cases} \varphi(t) - g(z_{\tau_1}, z_{\tau_2} \ldots, z_{\tau_q})(t), & t \in [-r, 0] \\ \mathcal{U}(t,0)\{\varphi(0) - g(z_{\tau_1}, z_{\tau_2} \ldots, z_{\tau_q})(0)\} & \\ + \int_0^t \mathcal{U}(t,s)[\mathsf{B}(s)(\Upsilon_0 \mathfrak{L}_0(z,u))(s) + f(s, z_s, u(s))]\,\mathrm{d}s, & t \in (0, t_1] \\ \mathsf{G}_i(t, z(t_i^-)), & t \in (t_i, s_i] \\ \mathcal{U}(t, s_i)\mathsf{G}_i(s_i, z(t_i^-)) + \int_{s_i}^t \mathcal{U}(t,s)\mathsf{B}(\Upsilon_i \mathfrak{L}_i(z,u))(s)\,\mathrm{d}s & \\ \quad + \int_{s_i}^t \mathcal{U}(t,s) f(s, z_s, u(s))\,\mathrm{d}s, & t \in (s_i, t_{i+1}]\,. \end{cases}$$

Observe that

- for $t \in [-r, 0]$,

$$\|y_i(t)\| = \left\|\varphi(t) - g(z_{\tau_1}, z_{\tau_2} \ldots, z_{\tau_q})(t)\right\| \leqslant \|\varphi\| + e\,\|z\|^\eta \leqslant A_1\,.$$

- For $t \in [0, t_1]$,

$$\begin{aligned} \|y_i(t)\| \leqslant{} & \left\|\mathcal{U}(t,0)\{\varphi(0) - g(z_{\tau_1}, z_{\tau_2} \ldots, z_{\tau_q})(0)\}\right\| \\ & + \left\|\int_0^t \mathcal{U}(t,s)[\mathsf{B}(s)\Upsilon_0\mathfrak{L}_0(z,u) + f(s, z_s, u(s))]\,\mathrm{d}s\right\| \\ \leqslant{} & M e^{wt}(\|\varphi\| + e\,\|z\|^\eta) \\ & + \int_0^t M e^{w(t-s)} \left(\|\mathcal{G}_{t_1}^*\| \left\| (\mathcal{G}_{t_1}\mathcal{G}_{t_1}^*)^{-1} \right\| \|\mathfrak{L}_0\|\,\|(z,u)\| \right) \mathrm{d}s \\ & + \int_0^t M e^{w(t-s)}\,\|f(\cdot, z, u)\|\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\leqslant Me^{wt}(\|\varphi\| + e\,\|z\|^{\eta}) \\
&\quad + Me^{wt}\left(\left\|\mathcal{G}^*_{t_1}\right\| \left\|\left(\mathcal{G}_{t_1}\mathcal{G}^*_{t_1}\right)^{-1}\right\| \|\mathfrak{L}_0\|\, T_D + \|f(\cdot, z, u)\|\right) \int_0^t e^{-ws}\,\mathrm{d}s \\
&\leqslant Me^{wt}(\|\varphi\| + e\,\|z\|^{\eta}) \\
&\quad + Me^{wt}\left(\left\|\mathcal{G}^*_{t_1}\right\| \left\|\left(\mathcal{W}_{[,[0,t_1]]}\right)^{-1}\right\| \|\mathfrak{L}_0\|\, T_D + \|f(\cdot, z, u)\|\right) \frac{1}{w} \\
&\leqslant M\left((\|\varphi\| + e\,\|z\|^{\eta}) + \left(\left\|\mathcal{G}^*_{t_1}\right\| T_2 T_D + T_1\right)\frac{1}{w}\right) \leqslant A_2.
\end{aligned}
$$

- If $t \in (t_i, s_i]$ and $i = 1, 2, 3, \ldots, N$,

$$
\|y_i(t)\| = \left\|\mathsf{G}_i(t, z(t_i^-))\right\| \leqslant T_{N+i+3} \leqslant \sum_{i=1}^{N} T_{N+i+3} \leqslant A_3\,.
$$

- Similarly, for $t \in [s_i, t_{i+1}]$ and $i = 1, 2, 3, \ldots, N$,

$$
\begin{aligned}
\|y_i(t)\| &\leqslant \left\|\mathcal{U}(t,0)\mathsf{G}_i(s_i, z(t_i^-))\right\| + \left\|\int_0^t \mathcal{U}(t,s)\mathsf{B}(s)\Upsilon_i\mathfrak{L}_i(z,u)\,\mathrm{d}s\right\| \\
&\quad + \left\|\int_0^t \mathcal{U}(t,s) f(s, z_s, u(s))\,\mathrm{d}s\right\| \\
&\leqslant Me^{wt}T_{N+i+3} + \int_0^t Me^{w(t-s)}\Big(\|f(\cdot, z, u)\|\Big)\mathrm{d}s \\
&\quad + \int_0^t Me^{w(t-s)}\left(\left\|\mathcal{G}^*_{t_{i+1}}\right\| \left\|\left(\mathcal{G}_{t_{i+1}}\mathcal{G}^*_{t_{i+1}}\right)^{-1}\right\| \|\mathfrak{L}_i\|\, \|(z,u)\|\right)\mathrm{d}s \\
&\leqslant Me^{wt}T_{N+i+3} + \left[Me^{wt}\left(\left\|\mathcal{G}^*_{t_{i+1}}\right\| \left\|\left(\mathcal{G}_{t_{i+1}}\mathcal{G}^*_{t_{i+1}}\right)^{-1}\right\| \|\mathfrak{L}_i\|\, T_D\right)\right. \\
&\quad \left. + Me^{wt}\left(\|f(\cdot, z, u)\|\right)\right] \int_0^t e^{-ws}\,\mathrm{d}s \\
&\leqslant Me^{wt}\left[T_{N+i+3} + \left\|\mathcal{G}^*_{t_{i+1}}\right\| \left\|\left(\mathcal{W}_{[s_i,t_{i+1}]}\right)^{-1}\right\| \|\mathfrak{L}_i\|\, T_D + \|f(\cdot, z, u)\|\right]\frac{1}{w}
\end{aligned}
$$

$$\leqslant M\left(T_{N+i+3} + \left(\left\|\mathcal{G}^*_{t_{i+1}}\right\| T_{i+2}T_D + T_1\right)\frac{1}{w}\right) \leqslant A_4\,.$$

Therefore,

$$\sup_{t\in[-r,\tau]} \|y_i(t)\| \leqslant \max\{A_1, A_2, A_3, A_4\} := A_5 < \infty$$

i.e.,

$$\|y_i(t)\| \leqslant A_5 \quad \text{for any } i \in \mathbb{N},\ t \in [-r,\tau],$$

hence, D_1 is uniformly bounded.

Now, take the subsequence $\left\{y_i|_{[-r,t_1]}\right\}_i \subset C([-r,t_1])$. Using the boundedness of D_1 together with *ii* restricted to the interval $[-r,t_1]$, Arzelá-Ascoli Theorem guarantees the existence of a subsequence $\{y_i^1\}_i = \{y_{i_k}\}_k \subseteq \{y_i\}_i$ convergent on $[-r,t_1]$. Under the same argument the new sequence restricted to the interval $[t_1,t_2]$, has a subsequence $\{y_i^2\}_i = \left\{y^1_{i_k}\right\}_k$ which is convergent on $[t_1,t_2]$, and thus on $[-r,t_2]$.Repeating the same reasoning on the intervals $[t_2,t_3],\cdots,[t_N,\tau]$, it yields that there exists a sequence $\{y_i^{N+1}\}_i = \left\{y^N_{i_k}\right\}_k \subseteq \{y_i\}_i$ which is convergent over $[-r,\tau]$. As a consequence, $\overline{D_1}$ is a metric sequentially compact, thus it is compact, and D_1 is relatively compact.

In a similar way, one can prove the existence of a convergent subsequence $\{v_{i_k}\}_k$ where $\{v_{i_k}\}_k \subseteq \{v_i\}_i$, and hence D_2 is relatively compact. Consequently, $\mathcal{S}(D)$ is, since $\overline{\mathcal{S}(D)} = \overline{D_1 \times D_2} = \overline{D_1} \times \overline{D_2}$ is compact.

iv. The following limit holds true

$$\lim_{\|(z,u)\|\to\infty} \frac{\|\mathcal{S}(z,u)\|}{\|(z,u)\|} = 0,$$

where $\|\cdot\|$ is the norm in the space $\widetilde{\mathcal{PC}}_{(t_1,\cdots,t_N)} \times \mathcal{PW}$. In fact, from the definition of $\mathfrak{L}_i$, the following estimates hold:
for $i = 0$

$$\|\mathfrak{L}_0(z,u)\| \leqslant N_0 + [D_0 + C_0]\{e\,\|z\|^\eta + a_0\,\|z\|^{\alpha_0} + b_0\,\|u\|^{\beta_0} + c_0\},$$

for $i = 1,2,3,\ldots,N$

$$\|\mathfrak{L}_i(z,u)\| \leqslant \|z_{t_{i+1}}\| + D_i\{a_0\,\|z\|^{\alpha_0} + b_0\,\|u\|^{\beta_0} + c_0\} + C_i\{a_i\,\|z\|^{\alpha_i} + c_i\},$$

where $N_0 = \|z^1\| + C_0\,\|\varphi(0)\|$, $C_i = Me^{\omega(t_{i+1}-s_i)}$, $D_i = \dfrac{M}{\omega}[e^{\omega(t_{i+1}-s_i)} - 1]$.

Moreover,

- for $t \in [0, t_1]$

$$\begin{aligned}\|\mathcal{S}_2(z,u)(t)\| \leqslant \|\mathsf{B}\|\, C_0 N_0 (\delta_0)^{-1} + \|\mathsf{B}\|\, C_0 [D_0 + C_0] (\delta_0)^{-1} \\ \times \{e\, \|z\|^{\eta} + a_0\, \|z\|^{\alpha_0} + b_0\, \|u\|^{\beta_0} + c_0\},\end{aligned}$$

- for $t \in (s_i, t_{i+1}]$,

$$\begin{aligned}\|\mathcal{S}_2(z,u)(t)\| \leqslant\ & \|\mathsf{B}\|\, C_i (\delta_i)^{-1} \|z_{t_{i+1}}\| \\ & + \ \|\mathsf{B}\|\, C_i D_i (\delta_i)^{-1} \{a_0\, \|z\|^{\alpha_0} + b_0\, \|u\|^{\beta_0} + c_0\} \\ & + \ \|\mathsf{B}\|\, C_i^2 (\delta_i)^{-1} \{a_i\, \|z\|^{\alpha_i} + c_i\}.\end{aligned}$$

Additionally,

- for $t \in [0, t_1]$

$$\begin{aligned}\|\mathcal{S}_1(z,u)(t)\| \leqslant\ & C_0\, \|\varphi(0)\| + \frac{M^2}{\omega}(e^{2\omega t_1} - 1)\, \|\mathsf{B}\|^2\, (\delta_0)^{-1} N_0 \\ & + \ [D_0 + C_0] E_0 \{e\, \|z\|^{\eta} + a_0\, \|z\|^{\alpha_0} + b_0\, \|u\|^{\beta_0} + c_0\},\end{aligned}$$

- for $t \in (s_i, t_{i+1}]$,

$$\begin{aligned}\|\mathcal{S}_1(z,u)(t)\| \leqslant\ & \frac{M^2}{\omega}(e^{2\omega(t_{i+1}-s_i)} - 1)\, \|\mathsf{B}\|^2\, (\delta_i)^{-1} \|z_{t_{i+1}}\|] \\ & + \ D_i E_i \{a_0\, \|z\|^{\alpha_0} + b_0\, \|u\|^{\beta_0} + c_0\} + C_i E_i \{a_i\, \|z\|^{\alpha_i} + c_i\}.\end{aligned}$$

- and for $t \in (t_i, s_i]$

$$\|\mathcal{S}_1(z,u)(t)\| \leqslant a_i\, \|z\|^{\alpha_i}_{\mathbb{R}^n} + c_i\,,$$

here $E_i = 1 + \dfrac{M^2\, \|\mathsf{B}\|^2}{\omega \delta_i} [e^{2\omega(t_{i+1}-s_i)} - 1]$, with $i = 0, 1, \cdots, N$.

As consequence,

- if $t \in [0, t_1]$

$$\begin{aligned}\|\mathcal{S}(z,u)(t)\| \leqslant\ & C_0 \|\varphi(0)\| + N_0 P_0 \\ & + \ (D_0 + C_0) L_0 \{e\, \|z\|^{\eta} + a_0\, \|z\|^{\alpha_0} + b_0\, \|u\|^{\beta_0} + c_0\},\end{aligned}$$

- if $t \in (s_i, t_{i+1}]$
$$\|\mathcal{S}(z,u)(t)\| \leqslant \|z_{t_{i+1}}\| P_i + D_i L_i \{a_0 \|z\|^{\alpha_0} + b_0 \|u\|^{\beta_0} + c_0\} \\ + \; C_i L_i \{a_i \|z\|^{\alpha_i} + c_i\},$$

- and if $t \in (t_i, s_i]$
$$\|\mathcal{S}(z,u)(t)\| \leqslant a_i \|z\|_{\mathbb{R}^n}^{\alpha_i} + c_i$$
with $P_i = \|\mathsf{B}\| (\delta_i)^{-1} \left\{ C_0 + \dfrac{M^2}{\omega}(e^{2\omega(t_{i+1}-s_i)} - 1) \|\mathsf{B}\| \right\}$ and $L_i = E_i + \|\mathsf{B}\| C_i (\delta_i)^{-1}$.

Therefore,

- for $t \in [0, t_1]$
$$\frac{\|\mathcal{S}(z,u)(t)\|}{\|(z,u)\|} \leqslant \frac{C_0 \|\varphi(0)\|}{\|z\| + \|u\|} + \frac{N_0 P_0}{\|z\| + \|u\|} + (D_0 + C_0) L_0 \\ \times \left\{ e \|z\|^{\eta - 1} + a_0 \|z\|^{\alpha_0 - 1} + b_0 \|u\|^{\beta_0 - 1} + \frac{c_0}{\|z\| + \|u\|} \right\}$$

- for $t \in (s_i, t_{i+1}]$
$$\frac{\|\mathcal{S}(z,u)(t)\|}{\|(z,u)\|} \leqslant \frac{\|z_{t_{i+1}}\| P_i}{\|z\| + \|u\|} + D_i L_i \left\{ a_0 \|z\|^{\alpha_0 - 1} + b_0 \|u\|^{\beta_0 - 1} + \frac{c_0}{\|z\| + \|u\|} \right\} \\ + \; C_i L_i \left\{ a_i \|z\|^{\alpha_i - 1} + \frac{c_i}{\|z\| + \|u\|} \right\},$$

- and for $t \in (t_i, s_i]$
$$\frac{\|\mathcal{S}(z,u)(t)\|}{\|(z,u)\|} \leqslant a_i \|z\|^{\alpha_i - 1} + \frac{c_i}{\|z\| + \|u\|}.$$

Thus, considering the hypothesis (a)–(b), with $0 < \alpha_i < 1$, $0 < \beta_0 < 1$, $i = 0, 1, 2, \cdots, N$, $0 < \eta < 1$, it follows that, for all $t \in [0, T]$

$$\lim_{\|(z,u)\| \to \infty} \frac{\|\mathcal{S}(z,u)\|}{\|(z,u)\|} = 0. \tag{15}$$

Hence, fixing $0 < \rho < 1$, there exists $r > 0$ big enough, such that the following inequality holds for all $\|(z,u)\| \geqslant r$.

$$\|\mathcal{S}(z,u)\| \leqslant \rho \|(z,u)\|.$$

In particularly, if take $\|(z,u)\| = r$, then $\|\mathcal{S}(z,u)\| \leqslant \rho r < r$.
Consequently, $\mathcal{S}(\partial B(0,r)) \subset B(0,r)$. □

Theorem 2 *If the assumptions (a)–(c) hold and the linear system* (6) *is controllable on any interval of the form* $[\alpha,\beta] \subseteq [0,T]$*, then the semilinear system with non-instantaneous impulses, delay, and nonlocal conditions* (3) *is controllable on* $[0,T]$*. Precisely, given* $\varphi \in \widetilde{\mathcal{PC}}_r$*,* $z^1 \in \mathbb{R}^n$ *and arbitrary points* $z^{t_{i+1}} \in \mathbb{R}^n$*,* $i = 0,1,2,\dots,N$ *there exists a control* $u \in \mathcal{PW}$ *such that the corresponding solution* $z(\cdot)$ *of* (3) *satisfies:*

$$z(0) + g(z_{\tau_1}, z_{\tau_2} \dots, z_{\tau_q})(0) = \varphi(0), \qquad z(t_{i+1}) = z^{t_{i+1}}, \quad i = 0,1,2,\dots,N$$

where $z(t_{N+1}) = z^{N+1} = z^1$*. Moreover, for all* $t \in (s_i, t_{i+1}]$ *and* $i = 0,1,2,\cdots,m$

$$u(t) = B^*(t)\mathcal{U}^*(t_{i+1},t)(\mathcal{W}_{[s_i,t_{i+1}]})^{-1}\mathfrak{L}_i(z,u),$$

with $\mathfrak{L}_i(z,u)$ *as in* (14).

Proof. Let $\varphi \in \widetilde{\mathcal{PC}}_r$, $z^1 \in \mathbb{R}^n$ and arbitrary points $z^{t_{i+1}} \in \mathbb{R}^n$, $\quad i = 0,1,2,\dots,N$. Then exists a control $u \in \mathcal{PW}$ given by Lemma 1 such that

$$u(t) = B^*(t)\mathcal{U}^*(t_{i+1},t)(\mathcal{W}_{[s_i,t_{i+1}]})^{-1}\mathfrak{L}_i(z,u),$$

for $t \in (s_i, t_{i+1}]$, $0 = 1,2,\cdots,N$. Replacing u into the solution (4), and evaluating it at $t = 0, t_1, t_{i+1}$ we obtain that:

$$\begin{aligned}
&z(0) + g(z_{\tau_1}, z_{\tau_2} \dots, z_{\tau_q})(0) = \varphi(0),\\
&z(t_1) = \mathcal{U}(t_1,0)[\varphi(0) - g(z_{\tau_1}, z_{\tau_2} \dots, z_{\tau_q})(0)] + \int_0^{t_1} \mathcal{U}(t_1,s) f(s, z_s, u(s))\,\mathrm{d}s\\
&\qquad + \int_0^{t_1} \mathcal{U}(t_1,s)\mathsf{B}(s)\mathsf{B}^*(s)\mathcal{U}^*(t_1,s)(\mathcal{W}_{[0,t_1]})^{-1}\Big\{z_{t_1} - \mathcal{U}(t_1,0)\times\\
&\qquad [\varphi(0) - g(z_{\tau_1}, z_{\tau_2} \dots, z_{\tau_q})(0)] - \int_0^{t_1} \mathcal{U}(t_1,v) f(v, z_v, u(v))\,\mathrm{d}v]\Big\}\,\mathrm{d}s\\
&\quad = \mathcal{U}(t_1,0)[\varphi(0) - g(z_{\tau_1}, z_{\tau_2} \dots, z_{\tau_q})(0)] + \int_0^{t_1} \mathcal{U}(t_1,s) f(s, z_s, u(s))\,\mathrm{d}s\\
&\qquad + (\mathcal{W}_{[0,t_1]})(\mathcal{W}_{[0,t_1]})^{-1}\{z_{t_1} - \mathcal{U}(t_1,0)[\varphi(0) - g(z_{\tau_1}, z_{\tau_2} \dots, z_{\tau_q})(0)]\\
&\qquad - \int_0^{t_1} \mathcal{U}(t_1,v) f(v, z_v, u(v))\,\mathrm{d}v\} := z^{t_1}
\end{aligned}$$

$$
\begin{aligned}
z(t_{i+1}) = {} & \mathcal{U}(t_{i+1}, s_i)\mathsf{G}_i(s_i, z(t_i^-)) + \int_{s_i}^{t_{i+1}} \mathcal{U}(t_{i+1}, s) f(s, z_s, u(s))\,\mathrm{d}s \\
& + \int_{s_i}^{t_{i+1}} \mathcal{U}(t_{i+1}, s)\mathsf{B}(s)\mathsf{B}^*(s)\mathcal{U}^*(t_{i+1}, s)(\mathcal{W}_{[s_i,t_{i+1}]})^{-1}\Big\{z_{t_{i+1}} - \\
& \mathcal{U}(t_{i+1}, s_i)\mathsf{G}_i(s_i, z(t_i^-)) \\
& - \int_{s_i}^{t_{i+1}} \mathcal{U}(t_1, v) f(v, z_v, u(v))\,\mathrm{d}v\Big\}\mathrm{d}s \\
= {} & \mathcal{U}(t_{i+1}, s_i)\mathsf{G}_i(s_i, z(t_i^-)) + \int_{s_i}^{t_{i+1}} \mathcal{U}(t_{i+1}, s) f(s, z_s, u(s))\,\mathrm{d}s \\
& + (\mathcal{W}_{[s_i,t_{i+1}]})(\mathcal{W}_{[s_i,t_{i+1}]})^{-1}\{z^{t_{i+1}} - \mathcal{U}(t_{i+1}, s_i)\mathsf{G}_i(s_i, z(t_i^-)) \\
& - \int_{s_i}^{t_{i+1}} \mathcal{U}(t_{i+1}, v) f(v, z_v, u(v))\,\mathrm{d}v\} := z^{t_{i+1}}.
\end{aligned}
$$

Observe that, if $i = N$, then $z(t_{N+1}) = z^{t_{N+1}} = z^1$, and since $t_{N+1} = T$, we get that $z(T) = z^1$. This complete the proof. □

4. Concluding remark

In this work, we proved that under some conditions a semilinear non-autonomous control system with non-instantaneous impulses, nonlocal conditions, and delay is exactly controllable, which was achieved using the uniform continuity of the evolution operator and Rothe's fixed point theorem. In fact, the uniform continuity of the evolution operator helped us to prove the equicontinuity and the uniform boundedness of a family of functions in the space of Cartesian product of the solutions space and the controls space. In infinite-dimensional Banach spaces to achieve the uniform continuity away from zero of the evolution operator, it must be assumed that it is compact, which implies that the linear control system governed by the evolution equation cannot be exactly controllable anymore, only approximately controllable. But, the approximate controllability of the semilinear system can be achieved also by applying Rothe's fixed point theorem to a family of operators and then using Canto's diagonalization process we can find a sequence of controls steering the system from an initial state to a neighborhood of the final state.

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