Eigenvalues assignment in uncontrollable linear systems

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Abstract. It is shown that in an uncontrollable linear system \( \dot{x} = Ax + Bu \) it is possible to assign arbitrarily the eigenvalues of the closed-loop system with state feedbacks \( u = Kx, K \in \mathbb{R}^{n \times n} \) if \( \text{rank} [A \ B] = n \). The design procedure consists of two steps. In step 1, a nonsingular matrix \( M \in \mathbb{R}^{n \times n} \) is chosen so that the pair \((MA, MB)\) is controllable. In step 2, the feedback matrix \( K \) is chosen so that the closed-loop matrix \( A_c = A - BK \) has the desired eigenvalues. The procedure is illustrated by a simple example.

Key words: controllability; eigenvalues; assignment; linear system; feedback; procedure component.

1. INTRODUCTION

The concepts of controllability and observability introduced by Kalman [1,2] have been the basic notions of the modern control theory. It is well-known that if the linear system is controllable then, by the use of state feedback, it is possible to modify the dynamical properties of the closed-loop systems [1–12]. If the linear system is observable, then it is possible to design an observer which reconstructs the state vector of the system [1–12]. Descriptor systems of integer and fractional order have been analyzed in [6,11]. The stabilization of positive descriptor fractional linear systems with two different fractional orders by decentralized controller has been investigated in [11].

In this paper, it will be shown that it is possible to assign arbitrarily the eigenvalues of the closed-loop system with state feedback if \( \text{rank} [A \ B] = n \). In Section 2 it will be shown that if \( \text{rank} [A \ B] = n \), then there exists a nonsingular matrix \( M \in \mathbb{R}^{n \times n} \) such that the pair \((MA, MB)\) is controllable. Two procedures for the computation of the matrix \( M \in \mathbb{R}^{n \times n} \) will be proposed and illustrated by simple numerical examples in Section 3. Concluding remarks will be given in Section 4.

The following notation will be used: \( \mathbb{R} \) – the set of real numbers, \( \mathbb{R}^{n \times m} \) – the set of \( n \times m \) real matrices, \( I_n \) – the \( n \times n \) identity matrix.

2. CONTROLLABILITY OF LINEAR SYSTEMS

Consider the linear continuous-time system

\[
\begin{align*}
\dot{x} &= Ax + Bu, \\
y &= Cx,
\end{align*}
\]

where \( x = x(t) \in \mathbb{R}^n, u = u(t) \in \mathbb{R}^m, y = y(t) \in \mathbb{R}^p \) are the state, input, and output vectors and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n} \).

Definition 1. [4, 5, 7, 9, 10, 13] The system (1) (the pair \((A, B)\)) is called controllable if there exists an input \( u(t) \in \mathbb{R}^m, t \in [0, t_f] \) which steers the state of the system from the initial state \( x(0) \in \mathbb{R}^n \) to the given final state \( x(t_f) \).

Theorem 1. The system (1a) (the pair \((A, B)\)) is controllable if and only if one of the following conditions is satisfied:

1. (Kalman condition)

\[
\text{rank} \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} = n,
\]

2. (Hautus condition)

\[
\text{rank} \begin{bmatrix} I_n & A & \ldots & A^{n-1} \end{bmatrix} = n,
\]

for \( s \in \mathbb{C} \) (the field of complex numbers). In the proof of the main result of this section, the following theorem will be used.

Theorem 2. (Kronecker–Capelly theorem, [13]) The equation

\[
P x = Q, \quad P \in \mathbb{R}^{n \times p}, \quad Q \in \mathbb{R}^{p \times q}, \quad n, p, q \geq 1
\]

has a solution \( x \in \mathbb{R}^{p \times q} \) if and only if

\[
\text{rank} \begin{bmatrix} P & Q \end{bmatrix} = \text{rank}[P],
\]

Theorem 3. If the pair \((A, B)\) is uncontrollable but satisfies the condition

\[
\text{rank} \begin{bmatrix} A & B \end{bmatrix} = n, \quad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m},
\]

then there exists a nonsingular matrix \( M \in \mathbb{R}^{n \times n} \) such that the pair \((\overline{A}, \overline{B})\),

\[
\overline{A} = MA, \quad \overline{B} = MB
\]

is controllable.

Proof. From (6) we have

\[
M \begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} \overline{A} & \overline{B} \end{bmatrix},
\]

where the pair \((\overline{A}, \overline{B})\) is controllable.

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From Theorem 2 applied to equation (7), it follows that there exists a nonsingular matrix $M$ satisfying (7) if the condition (5) holds.

To compute the desired matrix $M$ the following procedures can be recommended.

**Procedure 1.** By choosing the controllable pair $(\mathbf{A}, \mathbf{B})$ and post-multiplying equation (7) by the transposed matrix $[\mathbf{A} \mathbf{B}]^T$, we obtain

$$M[\mathbf{A}^T \mathbf{B}^T] = \mathbf{A} \mathbf{A}^T + \mathbf{B} \mathbf{B}^T.$$  

(8)

The matrix

$$[\mathbf{A} \mathbf{B}] \begin{bmatrix} \mathbf{A}^T \\ \mathbf{B}^T \end{bmatrix} = \mathbf{A} \mathbf{A}^T + \mathbf{B} \mathbf{B}^T$$  

(9)

is nonsingular since $\text{rank}[\mathbf{A} \mathbf{B}] = n$. From (8) we have the desired matrix

$$M = (\mathbf{A} \mathbf{A}^T + \mathbf{B} \mathbf{B}^T)(\mathbf{A} \mathbf{A}^T + \mathbf{B} \mathbf{B}^T)^{-1}.  

(10)

To find the desired matrix $M$ the following procedure can be also applied.

**Procedure 2.** Choose a nonsingular matrix $M \in \mathbb{R}^{n \times n}$ and compute the matrix $[\mathbf{A} \mathbf{B}]$. Check if the pair $[\mathbf{A} \mathbf{B}]^T$ is controllable if it is not the case then repeat the procedure for a new matrix $M$.

**Example 1.** Consider the uncontrollable system (1) with the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.  

(11)

From (11) it follows that it is impossible to stabilize the system by state feedback $u = \mathbf{K}x$, $\mathbf{K} = [k_1 \ k_2]$. The system with (11) satisfies condition (5) since

$$\text{rank}[\mathbf{A} \mathbf{B}] = \text{rank} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} = 2 = n.  

(12)

According to Procedure 1, we choose the controllable pair in the form

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.  

(13)

Using (10) and (13) we obtain

$$M = (\mathbf{A} \mathbf{A}^T + \mathbf{B} \mathbf{B}^T)(\mathbf{A} \mathbf{A}^T + \mathbf{B} \mathbf{B}^T)^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.  

(14)

According to Procedure 2, we choose the matrix $M$ for example in the form

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.  

(15)

In this case

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

(16)

The pair (16) is controllable since

$$\text{rank}[\mathbf{B} \mathbf{AB}] = \text{rank} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 2 = n.  

(17)

Note that if we choose

$$M \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix},  

(18)

then

$$\mathbf{A} = \mathbf{A}^2 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{B} = \mathbf{MB} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.  

(19)

and the pair $(\mathbf{A}, \mathbf{B})$ is uncontrollable. The above considerations for the pair $(\mathbf{A}, \mathbf{B})$ can be extended to the pair $(\mathbf{A}, \mathbf{C})$ of the system (1).

**Definition 2.** [4, 5, 7, 9, 10, 13] The linear system (1) is called observable if knowing its input $u(t) \in \mathbb{R}^m$ and its output $y(t) \in \mathbb{R}^p$ for $t \in [0 \ t_f]$ it is possible find its unique initial condition $x(0) \in \mathbb{R}^n$.

**Theorem 4.** If the pair $(\mathbf{A}, \mathbf{C})$ is unobservable but satisfies the condition

$$\text{rank} \begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = n, \quad \mathbf{A} \in \mathbb{R}^{n \times n}, \quad \mathbf{C} \in \mathbb{R}^{p \times n},  

(20)

then there exists a nonsingular matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$ such that the pair

$$(\mathbf{A}, \mathbf{C}), \quad \mathbf{A} = \mathbf{AM}, \quad \mathbf{C} = \mathbf{CM}$$

(21)

is observable.

**The proof** is similar (dual) to the proof of Theorem 3.

### 3. Stabilization of the Uncontrollable Linear Systems by State Feedbacks

Consider the linear system (1) with an uncontrollable pair $(\mathbf{A}, \mathbf{B})$. We are looking for the state feedback matrix $\mathbf{K}$ such that the closed-loop matrix

$$(\hat{\mathbf{A}}_c) = \mathbf{A} - \mathbf{B} \mathbf{K}$$

(22)

has the desired eigenvalues (Fig. 1).
First, we choose the matrix $M$ such that the pair
\[ \bar{A} = MA, \quad \bar{B} = MB \] (23)
is controllable and next using one of the well-known approaches [3, 5, 6, 9, 10, 12] of the eigenvalues assignment we choose the matrix $K$ such that the matrix $\hat{A}_c$ has the desired eigenvalues.

To solve the problem the following procedure can be applied.

**Procedure 3.**

**Step 1.** Using the approach of Section 2 find the matrix $M$ such that the pair (23) is controllable.

**Step 2.** Using one of the well-known approaches of the eigenvalues assignment find the matrix $K$ such that the closed-loop matrix (22) has the desired eigenvalues.

**Example 2.** (Continuance of Example 1) For the uncontrollable system with the matrices (11) find the state feedback matrix $K = [k_1 \ k_2]$ such that the close-loop matrix (22) has the eigenvalues $s_1 = -2, s_2 = -3$.

Using the procedure we obtain the following:

**Step 1.** For the matrix $M$ of the form (15), we have obtained the controllable pair
\[ \bar{A} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \] (24)

**Step 2.** In this case, the close-loop matrix has the form
\[ \hat{A}_c = \bar{A} - \bar{B}K = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} [k_1 \ k_2] = \begin{bmatrix} 2 - k_1 & -1 - k_2 \\ 3 - k_1 & -1 - k_2 \end{bmatrix} \] (25)

and
\[
\det \left[ I_2 s - \hat{A}_c \right] = \begin{vmatrix} s - 2 + k_1 & 1 + k_2 \\ -3 + k_1 & s + 1 + k_2 \end{vmatrix} = s^2 + (k_1 + k_2 - 1)s + k_2 + 1 = (s - s_1)(s - s_2) = s^2 - (s_1 + s_2)s + s_1s_2. \] (26)

From (26) for the desired eigenvalues $s_1 = -2, s_2 = -3$ we have $k_1 = 1, k_2 = 5$ and the desired state feedback matrix has the form $K = [k_1 \ k_2] = [1 \ 5]$.

**4. CONCLUDING REMARKS**

It has been shown that it is possible to assign arbitrarily the eigenvalues of the closed-loop system with state feedback if condition (5) is satisfied. It is also shown that if condition (5) is satisfied, then there exists a nonsingular matrix $M \in \mathbb{R}^{n \times n}$ such that the pair $(MA, MB)$ is controllable (Theorem 3). Two procedures for the computation of the matrix $M \in \mathbb{R}^{n \times n}$ have been proposed and illustrated by a simple numerical example. This approach can be extended to linear discrete-time linear systems and fractional orders linear systems.

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**REFERENCES**