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Optimal Demand-Driven Eco-Mechanisms Leading to Equilibrium in Competitive Economy

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Abstract

We examine new mechanisms that introduce environmentally friendly eco-changes involving the elimination of noxious commodities and take into account the structure of demand without a detrimental effect to agents' position. In the era of the fourth industrial revolution, these mechanisms allow eliminating unnecessary services or goods that are being replaced by modern technologies. We define optimal mechanisms under the criterion of distance minimization, when a small number of detrimental commodities is excluded from production processes as well as when producers are change-averse. The results have the form of theorems with rigorous proofs.

Keywords: mechanisms design, competitive economy, equilibrium, eco-change, optimal mechanism

JEL Classification: D5, L1, O1

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1 Introduction

In the current paper, a kind of new optimal mechanisms which result in introducing and adaptation of eco-changes is designed. Eco-change is an environmentally friendly change (see for instance: Arundel and Kemp 2009; Carrillo-Hermosilla et al. 2010). The elimination of noxious commodities or detrimental technologies from the production processes can serve us an example, and this kind of eco-changes is under our study. Eco-mechanism is understood as a mechanism which results in outcomes beneficial for the environment.

The new mechanisms presented in our study are considered in the conceptions of L. Hurwicz. An economic mechanism, by Hurwicz (1987), is a mathematical structure due to which institutions and economic activities can be formalized. Hurwicz mechanism consists of three components: a message space, a message correspondence and an outcome function (Hurwicz and Reiter 2006). The message space is the set of all information sent, consciously or unconsciously, by economic agents. The message correspondence links economic agents, characterized by so called economic environments, with the set of messages sent by the agents; the outcome function to every message assigns the outcome of activities of economic agents undertaken as a result of analysis of the message. The reasons for the implementation of a mechanism, the incentives for economic agents to take part in exactly that mechanism, the cooperation of economic agents with a partial or full access to information, the way of sending messages, innovativeness, the possibility of improvement of the agents' position and the economy as a whole (qualitative properties) described formally are the basis for designing the mechanism.

We consider a situation when at least one harmful commodity is to be eliminated from the market, under the proviso that the agents' economic positions do not deteriorate more than when nothing is done (compare to Lipieta and Malawski 2021). By the above, the mechanisms under study are not likely to bring an increase in profits. That is why it is assumed that producers are change-averse. However, they are forced to make the required changes by the structure of the demand. Our model describes the current economic situation of the European Union countries whose energy sources and transport are based on coal and petroleum. Any attempts to "transition" the economy to more ecological energy sources met with a fierce opposition from the public, mainly miners, expressed, for example, through the strike in Great Britain in 1984 during the government of Prime Minister Margaret Thatcher, who, as the prime minister but also as a chemist, began environmental economic transformation in GB. The reduction of mined coal and extracted oil is still opposed by the representatives of other industries. Only EU Directive, Directive (EU) 2018/410 of the European Parliament and of the Council of 14 March 2018 amending Directive 2003/87/EC to enhance cost-effective emission reductions and low-carbon investments, and Decision (EU) 2015/1814, and earlier Paris Climate Agreement in 2015, brought about the first universal, legally binding agreement of 192 countries in the field of climate.

In December 2019, the European Commission, in the communication "The European Green Deal", proposed further goals to reduce greenhouse gas emissions by 2030, according to two scenarios – emission reduction by 50% and by 55% compared to the base year (1990). The most far-reaching reduction target for 2030, according to the long-term goals presented in the Conclusions of the Council of the European Union of December 2019, could lead to the achievement of net zero greenhouse gas emissions in the European Union in 2050. International activities to protect the environment made the governments of many countries announce that they would attempt to close mines and restructure the mining industry.

Taking the above under consideration, the criterion for the choice of an optimal mechanism is change minimization. Change minimization in fact means minimization of the distance in the considered norm between the initial and modified objects and processes, adequately. The mechanisms, which carry improvement of the state of the environment but which do not make agents' economic positions worse, are worth studying, especially nowadays, when reduction and prevention of the effect of the changes in the environment induced by human activities, are a necessity. The novelty of our research is that some kinds of optimal mechanisms, under the criterion of distance minimization with respect to two norms, in case when a small number of commodities is excluded from the production processes, are designed in the paper.

The new research methods used in the paper rely on the analysis of mappings with the minimal operator norm (see for instance Cheney 1966; Lipieta 1999; Denkowska 2013). The approximation theory supplies the tools for indicating (computing) the best economic objects, under the criterion of distance minimization with respect to a given norm in the considered space (see Cheney 1966; Lewicki and Odyniec 1990).

A typical best approximation problem consists in finding for a given point x in a metric space W the closest element $v \in V$ in a given set $V \subset W$. This can be restated in terms of a multivalued projection from W to V. In the case of a normed space W and its linear subspace V, a further restatement of the problem involves looking for a linear projection $W \to V$ with a minimal norm. Any continuous linear projection $Q : W \to V$ has norm $\|Q\| \geq 1$, provided $V \neq \{0\}$. If W is unitary, or better a Hilbert space, there is always a minimal projection i.e. having norm 1 – it is the orthogonal projection. In an infinite dimensional Banach space, there may be no norm 1 projections onto V; subspaces that have such projections are called one-complemented. Even in a finite dimension (from $n = 3$), not every subspace is one-complemented as we know from the Kakutani Theorem (Kakutani 1939). It says that every two-dimensional subspace of Banach space X is one-complemented if and only if X is a Hilbert space. Thus already in $\mathbb{R}^3$ with a norm not satisfying the parallelogram condition, there is a plane that is not one-complemented. We may however refine the problem: given a subspace $V \subset \mathbb{R}^l$ we consider not all the linear complements $\widetilde{V}$ such that $V \oplus \widetilde{V} = \mathbb{R}^l$, but those complements that satisfy some additional requirements such as e.g. (as in the present article) being orthogonal to a given vector $p \in \mathbb{R}^l$. Then we look for the projection with minimal possible norm.

The question makes sense for the projections on both subspaces: V and the (variable) complement $\widetilde{V}$.
This survey relies on examining the relationships between quantities of goods and quantities of the productive factors used to produce them, as well as the analysis of conditions for the existence of states of equilibrium in a transformation of the economy under study. In our approach, producers and consumers spend their time on observing local organization environments, on transmitting messages, on computing, storing and retrieving information (see Arrow and Intriligator 1987). The mentioned analysis could help to determine, among others, the producers' and consumers' optimal plans as well as the best agents' activities with respect to the given criterion. In the problem under study, it is essential to answer the following questions: who guides a given transformation? how to encourage the economic agents to take part in it? how does the designer force the agents to choose the desired mechanisms? do there exist optimal mechanisms and what do they look like?
In the model presented, agents can acquire information by observation of other agents' market activities. It is assumed that producers and consumers have full access to knowledge about the market which reflects the fact that as a result of the Industry 4.0 nowadays information is transformed almost immediately. Analysed innovations are technological and endogenous, while agents are heterogeneous.
The results have the form of theorems with rigorous proofs. More specifically, in Theorem 1 we prove the existence of a mechanism that results in equilibrium in such a transformation of initial competitive economy, which is determined by a specific linear mapping. To do this, in the proof of that theorem, a set of mechanisms, determined by some kind of linear mappings, is constructed. In Theorems 3 and 4 as well as in Lemmas 5 and 6 we present the mechanisms, which are optimal due to the criterion of changes minimization, among the mechanisms determined in the proof of Theorem 1, depending on initial conditions.
The paper consists of seven parts. The second part contains the literature review. In the third part, a model of the economy as well as the idea of the Hurwicz mechanism are presented. The fourth part is devoted to modelling eco-mechanisms resulting in equilibrium within the economic evolution, while the fifth part deals with specification of optimal mechanisms with respect to a given criterion in the set of mechanisms presented. The sixth part of the paper is devoted to discussion while the seventh part presents conclusions.

2 Literature Review

Increasing pollution combined with the increased awareness of the developed countries societies of the importance and harmfulness to our existence of waste and environment pollution, led in the 1960s to the development of a new field in economic theory called environmental economics.
The beginning of environmental economics in the 1970s was within the neo-classical

paradigm (https://www.soas.ac.uk/cedep-demos/000_P570_IEEP_K3736-Demo/unit1/page_12.htm). Sustainable economic growth needs the use of natural resources, hence the economic policy should also focus on the activities providing a hedge against depletion of natural resources and protection of the environment as a whole. For the environmental economists, the environment is a form of natural capital which provides, above all, life support and other functions that cannot be supplied by the economy (Barbier 2019). Then the most popular subjects of that environmental approach in economics were the studies on market failures, inappropriate resource allocation and the management of public goods (see also van der Straaten and Gordon 1995). As we can see, relationships between the economy and the environment or the influence of the economic processes on the environments were not in the core of interest of that approach at that time. The lack of such surveys led to developing ecological economics (see also Polasky et al. 2019).

The ecological economics was established in the late 1980s, when a new economic thinking on the social and economic systems and their roles in the biophysical world started to develop dynamically (Rigo et al. 2020). The roots of ecological economics can be found in biophysical understanding of economics (Odum 1971; Daly 1977; Jansson 1984; Martinez-Alier 1987). One of the most important subjects of ecological approach in economics is the analysis of efficient allocation, which contributes to human well-being and sustainability (see https://insights.som.yale.edu/insights/what-is-ecological-economics). Hence, ecological economics aims, among others, at analyzing problems related to governing economic activity to satisfy human well-being and sustainable development (Dorninger et al. 2021; Bloomfield 2021; Macdonald 2020).

Joseph Schumpeter (1942) defined a mechanism, understood as a set of rules and regularities, clarifying the structure of the processes of economic development, which he called creative destruction. According to Schumpeter, creative destruction is the synthesis of two opposite processes: introducing innovations and processes of elimination of existing, outdated solutions: commodities, firms, technologies etc. (see also Schumpeter 1912; Hanusch and Pyka 2007; Bolton 1993).

The studies on economic mechanisms within the framework used in the current paper have their origin in Leonid Hurwicz's paper (1960). Designing of mechanisms resulting in requisite outcomes, which is a natural extension of the analysis on the behaviour of the economic agents, remains in the core of interest of the economic theory and is applied in many areas of life (see Abdulkadiroğlu and Sönmez 2003; Bessen, Maskin 2009; Maskin et al. 2000; Myerson 1983; Pycia and Ünver 2017; Lipieta and Malawski 2016, 2021).

The exploration of the rules that govern economic life with a special focus on the role of innovativeness within the evolution of the economy, is the subject of investigation of the theory of economic development as well as of the theory of economic growth (see for instance Schumpeter 1912, 1934; Acemoglu 2009; Aghion and Howitt 1992, 1998; Ciałowicz and Malawski 2011, 2016; Nelson 2016; Nelson and Winter 1982; Segerstrom 1990; Shenkar 2010).

It should be noted here that, generally, the analysis of the mechanism resulting in equilibrium is not the object of study of either growth models, or the theory of economic development (Foster 2011). The theory of economic growth focuses on modelling of innovative processes and analysis of their quantitative properties while the theory of economic development deals with their qualitative features. In the theory of economic development and in economic growth theory, innovative processes are perceived as the driving force of the structural changes and disequilibrium. However, according to Schumpeter, the founder of the theory of economic evolution, there is no analysis of economic dynamics without an analysis of statics (Schumpeter 1912) since, for him, economy in equilibrium is both a starting and an ending point of innovative processes (Andersen 2009; Shionoya 2007). Additionally, in an economy in equilibrium, economic agents can realise their optimal plans of action, and there are no surpluses of commodities on the market. Hence economic agents do not have a motivation to change their plans of action. That is why the economy can last in equilibrium for any spell of time. In cases when a global or a regional economy is in a crisis or a crisis is starting, decision makers could consider implementing a mechanism leading to equilibrium to provide, in given conditions, maximal satisfaction for the economic entities. Among others, for the above reason it is worth studying economic mechanisms resulting in equilibrium in various economic models and under various initial conditions.

The methods and mathematical tools used in the current paper have rarely been used so far in the context of modelling of eco-changes. Some examples can be found in papers (Lipieta 2015b; 2015c; 2016). In those articles, the problems connected with specifying a mapping which could determine an optimal adjustment process, i.e. also an optimal mechanism, were analyzed. More specifically, in the paper (Lipieta 2015b), a criterion of the choice of an optimal adjustment process in a competitive economy was presented as well as an optimal adjustment process, which under the initial conditions considered, transforms a competitive economy in equilibrium into its transformation being also in equilibrium, and results in eliminating a harmful commodity or an obsolete and detrimental technology from the market. In the papers (Lipieta 2015c, 2016), the same problem was studied, however for other initial conditions. In contrast to the above papers, in the current work, we work on mechanisms which transform a competitive economy in which there is no equilibrium, into its equilibrated form and which results in the same eco-changes as were considered in papers (Lipieta 2015b, 2015c, 2016).

3 Model

Time in the model is considered as a discrete variable. Two disjoint periods of time, in which agents' economic activities do not change, are under our consideration. Hence without loss of generality we assume that $t = 0$ and $t = 1$. Let

i) $A_t = \{a_1, a_2, \ldots, a_{m_t}\}$ be a finite set of consumers at time t,

ii) $B_t = \{b_1, b_2, \ldots, b_{n_t}\}$ be a finite set of producers at time t,

m_t, $n_t \in \{1, 2, \ldots\}$. As some agents can enter the market at time $t = 1$, it is assumed that

$$A_0 \subset A_1, \quad B_0 \subset B_1 \quad \text{and consequently} \quad m_1 \geq m_0 \quad \text{and} \quad n_1 \geq n_0.$$

Denote by $\ell_t \in \mathbb{N}_+ \stackrel{\text{def}}{=} \{1, 2, \ldots\}$ the number of the commodities, which are produced and consumed in the economy at time t or which were produced and consumed earlier. Hence $\ell_0 \leq \ell_1$. It is obvious that

$$\mathcal{R}^{\ell_0} \stackrel{\text{def}}{=} \mathbb{R}^{\ell_0} \times \{0\} \times \{0\} \times \ldots \times \{0\} \subset \mathbb{R}^{\ell_1}.$$

In this part of the paper, the linear space $\mathbb{R}^{\ell_1}$ is also denoted by $\mathcal{R}^{\ell_1}$. The space $\mathcal{R}^{\ell_t}$ is interpreted as the commodity-price space in the economy at time $t \in \{0, 1\}$. If $\ell_0 < \ell_1$, then at time $t = 0$ every coordinate $l \in \{\ell_0 + 1, \ldots, \ell_1\}$ is the number of a potential future good at time $t = 1$. The description of the commodity-price space $\mathcal{R}^{\ell_0}$ presented above simplifies the description of processes in which the set of commodities as well as the number of economic agents can be changed in time.
Below the characteristics of economic agents are presented. A production activity of producer b_j at time t ($b_j \in B_t$), feasible with respect to technologies, is described by a vector $y^{b_j}(t) \in \mathcal{R}^{\ell_t}$, which is called the producer's b_j production plan. All production plans of producer b_j at time t form the production set $Y^{b_j}(t) \subset \mathcal{R}^{\ell_t}$ under the assumption that, if $n_1 > n_0$, then

$$\forall j \in \{n_0 + 1, \ldots, n_1\} \; Y^{b_j}(0) \stackrel{\text{def}}{=} \{0\}$$

as well as, for $t \in \{0, 1\}$,

$$\exists y^{b_j}(t) \in Y^{b_J}(t) \; y^{b_j}_{\ell_t} \neq 0.$$

Every producer b_j, for $j \in \{n_0 + 1, \ldots, n_1\}$, is called inactive at time $t = 0$; similarly if, for some $j \in \{1, \ldots, n_0\}$, $Y^{b_j}(1) = \{0\}$, then producer b_j is inactive at time $t = 1$. In the same way, consumers' characteristics are defined. Let

i) $X^a(t) \subset \mathcal{R}^{\ell_t}$ be the consumption set of consumer $a \in A_t$, under the assumption that, if $m_1 > m_0$, then

$$\forall i \in \{m_0+1,\dots,m_1\}\ X^{a_i}(0) \stackrel{\text{def}}{=} \{0\}\,,$$

which means that, for $i \in \{m_0+1,\dots,m_1\}$, every consumer $a_i \in A_1$ is inactive; if, for some $i \in \{1,\dots,m_0\}$, $X^{a_i}(1) = \{0\}$, then consumer $a_i \in A_1$ is inactive at time $t = 1$,

ii) Ξ_t be the set of preference relations in space $\mathcal{R}^{\ell_t}$,

iii) $\preccurlyeq_t^a \subset X^a(t) \times X^a(t)$ be the preference relation of consumer $a \in A_t$ at period t,

iv) $\omega^a(t) \in X^a(t)$ be the initial endowment of consumer $a \in A_t$ at period t,

v) $\omega(t) = \sum_{a\in A_t} \omega^a(t) \in \mathcal{R}^{\ell_t}$ be the total endowment of the economy at period t,

vi) $\theta_t(a,b) \in [0,1]$, for every $a \in A_t$ and $b \in B_t$, be the share of consumer a in the profit of producer b at period t under the assumption that $\sum_{a\in A_t} \theta_t(a,b) = 1$, for every $b \in B_t$; function $\theta_t : A_t \times B_t \to [0,1]$ is called the share function.

Let $K_t \stackrel{\text{def}}{=} A_t \cup B_t$. For $k \in K_t$, the sequence

$$e^k(t) = \Big(Y^k(t), X^k(t), \omega^k(t), \widetilde{\varepsilon}_t(k), \widetilde{\theta}_t(k,\cdot)\Big)\,, \tag{1}$$

where:

i) $Y^k(t) = \{0\}$ for $k \notin B_t$,

ii) $X^k(t) = \{0\}$ for $k \notin A_t$,

iii) $\omega^k(t) = 0$ for $k \notin A_t$,

iv) $\widetilde{\varepsilon}_t(k) = \preccurlyeq_t^a$ for $k \in A_t$, $\widetilde{\varepsilon}_t(k) = \{\varnothing\}$ for $k \notin A_t$,

v) the mapping $\widetilde{\theta}_t : K_t \times K_t \to [0,1]$ is the extension of mapping θ_t onto set $K_t \times K_t$ in such a way that $\widetilde{\theta}_t(k,\cdot) \equiv 0$ for $k \notin A_t$, $\widetilde{\theta}_t(\cdot,k) \equiv 0$ for $k \notin B_t$, $\widetilde{\theta}_t(a,b) = \theta_t(a,b)$ for $a \in A_t$ and $b \in B_t$,

is called the economic environment of agent k at time t (called also: the environment of agent k at time t; compare to Arrow, Intriligator 1987; Lipieta, Malawski 2021). It is obvious that

$$e^k(t) \in E^k(t) \stackrel{\text{def}}{=} P(\mathcal{R}^{\ell_t}) \times P(\mathcal{R}^{\ell_t}) \times \mathcal{R}^{\ell_t} \times P(\mathcal{R}^{\ell_t} \times \mathcal{R}^{\ell_t}) \times \mathcal{F}\left(K_t,[0,1]\right)\,,$$

where $P\left(\mathcal{R}^{\ell_t}\right)$ is the set of all subsets of space $\mathcal{R}^{\ell_t}$, while $\mathcal{F}\left(K_t,[0,1]\right) \stackrel{\text{def}}{=} \{f|\ f : K_t \to [0,1]\}$. Set $E^k(t)$ is the set of all feasible economic environments of agent k at time t. Suppose that $K_t = \{k_1, \ldots, k_{\kappa_t}\}$, where $\kappa_t \leq m_t + n_t$. The set

$$E(t) \overset{\text{def}}{=} E^{k_1}(t) \times \ldots \times E^{k_{\kappa_t}}(t) \tag{2}$$

is called the set of economic environments at time t. The vector

$$e(t) = (e^{k_1}(t), \ldots, e^{k_{\kappa_t}}(t)) \in E(t) \tag{3}$$

is called the economic environment at time t. Under the previous arrangements, the components of the environment $e^k(t)$ are not changed at time t.
It is easy to see that components of the environment $e(t)$ defined in (1), form a private ownership economy (compare to Arrow, Debreu 1954; Debreu 1959; Mas-Colell et al. 1995; Lipieta 2018), denoted further by $\mathcal{E}(t)$, with space $\mathcal{R}^{\ell_t}$ as the commodity-price space. Recall that, if in economy $\mathcal{E}(t)$ there exists a sequence:

$$(x^*(t), y^*(t), p(t)),$$

where $x^*(t) = \left(x^{k_1*}(t), \ldots, x^{k_{\kappa_t}*}(t)\right)$, $y^*(t) = \left(y^{k_1*}(t), \ldots, y^{k_{\kappa_t}*}(t)\right)$, $p(t) \in \mathcal{R}^{\ell_t}$ such that

i) $y^{k*}(t)$ maximizes the profit of producer k at price vector $p(t)$ on set $Y^k(t)$, if $k \in B_t$; $y^{k*}(t) = 0$, if $k \notin B_t$,

ii) $x^{k*}(t)$ maximizes the preferences of consumer k on nonempty budget set

$$\beta_t^a(p(t)) \overset{\text{def}}{=} \Bigg\{ x^a(t) \in X^a(t) : \\ p(t) \circ x^a(t) \leq p(t) \circ \omega^a(t) + \sum_{b \in B_t} \theta_t(a,b) \cdot \left(p(t) \circ y^{b*}(t)\right) \Bigg\}, \tag{4}$$

if $k \in A_t$, as well as $x^{k*}(t) = 0$, if $k \notin A_t$,

iii) $\sum_{k \in K_t} x^{k*}(t) - \sum_{k \in K_t} y^{k*}(t) = \omega(t)$,

then it is called the state of equilibrium in the economy $\mathcal{E}(t)$ (see Arrow, Debreu 1954; Mas-Colell et al. 1995), while the economy $\mathcal{E}(t)$ is said to be in equilibrium. Let us emphasize that, if the economy $\mathcal{E}(t)$ is in equilibrium, then economic agents realize their plans of actions which, at prices $p(t)$, form a state of equilibrium.
In competitive economies the economic agents do not cooperate and do not directly communicate. However, the agents' plans of action can be interpreted as messages which economic agents sent to themselves (see Hurwicz 1987). The sequence

$$m^k(t) \overset{\text{def}}{=} (p(t), \check{y}^k(t), \check{x}^k(t)) \tag{5}$$

where:

i) $\check{x}^k(t) \in X^k(t)$ is the plan of action of consumer $k \in A_t$ at time t; $\check{x}^k(t) = 0 \in \mathcal{R}^{\ell_t}$ for $k \notin A_t$,

ii) $\check{y}^k(t) \in Y^k(t)$ is producer's $k \in B_t$ plan of action at time t; $\check{y}^k(t) = 0 \in \mathcal{R}^{\ell_t}$ for $k \notin B_t$,

is called the message of agent $k \in K_t$ at time t. The set of all messages of the form (5) is contained in set $\mathcal{R}^{\ell_t} \times \mathcal{R}^{\ell_t} \times \mathcal{R}^{\ell_t}$ and is denoted by $M^k(t)$. Vector

$$m(t) \stackrel{\text{def}}{=} \left(m^{k_1}(t), \ldots, m^{k_{\kappa_t}}(t)\right) \in M^{k_1}(t) \times \ldots \times M^{k_{\kappa_t}}(t) \tag{6}$$

is called the message at time t. Assume that

$$M(t) \subset M^{k_1}(t) \times \ldots \times M^{k_{\kappa_t}}(t) \text{ and } M(t) \neq \emptyset.$$

Now we recall the definition of the economic mechanism in the Hurwicz sense.

Definition 1 (See for example Hurwicz and Reiter 2006, Lipieta and Malawski 2016). *The triple* $\Gamma_t = (M(t), \mu_t, h_t)$, *where*

i) $\mu_t : E(t) \to M(t)$ *is the message correspondence,*

ii) $h_t : M(t) \to Z$ *is the outcome function*

is called the mechanism in the sense of Hurwicz (or the Hurwicz mechanism).

Let us present the economic interpretation of the components of the Hurwicz mechanisms (see also Lipieta 2015, 2016b). Outcome function h_t links every message $m(t) \in M(t)$ with the allocations which are the result of the analysis of message $m(t)$ by economic agents. Message correspondence μ_t, to every economic environment $e(t)$ assigns the set of messages, consciously or unconsciously sent by economic agents at time t.

Under the above arrangements and notion, the economy $\mathcal{E}(1)$ can be interpreted as the transformation of economy $\mathcal{E}(0)$.

4 Demand-driven eco-mechanisms leading to equilibrium in the private ownership economy

The fourth industrial revolution, initiated by artificial intelligence (AI), Internet of Things (IoT), robotics and big data, is transforming global economic systems. Thanks to modern production solutions, cheap labor is more and more often eliminated from many industries. Models of production and distribution of public goods and services are continually changing. Due to demographic, immigration and ecological disruptions or because of a pandemic, there are many unfavorable economic, economic and social phenomena that cause instability.

In this part of the paper we determine a group of mechanisms resulting in equilibrium in the private ownership economy. We consider a situation, when consumers, aware of the harmfulness for living beings of some goods (inputs) used in production processes or of some side effects (such as carbon dioxide), make producers changing the ways of manufacturing of some commodities in order to eliminate noxious inputs and outputs. Such consumers can be regarded as eco-consumers. As a result of the consumers' impact, the producers would modify their activities on the market what would be seen in their realized production plans. It leads us to modelling an economy which will be a transformation of economy $\mathcal{E}(0)$ in which the production sector will be altered. Mechanisms resulting in equilibrium in such an economy can be also specified. Assume that $\ell_0 > 1$. Suppose that in economy $\mathcal{E}(0)$ there is $d \in \{1,\ldots,\ell_0\}$ noxious commodities $l_1,\ldots,l_d \in \{1,\ldots,\ell_0\}, l_1 < \ldots < l_d$ to be reduced from producers' processes. Without loss of generality we can assume that $l_1 = 1,\ldots,l_d = d$. Put

$$D = \{1,\ldots,d\}.$$

Consider price system $p \in \mathcal{R}^{\ell_0}$ that can be, but does not have to be, the market price system at time $t = 0$, denoted by $p(0) = p$ or $p(0) \neq p$. Assume that there exists an allocation

$$\left(\left(x^{k*}(0)\right)_{k\in K_0}, \left(y^{k*}(0)\right)_{k\in K_0}\right) \tag{7}$$

in which:

i) $y^{k*}(0)$ maximizes the profit of producer k at price vector p on set $Y^k(0)$, if $k \in B_0$; $y^{k*}(0) = 0$, if $k \in K_0 \backslash B_0$,

ii) $x^{k*}(0)$ maximizes the preferences of consumer k on nonempty set

$$\beta_0^a(p) = \left\{x^a(0) \in X^a(0) : p \circ x^a(0) \leq p \circ \omega^a(0) + \sum_{b\in B_0} \theta_0(a,b) \cdot p \circ y^{k*}(0)\right\} \tag{8}$$

if $k \in A_0$ (compare to $\beta_0^a(p(0))$ in (4)),

$$x^{k*}(0) = 0, \text{ if } k \in K_0 \backslash A_0.$$

Let

$$\zeta(0) \overset{\text{def}}{=} \sum_{k\in K_0} x^{k*}(0) - \sum_{k\in K_0} y^{k*}(0) - \sum_{k\in K_0} \omega^k(0). \tag{9}$$

It is easy to see that if $p(0) = p$, then $\beta_0^a(p) = \beta_0^a(p(0))$. If $p(0) = p$ and $\zeta(0) \neq 0$, then sequence $\left(\left(x^{k*}(0)\right)_{k\in K_0}, \left(y^{k*}(0)\right)_{k\in K_0}, p(0)\right)$ is not the state of equilibrium in economy $\mathcal{E}(0)$. We show that under some assumptions, which reflect the consumers' willingness and need to eliminate noxious commodities, some Hurwicz mechanisms (see Definition 1) leading to equilibrium in a transformation of the economy $\mathcal{E}(0)$ can be determined. The considerations will aim at the extension of the results obtained in (Lipieta 2015b, 2015c, 2016) where the adjustment processes which kept equilibrium in competitive economy were analysed. Assume that

$$\forall k \in K_0 \;\; X^k(0) \subset V, \tag{10}$$

where

$$V = \left\{x \in \mathcal{R}^{\ell_0} :\; x_1 = \ldots = x_d = 0\right\}. \tag{11}$$

Assumption (10) reflects the consumers' wishes not to consume the first d commodities. If $\zeta(0) \notin V$, then some producers use in their processes forming vector $y^*(0)$ (see (7)) at least one commodity from set D. Additionally, let us notice that set $V_{|\mathbb{R}^{\ell_0}}$ is a linear subspace of $\mathbb{R}^{\ell_0}$ of dimension $\ell_0 - d$. Consequently V is a linear, final dimensional subspace of $\mathcal{R}^{\ell_0}$.
Denote $\zeta = \zeta(0)$ and assume that

$$p \circ \zeta = 0 \text{ and } \zeta \notin V. \tag{12}$$

Some results on the existence of mechanisms leading to equilibrium, when assumption (12) is not satisfied, can be found in (Lipieta 2015a, 2015c, 2016).
By (12), vector ζ has at least one coordinate from the set D not equal zero. Without the loss of generality we can assume that

$$\zeta_1 \neq 0 \quad \text{and} \quad |\zeta_1| = \min\left\{|\zeta_s| :\; s \in D \wedge \zeta_s \neq 0\right\} \tag{13}$$

Put

$$D_0 = \{s \in D :\; \zeta_s = 0\} \quad \text{and} \quad \widetilde{D} = \{2, \ldots, d\}. \tag{14}$$

Theorem 1. *If condition* (10) *is satisfied with a subspace of the form* (11) *as well as* (12) *is valid, then there exists a Hurwicz mechanism, which transforms economy* $\mathcal{E}_p(0)$ *into economy* $\mathcal{E}_p(1)$*, in which there is a state of equilibrium at price* $p(1) = p$ *as well as* $Y^k(1) \subset V$*, for every* $k \in K_1 = K_0$*. Moreover in the economy* $\mathcal{E}_p(1)$ *the following are satisfied:*

i) $\ell_1 = \ell_0$, $n_1 = n_0$, $m_1 = m_0$,

as well as for every $k \in K_1 = K_0$

ii) $X^k(1) = X^k(0)$, $\omega^k(1) = \omega^k(0)$, $\widetilde{\varepsilon}_1(k) = \widetilde{\varepsilon}_0(k)$, $\widetilde{\theta}_1(k,\cdot) = \widetilde{\theta}_0(k,\cdot)$.

Proof. Let vectors $g^s \in \mathcal{R}^{\ell_0}$, for $s \in D$, be defined by the formula:

$$g^1_l = \begin{cases} 1 \text{ for } l = 1, \\ 0 \text{ for } l \neq 1, \end{cases} \quad g^s_l = \begin{cases} 1 \text{ for } l = s, \\ 0 \text{ for } l \neq s \end{cases} \quad \text{if } \zeta_s = 0,\; g^s_l = \begin{cases} \frac{1}{\zeta_1} \text{ for } l = 1, \\ -\frac{1}{\zeta_s} \text{ for } l = s, \\ 0 \text{ for } \;\notin \{1, s\} \end{cases} \quad \text{if } \zeta_s \neq 0. \tag{15}$$

Then subspace V of the form (11) satisfies the following

$$V = \bigcap_{s=1}^{d} \ker \widetilde{g}^s, \tag{16}$$

where $\widetilde{g}^s$, for $s \in D$, is a mapping of the form:

$$\widetilde{g}^s : \mathcal{R}^{\ell_0} \ni x \to \sum_{l=1}^{\ell_0} g_l^s x_l \in \mathbb{R},$$

and

$$\ker \widetilde{g}^s = \left\{ x \in \mathcal{R}^{\ell_0} : \widetilde{g}^s(x) = 0 \right\}.$$

Notice that every mapping $\widetilde{g}^s$ is linear and continuous. Moreover

$$\widetilde{g}^s(\zeta) = \begin{cases} 1 & \text{if } s = 1, \\ 0 & \text{if } s \in \widetilde{D}, \end{cases}$$

where $\widetilde{D}$ is defined in (14). Let $V^\perp = \{x \in \mathcal{R}^{\ell_0} : \forall v \in V \; x \circ v = 0\}$.
Below two cases are considered.
1) $p \notin V^\perp$. Then vectors $g^1, \ldots, g^d, p$ are linearly independent. Due to the foregoing, the following system of equalities:

$$\widetilde{g}^s\left(q^r\right) = \delta^{sr} \quad \text{for} \quad s, r \in \{2, \ldots, d\}, \tag{17}$$

$$p \circ q^s = 0, \tag{18}$$

where

$$\delta^{sr} = \begin{cases} 1 & \text{if } s = r, \\ 0 & \text{if } s \neq r, \end{cases}$$

is the Kronecker delta, has a solution.
Consequently mapping $Q : \mathcal{R}^{\ell_0} \to V$ of the form

$$Q(x) = x - \sum_{s=1}^{d} \widetilde{g}^s(x) \cdot q^s, \tag{19}$$

where $q^1 = \zeta$, is a linear and continuous projection on the subspace V, determined by vectors $q^1, \ldots, q^d$ (see Cheney 1966) satisfying (by (12), (18) and (19))

$$\forall x \in \mathcal{R}^{\ell_0} : p \circ x = p \circ Q(x).$$

By the above, we get that

$$p \circ \omega^a(0) + \sum_{b \in B_0} \theta_0(a,b) \cdot (p \circ y^{b*}(0)) = p \circ \omega^a(0) + \sum_{b \in B_0} \theta_0(a,b) \cdot (p \circ Q(y^{b*}(0))). \tag{20}$$

2) $p \in V^{\perp}$.
Then for every mapping Q of form (19), for every $x^a(0) \in X^a(0)$, $p \circ (x^a(0) - \omega^a(0)) = 0$ (by assumption (10) and the fact that, for every $a \in A_t$ and t, $\omega^a(t) \in X^a(t)$). Hence, mapping Q of form (19), for every $x^a(0) \in X^a(0)$, satisfies:

$$0 = p \circ (x^a(0) - \omega^a(0)) \leq \sum_{b \in B_0} \theta_0(a,b) \cdot (p \circ y^{b*}(0)).$$

Moreover

$$\forall v \in V : p \circ v = p \circ Q(v) = 0$$

which gives that

$$\sum_{b \in B_0} \theta_0(a,b) \cdot (p \circ y^{b*}(0)) \geq 0.$$

By the above, for every projections of form (19), for which $q^1 = \zeta$ and $q^1, \dots, q^d$ satisfy (17),

$$0 = p \circ (x^a(0) - \omega^a(0)) \leq \sum_{b \in B_0} \theta_0(a,b) \cdot (p \circ Q(y^{b*}(0))). \tag{21}$$

In both cases (i.e. for every p) we put

$$p(1) = p \quad \text{and} \quad Y^k(1) = Q\left(Y^k(0)\right) = \left\{Q\left(y^k(0)\right) : \ y^k(0) \in Y^k(0)\right\}.$$

By (20) and (21), respectively, we get that the sequence:

$$\left(\left(x^{k*}(1)\right)_{k \in K_1}, \left(y^{k*}(1)\right)_{k \in K_1}, p(1)\right),$$

where $x^{k*}(1) = x^{k*}(0)$, $y^{k*}(1) = Q(y^{k*}(0))$, $p(1) = p$ is the state of equilibrium in economy $\mathcal{E}(1)$ defined in the thesis of the theorem, in which additionally $Y^k(1) = Q(Y^k(0))$ for every $k \in K_0$.
At the end of the proof, components of mechanism Γ_0 are defined:

i) environment at time $t = 0$ of the form (3); consequently $E(0)$ of the form (2),

ii) every message of agent k at time $t = 0$ of form (5), where $\check{y}^k(0) = y^{k*}(0)$, as well as $\check{x}^k(0) = x^{k*}(0)$,

iii) $Z = \left\{ \begin{array}{c} (x^*(1), y^*(1), p) : \\ (x^*(1), y^*(1), p) \text{ is a state of equilibrium in economy } \mathcal{E}(1) \end{array} \right\}$,

iv) $\mu_0 : \ E(0) \to M(0)$, where $\mu\,(e(0)) = m(0)$,

v) $h_0 : M(0) \to Z$, $h\,(m(0)) = (x(1), y(1), p(1))$,
where $x(1) = x(0)$, $y(1) = Q(y(0))$, $p(1) = p$.

On the basis of the above, structure $\Gamma_0 = (M(0), \mu_0, h_0)$ is the economic mechanism in the sense of Hurwicz resulting in equilibrium in economy $\mathcal{E}(1)$. □

Remark 1. *By conditions* (15), *we get, for* q^s, $s \in \{2, \dots, d\}$, *satisfying* (17), *the following conditions:*

$$\text{if } \zeta_s = 0 \text{ then } q_l^s = \begin{cases} 1 \text{ for } l = s, \\ 0 \text{ for } l \in D \setminus \{s\}; \end{cases} \qquad \text{if } \zeta_s \neq 0 \text{ then } q_l^s = \begin{cases} -\zeta_s \text{ for } l = s, \\ 0 \text{ for } l \in D \setminus \{s\} \end{cases} \tag{22}$$

are valid.

Mechanism Γ_0 defined in the proof of the Theorem 1 is said to be determined by projection Q.
If $d = 1$, then there exists exactly one projection of the form (19) which determines the mechanism Γ_0 which transforms the economy $\mathcal{E}_p(0)$ into an economy $\mathcal{E}_p(1)$ characterized in the thesis of Theorem 1. The same is reflected to $d = \ell - 1$ if $p \notin V^\perp$.
Assumption (10) reflects the case, when consumers' demand for commodities $1, \dots, d$ are fulfilled by their total endowments or consumers do not want to consume these commodities. Consumers taking care of the environment and natural resources do not want to consume harmful or unnecessarily manufactured goods. To sum up, there is a strong social and economic pressure on producers to remove these commodities from the production processes. Theorem 1 shows that in the above case a mechanism, in the result of which all commodities from set D are eliminated from the production processes, could be implemented. That mechanism, since it is forced by the consumers, is said to be the demand-driven mechanism. Through Assumption (10) the consumers indicate to the producers which commodities are unwanted and should be eliminated from production.

Remark 2. *Taking the results of (Lipieta, Malawski 2021) into consideration, we can say that mechanism* Γ_0 *is imitative if, for every* $k \in K_1$, $Y^k(1) \subset \bigcup_{k \in K_0} Y^k(0)$. *If mechanism* Γ_0 *is imitative and additionally, for every* $k \in K_0$, $Y^k(0) \subset Y^k(1)$, *then that mechanism* Γ_0 *is cumulative. If, for at least one* $k \in K_0$, $Y^k(1) \nsubseteq \bigcup_{k \in K_0} Y^k(0)$, *then mechanism* Γ_0, *defined in the proof of Theorem 1, is innovative.*

The concepts of innovative and imitative mechanisms have their roots in the Schumpeter's theory of the economic development. They were also widely explored in (Lipieta, Malawski 2016, 2021). Imitative mechanisms, in contrast to innovative mechanisms, do not bring significant changes into the economic system. Only earlier known and used solutions, commodities or technologies, can be their results. So, if every point on the timeline is determined by any change in the economic system, then equilibrium can be the result of an imitative mechanism or of such an innovative mechanism in which the innovations are new technologies or new organizational structures, but the set of commodities remain the same as at the previous period of time. Due to the fact that $Y^k(1) \subset V$, for every $k \in K_1$, where V is of form (11), mechanism Γ_0 defined in Theorem 1 results in at least one technological eco-innovation.

5 Optimal demand-driven eco-mechanisms leading to equilibrium in the private ownership economy

The recipe for an optimal mechanism depends on the model under study, on a criterion of the choice as well as on the initial conditions. Below we formulate criteria for specifying optimal mechanisms within the mechanisms defined in the third part of the paper.

We focus on specifying optimal mechanisms in the set of mechanisms defined in the proof of Theorem 1. In economy $\mathcal{E}(1)$ defined in the proof of Theorem 1, the set of commodities is the same as in economy $\mathcal{E}(0)$. Therefore, there is no need to consider two commodity-price spaces in the further investigations, especially that in that case $\ell = \ell_0 = \ell_1$ and consequently $\mathcal{R}^{\ell_1} = \mathcal{R}^{\ell_0} = \mathbb{R}^\ell$. From now, in both economies $\mathcal{E}(0)$ and $\mathcal{E}(1)$, the commodity-space is denoted as $\mathbb{R}^\ell$. The same concerns sets A_0 and B_0. Consequently $m = m_0 = m_1$, $n = n_0 = n_1$ and $\kappa = \kappa_0 = \kappa_1$. Hence from now we assume that $A = A_0 = A_1$, $B = B_0 = B_1$ and $K = K_0 = K_1$. Below, all vectors and sets considered in the economies $\mathcal{E}(0)$ and $\mathcal{E}(1)$, although denoted in the same way as earlier, are considered as vectors and sets, adequately, in space $\mathbb{R}^\ell$.

The starting point of our modelling is the assumption that in the case when a change in economic system does not give enough high profits, the producers are change-averse. If they have to, they will introduce the smallest possible changes in their market activities. Hence, the criterion under study is the minimisation of the distance i.e. we have to minimize, for every $x \in \mathbb{R}^\ell$, number $\|x - Q(x)\|$, out of all projections Q defined in the proof of Theorem 1, in a given norm. Minimization of the distance between the initial and modified objects or processes, adequately, means the changes minimization under the considered criterion which reflects our starting assumption.

Let $g^1, \ldots, g^d \in \mathbb{R}^\ell$ satisfy (15). Consider subspace $V \subset \mathbb{R}^\ell$ satisfying (16). Let $Q : \mathbb{R}^\ell \to V$ be a mapping of t form (19) determined by vectors $q^1, \ldots, q^d \in \mathbb{R}^\ell$ calculated by (17). It is well known (see Cheney1966) that

$$\operatorname{dist}(x, V) \leq \|(Id - Q)(x)\| \leq \|Id - Q\| \operatorname{dist}(x, V) \leq (1 + \|Q\|) \operatorname{dist}(x, V), \qquad (23)$$

where $\|\cdot\|$ is a norm on space $\mathbb{R}^\ell$ and

$$\|Id - Q\| = \sup \left\{ \|(Id - Q)(x)\| : \ x \in \mathbb{R}^\ell \wedge \|x\| \leq 1 \right\} \geq 1 \qquad (24)$$

By (23), it is easy to see that, if norm $\|Id - Q\|$ or $\|Q\|$ is "small", then the production plans and their modifications directed by the projection Q are close in terms of the distance. This is the reason for which the mechanism determined by the mapping Q specified in the proof of Theorem 1 with the smallest number $\|Id - Q\|$, if it exists, is the optimal producers' mechanism under the criterion of distance minimization. Especially, if subspace V is closed and $\|Id - Q\| = 1$, then

$$\|x - Q(x)\| = \operatorname{dist}(x, V).$$

We analyze two norms in space $\mathbb{R}^\ell$:

$$\|x\|_\infty = \max\{|x_l| : \ l \in \{1, 2, \ldots, \ell\}\} \quad \text{and} \quad \|x\|_1 = \sum_{i=1}^{\ell} |x_l|\,. \tag{25}$$

If size minimization of the absolute value of the maximal change is an additional criterion of the choice of an optimal eco-mechanism, then the mechanism Γ_0 specified in Theorem 1, determined by the projection Q_0 with minimal norm $\|Id - Q_0\|$ (see (24)) and, where on the space $\mathbb{R}^\ell$ the norm $\|\cdot\|_\infty$ is considered (see (25)), can be the optimal eco-mechanism under the above listed criteria. If size minimization of the sum of absolute values of all the changes is an additional criterion of the choice of an optimal mechanism, then the mechanism Γ_0 specified in Theorem 1, determined by projection Q_0 with minimal norm $\|Id - Q_0\|$ (see (24)) where norm $\|\cdot\|_1$ on $\mathbb{R}^\ell$ is considered (see (25)), is the required optimal mechanism.
Let $\mathcal{E}(0)$ be a private ownership economy, $p \in \mathbb{R}^\ell$ be a price system satisfying:

$$p(0) = p \quad \text{or} \quad p(0) \neq p.$$

Consider allocation (7) in which, as earlier:

i) $y^{k*}(0)$ maximizes the profit of producer k at price vector p on set $Y^k(0)$, if $k \in B$; $y^{k*}(0) = 0$, if $k \in K \backslash B$,

ii) $x^{k*}(0)$ maximizes the preferences of consumer k on nonempty set (8) if $k \in A$; $x^{k*}(0) = 0$, if $k \in K \backslash A$.

Let ζ be of form (9) and satisfy (12) and (13). If, for some $l \in \{1, \ldots, \ell\}$, $\zeta_l > 0$, then it would mean that the total amount of commodity l consumed at time $t = 0$ by consumers is greater than the amount of this commodity feasible for realization, namely the sum of the amount of commodity l that has existed in the economy so far and the amount of commodity l supplied at time $t = 0$ by producers. Hence, moreover, it is assumed that

$$\zeta \neq 0 \Rightarrow \zeta_l \leq 0 \quad \text{for} \quad l \in \{1, \ldots, \ell\}. \tag{26}$$

Let the subspace $V \subset \mathbb{R}^\ell$ be of form (16), where $g^1, \ldots, g^d \in \mathbb{R}^\ell$ satisfy (15). We put

$$\mathcal{P}\left(\mathbb{R}^\ell, g^1, \ldots, g^d, \zeta\right) \overset{\text{def}}{=} \{Q : \ Q \text{ is of the form (19) and } q^1 = \zeta\} \tag{27}$$

$$\mathcal{P}\left(\mathbb{R}^\ell, g^1, \ldots, g^d, \zeta, p\right) \overset{\text{def}}{=} \{Q \in \mathcal{P}\left(\mathbb{R}^\ell, g^1, \ldots, g^d, \zeta\right) : \ q^2, \ldots, q^d \text{ satisfy (18)}\}. \tag{28}$$

The elimination of even one commodity from the economic processes can appear to be difficult due to technological possibilities. Elimination of, or even limitation, of the emission of carbon dioxide is an example. Hence further, we focus on modelling an optimal mechanism in the group of mechanisms defined in the proof of Theorem 1, within which a small number of commodities is removed from the market. As it was mentioned earlier, if exactly one commodity is to be rejected from the economic processes, then there is only one mechanism Γ_0 determined by a projection of form (19) which would lead to equilibrium in economy $\mathcal{E}(1)$ satisfying the requirements listed in the thesis of Theorem 1. Therefore, in the further part of the paper we restrict our considerations to the case when $d \in \{1, 2\}$. However, some results for $p \notin V^\perp$ and $d > 2$ also will be presented.
To find an optimal mechanism in the set of mechanisms defined in the proof of Theorem 1, we aim at determining projection $Q_0 \in \mathcal{P}\left(\mathbb{R}^\ell, g^1, \ldots, g^d; p, \zeta\right)$ (see (28)) satisfying

$$\|Id - Q_0\|_\infty = \inf\left\{\|Id - Q\|_\infty : \ Q \in \mathcal{P}\left(\mathbb{R}^\ell, g^1, \ldots, g^d; p, \zeta\right)\right\} \tag{29}$$

or

$$\|Id - Q_0\|_1 = \inf\left\{\|Id - Q\|_1 : \ Q \in \mathcal{P}\left(\mathbb{R}^\ell, g^1, \ldots, g^d; p, \zeta\right)\right\} \tag{30}$$

if $p \notin V^\perp$ as well as projection $Q_0 \in \mathcal{P}\left(\mathbb{R}^\ell, g^1, \ldots, g^d; \ \zeta\right)$ (see(27)) satisfying

$$\|Id - Q_0\|_\infty = \inf\left\{\|Id - Q\|_\infty : \ Q \in \mathcal{P}\left(\mathbb{R}^\ell, g^1, \ldots, g^d; \zeta\right)\right\} \tag{31}$$

or

$$\|Id - Q_0\|_1 = \inf\left\{\|Id - Q\|_1 : \ Q \in \mathcal{P}\left(\mathbb{R}^\ell, g^1, \ldots, g^d; \zeta\right)\right\} \tag{32}$$

if $p \in V^\perp$.
Now, we present a technical lemma which will be in further use.

Lemma 2. *If $d \geq 2$, $Q \in \mathcal{P}\left(\mathbb{R}^\ell, g^1, \ldots, g^d; \zeta\right)$, then*

1. *we get*

$$\|Id - Q\|_\infty = \max\{1, M_{Q,l} : l \in \{d+1, \ldots, \ell\}\}, \tag{33}$$

where for $l \in \{d+1, \ldots, \ell\}$

$$M_{Q,l} = \left|\frac{\zeta_l}{\zeta_1} + \sum_{s \in \widetilde{D} \setminus D_0} \frac{q_l^s}{\zeta_1}\right| + \sum_{s \in D_0} |q_l^s| + \sum_{s \in \widetilde{D} \setminus D_0} \left|\frac{q_l^s}{\zeta_s}\right|; \tag{34}$$

2. *we get*

$$\|Id - Q\|_1 = 1 + \max\left\{N_{Q,s} : s \in D\right\}, \tag{35}$$

where

$$\begin{cases} N_{Q,1} = \sum_{l=d+1}^{\ell} \left| \left(\frac{\zeta_l}{\zeta_1} + \sum_{s\in \widetilde{D}\setminus D_0} \frac{q_l^s}{\zeta_1} \right) \right| & \\ N_{Q,s} = \sum_{l=d+1}^{\ell} \left|q_l^s\right| & \text{for } s \in D_0, \\ N_{Q,s} = \sum_{l=d+1}^{\ell} \left|\frac{q_l^s}{\zeta_s}\right| & \text{for } s \in \widetilde{D}\setminus D_0. \end{cases} \tag{36}$$

Proof. See Appendix. □

Let us notice that generally there are infinitely many projections $Q_0 \in \mathcal{P}\left(\mathbb{R}^\ell, g^1, \ldots, g^d; \zeta\right)$ (see (27)) satisfying (31) or (32). Below some examples are presented.

Example 1. *Let $\ell = 4$, $d = 2$. There are infinitely many projections $Q_0 \in \mathcal{P}\left(\mathbb{R}^\ell, g^1, g^2; \zeta\right)$ satisfying (31), where*

$$\|Id - Q_0\|_\infty = \max\left\{1, \left|\frac{\zeta_3}{\zeta_1}\right|, \left|\frac{\zeta_4}{\zeta_1}\right|\right\}.$$

For projection Q_0 satisfying (31) the following is valid:

i) $\|Id - Q_0\|_\infty = 1$ if and only if, $|\zeta_3| \le |\zeta_1|$ and $|\zeta_4| \le |\zeta_1|$.

ii) if $|\zeta_3| = |\zeta_1|$ and $|\zeta_4| = |\zeta_1|$, then $\|Id - Q_0\|_\infty = 1$ if and only if $q_3^2 = q_4^2 = 0$.

If, $|\zeta_3| \le |\zeta_1|$ and $|\zeta_4| < |\zeta_1|$ or $|\zeta_3| < |\zeta_1|$ and $|\zeta_4| \le |\zeta_1|$, then there are infinitely many projections satisfying (31) for which $\|Id - Q_0\|_\infty = 1$.

Proof. See Appendix. □

Example 2. *Let $\ell = 4$, $d = 2$. If $\zeta_2 = 0$, then $\|Id - Q\|_1 \ge 1 + \frac{|\zeta_3|+|\zeta_4|}{|\zeta_1|}$ and either $\zeta_3 = \zeta_4 = 0$, in which case there is a unique projection of norm 1 given by ζ and $q_3^2 = q_4^2 = 0$, or ζ_3, ζ_4 are not both zero and then there are infinitely many projections $Q_0 \in \mathcal{P}\left(\mathbb{R}^\ell, g^1, g^2; \zeta\right)$ satisfying (32) and given by ζ and any q^2 with $\left|q_3^2\right| \ge \frac{|\zeta_3|}{|\zeta_1|}$ and $\left|q_4^2\right| \ge \frac{|\zeta_4|}{|\zeta_1|}$, and then $\|Id - Q_0\|_1 = 1 + \frac{|\zeta_3|+|\zeta_4|}{|\zeta_1|}$.*
If $\zeta_2 \ne 0$ there is always a projection of norm 1. It is given by ζ and $q_3^2 = q_4^2 = 0$.

Proof. See Appendix. □

Theorem 3. *Let $d = 1$.*

1. *If $p \notin V^\perp$, then there is exactly one projection $Q_0 \in \mathcal{P}\left(\mathbb{R}^\ell, g^1; p, \zeta\right)$ (see (28)) satisfying both (29) and (30).*

2. *If $p \in V^{\perp}$, then there is exactly one projection $Q_0 \in \mathcal{P}\left(\mathbb{R}^{\ell}, g^1; \zeta\right)$ (see (27)) satisfying both (31) and (32).*

Proof. It is not difficult to check (by using Assumption (12) and Equation (19)) that only the projection

$$Q_0(x) = x - \widetilde{g}^1(x) \cdot \zeta$$

satisfies the thesis of the theorem. Then (see (34) and (35))

$$\|Id - Q_0\|_{\infty} = max\left\{1, \frac{|\zeta_l|}{|\zeta_1|} : \ l \in \{1, \dots, \ \ell\}\right\} \quad \text{and} \quad \|Id - Q_0\|_1 = 1 + \sum_{l=1}^{\ell} \frac{|\zeta_l|}{|\zeta_1|}.$$

□

Theorem 4. *Let $d = \ell - 1$ and $p \notin V^{\perp}$. There is exactly one projection $Q_0 \in \mathcal{P}\left(\mathbb{R}^{\ell}, g^1, \dots, g^{\ell-1}; p, \zeta\right)$ (see (27)) satisfying both (29) and (30). Additionally*

$$\|Id - Q_0\|_{\infty} = \max\left\{1, \sum_{s=1}^{\ell-1}\left|\frac{p_s}{p_\ell}\right|\right\},$$

as well as

$$\|Id - Q_0\|_1 = 1 + \max\left\{\left|\frac{p_s}{p_\ell}\right| : s \in \{1, \dots, \ell-1\}\right\}.$$

Proof. Due to conditions (17), (18) and (22), by Appendix we get that

$$\text{if } \zeta_s = 0 \text{ then } q_l^s = \begin{cases} 1 \text{ for } l = s, \\ 0 \text{ for } l \in D \setminus \{s\}, \\ \frac{-p_s}{p_l} \text{ for } l = \ell; \end{cases} \quad \text{if } \zeta_s \neq 0 \text{ then } q_l^s = \begin{cases} -\zeta_s \text{ for } l = s, \\ 0 \text{ for } l \in D \setminus \{s\}, \\ \frac{p_s \zeta_s}{p_l} \text{ for } l = \ell. \end{cases} \tag{37}$$

Hence, set $\mathcal{P}\left(\mathbb{R}^{\ell}, g^1, \dots, g^{\ell-1}; p, \zeta\right)$ consists of only one element – projection Q_0 determined by vectors $q^1, q^2, \dots, q^{\ell-1}$ where $q^1 = \zeta$ while $q^2, \dots, q^{\ell-1}$ are of form (37). By Lemma 2 (see (34) and (35)), it is immediately seen that

$$\|Id - Q_0\|_{\infty} = \max\left\{1, \left|\frac{\zeta_\ell}{\zeta_1} + \sum_{s \in D} \frac{p_s \zeta_s}{p_\ell \zeta_1}\right| + \sum_{s \in D}\left|\frac{p_s}{p_\ell}\right|\right\},$$

while

$$\|Id - Q_0\|_1 = 1 + \max\left\{\left|\frac{\zeta_\ell}{\zeta_1} + \sum_{s \in D} \frac{p_s \zeta_s}{p_\ell \zeta_1}\right|, \left|\frac{p_s}{p_l}\right| : s \in D\right\}.$$

Since $p \circ \zeta = 0$ (Assumption (12)) $\frac{\zeta_\ell}{\zeta_1} + \sum_{s \in D} \frac{p_s \zeta_s}{p_\ell \zeta_1} = \frac{1}{\zeta_1} \cdot \left(\zeta_\ell + \sum_{s \in D} \frac{p_s \zeta_s}{p_\ell}\right) = 0$. Hence

$$\|Id - Q_0\|_\infty = \max\left\{1, \sum_{s=1}^{\ell-1} \left|\frac{p_s}{p_\ell}\right|\right\},$$

as well as

$$\|Id - Q_0\|_1 = 1 + \max\left\{\left|\frac{p_s}{p_\ell}\right| : s \in \{1, \ldots, \ell - 1\}\right\}.$$

□

The immediate consequence of Theorem 1 and the finite dimension of the commodity space is the following

Remark 3. *If condition* (10) *is satisfied with subspace of form* (11) *as well as conditions* (12) *is valid, then there is a Hurwicz mechanism* Γ_0 *determined by projection* Q_0 *satisfying* (29) *or* (30) *which transforms the economy* $\mathcal{E}_p(0)$ *into economy* $\mathcal{E}_p(1)$*, in which there is a state of equilibrium at price* $p(1) = p$ *as well as* $Y^k(1) \subset V$*, for every* $k \in K_1 = K_0$*. Moreover in the economy* $\mathcal{E}_p(1)$ *the following are satisfied:*

i) $\ell_1 = \ell_0$, $n_1 = n_0$, $m_1 = m_0$,

as well as for every $k \in K_1 = K_0$

ii) $X^k(1) = X^k(0)$, $\omega^k(1) = \omega^k(0)$, $\widetilde{\varepsilon}_1(k) = \widetilde{\varepsilon}_0(k)$, $\widetilde{\theta}_1(k, \cdot) = \widetilde{\theta}_0(k, \cdot)$.

Let $d = 2$ and $p \notin V^\perp$. The latter means that two outputs, $l_1 = 1$ and $l_2 = 2$ are to be eliminated. We assume additionally that commodities l_1 and l_2 are harmful for the environments and consumers. It means (see Mas-Colell et al. 1995) that

$$p_1, p_2 < 0 \quad \text{and} \quad p_3, \ldots, p_\ell > 0. \tag{38}$$

Let us notice that, if $\zeta_l \neq 0$, then, by (26), $\frac{\zeta_l}{\zeta_1} > 0$. Below, some examples of a projection Q_0 satisfying (29) or (30) are presented.

Lemma 5. *If* $d = 2$*, condition* (38) *is satisfied as well as additionally, for every* $l \in \{3, \ldots, \ell\}$*,*

$$\frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j} \geq \frac{\zeta_l}{\zeta_1}, \tag{39}$$

then there is projection $Q_0 \in \mathcal{P}\left(\mathbb{R}^\ell, g^1, g^2; p, \zeta\right)$ *(see* (27)*) satisfying* (29)*, where*

$$\|Id - Q_0\|_\infty = \max\left\{1, \frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j}\right\}. \tag{40}$$

Projection Q_0 is determined by vectors $q^1 = \zeta$ and q^2, where

$$q_l^2 = \frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j} - \frac{\zeta_l}{\zeta_1},\ q_1^2 = 0,\ q_2^2 = 1 \text{ if } \zeta_2 = 0$$

while

$$q_l^2 = \left(\frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j} - \frac{\zeta_l}{\zeta_1} \right) \cdot \left(\frac{1}{\zeta_1} + \frac{1}{\zeta_2} \right)^{-1},\ q_1^2 = 0,\ q_2^2 = -\zeta_2 \text{ if } \zeta_2 \neq 0.$$

Proof. See Appendix. □

Let us notice that if $d = 2$, (38) is valid but (39) is not satisfied, then

$$\|Id - Q_0\|_\infty \geq \max \left\{ 1, \frac{-p_1 - p_2}{\sum_{l=3}^{\ell} p_l} \right\}.$$

Remark 4. *If $d = 2$, conditions* (38) *and* (39) *are satisfied as well as additionally,*

$$\zeta_3 = \ldots = \zeta_\ell = 0, \tag{41}$$

then the projection Q_0 by Lemma 5 is determined by vectors q^1 and q^2, where

$$q^1 = \zeta,\ q_1^2 = 0,\ q_2^2 = 1,\ q_3^2 = \ldots = q_\ell^2 = \frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j} \text{ if } \zeta_2 = 0,$$

while

$$q^1 = \zeta,\ q_1^2 = 0,\ q_2^2 = -\zeta_2,\ q_3^2 = \ldots = q_\ell^2 = \left(\frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j} \right) \cdot \left(\tfrac{1}{\zeta_1} + \tfrac{1}{\zeta_2} \right)^{-1}, \text{ if } \zeta_2 \neq 0.$$

Condition (41) *occurs, if the there is no equilibrium only on the markets of unwanted/harmful commodities. Hence Remark 4 presents in that case the formula for Hurwicz mechanism Γ_0 determined by projection Q_0 satisfying* (29)*.*

Remark 5. *If $\ell = 3$, $d = 2$, condition* (38) *is satisfied, then projection Q_0 is unique and $\|Id - Q_0\|_\infty = \max \left\{ 1, \frac{-p_1 - p_2}{p_3} \right\}$. It is the immediate consequence of the proof of Lemma 5.*

Lemma 6. *Let $d = 2$ and* (38) *be satisfied. For any projection $Q \in \mathcal{P}\left(\mathbb{R}^\ell, g^1, g^2; p, \zeta\right)$ (see* (27)*) satisfying* (30)*)*

$$\|Id - Q\|_1 \geq 1 + \frac{\max\{-p_1, -p_2\}}{\max\{p_j : j \in \{3, \ldots, \ell\}\}}.$$

Additionally, if $p_2 \leq p_1$, there exists projection Q_0 such that

$$\|Id - Q_0\|_1 = 1 + \frac{\max\{-p_1, -p_2\}}{\max\{p_j : j \in \{3, \ldots, \ell\}\}}.$$

Projection Q_0 is determined by vectors ζ and q^2 in the following way:

i) if $\zeta_2 = 0$*, then* $q_1^2 = 0$*,* $q_2^2 = 1$*,* $q_k^2 = 0$ *for* $k \neq j_0$*, and* $q_{j_0}^2 = \frac{-p_2}{p_{j_0}}$ *for a fixed* j_0 *such that* $p_{j_0} = \max\{p_j : j \in \{3, \ldots, \ell\}\}$*,*

ii) if $\zeta_2 \neq 0$*, then* $q_1^2 = 0$*,* $q_2^2 = -\zeta_2$*,* $q_k^2 = 0$ *for* $k \neq j_0$*, and* $q_{j_0}^2 = \frac{\zeta_2 p_2}{p_{j_0}}$ *for fixed* j_0 *for which* $p_{j_0} = \max\{p_j : j \in \{3, \ldots, \ell\}\}$*.*

Proof. See Appendix. □

6 Discussion

This research links some areas of theoretical economics such as ecological economics, environmental economic, economic development, and organization design. Moreover, to obtain satisfactory results, we have to resolve some theoretical problems concerning linear operators. That is why many additional problems, which are rooted in the specific area of the economic theory but not considered in the others, arise.

Eco-mechanisms may be difficult to be implemented in the economy, even if the society as a whole believes that harmful technologies and commodities should be eliminated from the production processes. In many cases, economic considerations outweigh the common sense thinking about the industrial waste and environmental pollution and their impact on living conditions of living creatures. A decision maker has to face at least two following challenges: how to force the economic agents to change their market activities to be ecological and how to make the producers take part in the same eco-mechanism? Introducing Pigouvian taxation on firms to regulate negative externalities can be the answer to the first question. The society's increasing ecological awareness and the fashion for a life style that is simultaneously ecological and technologically advanced, profiting from all the good brought along by the Industry 4.0, may act as an incentive for producers to introduce changes. Therefore, to answer the second question, the problem of incentives should be analyzed. In this case a positive solution can be an attempt to implement an eco-mechanism which would be optimal from the point of view of both the members of the society and the decision maker. New mechanisms analyzed in the current paper, on the one hand, aim at introducing eco-changes and do not make the firm's profits decrease; on the other hand, they minimize the "size of the change" in the sense of minimizing the distance between a trajectory determining the necessary change in producers' activities and the identity mapping which describes the unchanged producers' activities. It should be emphasized that the mechanisms presented in the paper concern an economy in which the producers are change-averse. If producers were not change-averse, then the size of changes to be introduced could be higher. However, in such a case, according to Schumpeter's theory, the profits of some producers could be decreased, or some producers could be eliminated from the market. Such a mechanism would not be "good" for all producers especially those in our approach. Economic agents do not cooperate and act on their own in such a way that their economic positions are not decreased. It is worth adding that the driver of our mechanism is not defined in our approach. It could be a decision maker, government etc.

The results presented in the paper show that, if the consumers are aware of the consequences of the use of harmful commodities or technologies in the production processes and, consequently, they do not want to consume the goods manufactured by the use of any of them, then the producers to have their outputs sold will have to change their technologies. In fact it means that then the produces will follow a mechanism which will result in the elimination of the unwanted technologies or goods from the market. Thus, due to the structure of the demand, i.e. the consumers' market activities, the consumers force the producers to change their market activities. In that meaning, the consumers' plans of actions, through which they show which commodities are unwanted, incline the producers to implement a mechanism in the result of which the unwanted commodities or technologies will be eliminated. On the other hand, it is obvious that, if the producers do not expect additional profits due to the introduction of some changes in the production, then they will want a negligible change of their activities. Hence the concept of the optimal mechanism presented in the paper seems to be reasonable in this situation. Let us notice that in this case the main role of a decision maker is making the consumers aware of the danger of producing and consuming harmful goods and showing the positive effects of eco-changes and eco-mechanisms.

An example of the foregoing situation is the production of infant foods. It is obvious that the food prepared on the basis of not healthy commodities or technologies can result in many small babies falling ill or even dying. The firms which would cause such a situation might lose confidence of parents, which can lead to a significant fall in profits or even to failure. Therefore those kinds of foods, i.e. powdered infant milk, porridges for kids etc. are prepared only with the use of safe, high quality inputs. Should the technology appear to be harmful, then it is instantly eliminated.

The main difficulty of this research was to determine a proper criterion for the choice of an optimal mechanism. Nonetheless, specifying the optimal mechanism, in fact an adequate projection, under the proposed criterion is also a challenge since, in many cases, the problem of specifying a projection closest to the identity mapping has not been resolved yet. By the results presented in (Cheney 1966), it is known that in finite dimensional spaces there exists a projection which is closest, in the sense of the norm, to the identity mapping, but generally it is not known what such a projection looks like, and whether it is unique. The same concerns a projection satisfying the additional requirements such as those considered in the current paper. Let us recall that we present the exact formula for a projection in some cases (for considered subspaces of space $\mathbb{R}^\ell$ of dimension equals to $d = 1, \ell - 1$ or in some cases for $d = 2$). The problem of uniqueness and specifying such a projection on a space of dimension $d = 2$, for which condition (39) is not satisfied, remains under our considerations.

7 Conclusions

To sum up: in the current paper, we specified formulas for optimal eco-mechanisms which result in the elimination of at least one harmful commodity or noxious technology, under the criterion of distance minimization and assumptions interpreted in the economic theory. Possible applications are important in view of the struggle against climate changes. Every presented optimal eco-mechanism is determined by a linear and continuous projection and results in equilibrium in a transformation of the initial economy.

The formula for the linear projection closest to the identity mapping seems to be quite complicated, especially if more than one commodity is to be eliminated from the market. However, the existence of the optimal mechanism under the criterion of changes minimization shows the possibility of designing a mechanism in which the economic positions of market participants are not worse than when the mechanism is not implemented.

The eco-producers for whom the distance minimization and lack of losses are the main criteria for the choice of the eco-mechanism, operating in their own interest take advantage of the implementation of the presented optimal mechanism.

The mechanisms considered in the paper emphasize the significant role of information and the ways of exchanging messages during economic processes. In the mechanisms under study, the effects of creative destruction are also revealed: harmful products and technologies disappear from the market.

Axiomatization of mechanisms of the evolution of the economy by the use of Hurwicz's apparatus exposed the positive, from the producers' and consumers' points of view, qualitative properties of the optimal mechanism.

The mechanisms presented in the paper are, in many cases, mechanisms of creative destruction (see Schumpeter 1912; Aghion, Howitt 1992), since on the one hand they result in the elimination of some commodities or technologies from the market, while on the other hand, within them, new technologies are introduced by some producers The results obtained in the current paper can be useful in the economic and ecological theorizing to understand the nature of economic processes, especially if the access to empirical data is impossible or limited.

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Appendix

Proof of Lemma 2. Recall that,

$$\|Id - Q\|_\infty = \sup\left\{\left\|\sum_{s=1}^{d}\widetilde{g}^s(x)\cdot q^s\right\|_\infty : \|x\|_\infty = 1\right\},$$

as well as

$$\|Id - Q\|_1 = \sup\left\{\left\|\sum_{s=1}^{d}\widetilde{g}^s(x)\cdot q^s\right\|_1 : \|x\|_1 = 1\right\}.$$

Ad. 1) For every $x \in \mathbb{R}^\ell$

$$\|(Id - Q)(x)\|_\infty = \max\left\{\left|\sum_{s=1}^{d}\widetilde{g}^s(x)\cdot q^s_l\right| : \ l \in \{1, 2, \ldots, \ell\}\right\}.$$

Using Assumption (15), properties (22) as well as the fact that $q^1 = \zeta$, by elementary calculations, we get that

$$\begin{aligned}
\left|\sum_{s=1}^{d}\widetilde{g}^s(x)\cdot q^s_l\right| &= \left|\left(\frac{x_1}{\zeta_1}\cdot\zeta_l + \sum_{s\in D_0} x_s q^s_l + \sum_{s\in\widetilde{D}\setminus D_0}\left(\frac{x_1}{\zeta_1} - \frac{x_s}{\zeta_s}\right) q^s_l\right)\right| = \\
&= \left|\left(\frac{\zeta_l}{\zeta_1} + \sum_{s\in\widetilde{D}\setminus D_0}\frac{q^s_l}{\zeta_1}\right) x_1 + \sum_{s\in D_0} x_s q^s_l - \sum_{s\in\widetilde{D}\setminus D_0} x_s \frac{q^s_l}{\zeta_s}\right| \leq \\
&\leq \left(\left|\frac{\zeta_l}{\zeta_1} + \sum_{s\in\widetilde{D}\setminus D_0}\frac{q^s_l}{\zeta_1}\right| + \sum_{s\in D_0}|q^s_l| + \sum_{s\in\widetilde{D}\setminus D_0}\left|\frac{q^s_l}{\zeta_s}\right|\right)\times \\
&\quad \times \max\{|x_1|, \ldots, |x_d|\}.
\end{aligned}$$

Put $M_{Q,l}$ of form (34) and

$$M_Q = \max\{M_{Q,l} : l \in \{1, 2, \ldots, \ell\}\}.$$

It is obvious that (see (25)),

$$\|(Id - Q)(x)\|_\infty \leq M_Q \text{ for } x \in \mathbb{R}^\ell \text{ for which } \|x\|_\infty = 1.$$

Suppose that $\widetilde{l} \in \{1, 2, \ldots, \ell\}$ satisfies $M_Q = M_{Q,\widetilde{l}}$. Putting

$$x_1 = \operatorname{sgn}\left(\frac{\zeta_{\widetilde{l}}}{\zeta_1} + \sum_{s \in \widetilde{D} \setminus D_0} \frac{1}{\zeta_1} q^s_{\widetilde{l}}\right), \; x_s = \operatorname{sgn} q^s_{\widetilde{l}} \text{ if } \zeta_s = 0 \text{ and } x_s = \operatorname{sgn} \frac{q^s_{\widetilde{l}}}{\zeta_s} \text{ if } \zeta_s \neq 0,$$

we get that $\|(Id - Q)(x)\|_\infty = M_Q$, which means that $\|(Id - Q)\|_\infty = M_Q$.
Let us notice that by (15) and (22) the following is valid:

$$M_{Q,l} = 1 \text{ for } l \in D.$$

By the above we get (33).
Ad. 2) For every $x \in \mathbb{R}^\ell$

$$\|(Id - Q)(x)\|_1 = \sum_{l=1}^{\ell} \left| \sum_{s=1}^{d} \widetilde{g}^s(x) \cdot q^s_l \right|.$$

Hence, by (15), (22) as well as the fact that $q^1 = \zeta$,

$$\begin{aligned}
\|(Id - Q)(x)\|_1 &= \sum_{l=1}^{\ell} \left| \left(\frac{x_1}{\zeta_1} \cdot \zeta_l + \sum_{s \in D_0} x_s q^s_l + \sum_{s \in \widetilde{D} \setminus D_0} \left(\frac{x_1}{\zeta_1} - \frac{x_s}{\zeta_s} \right) q^s_l \right) \right| = \\
&= \sum_{l=1}^{\ell} \left| \left(\frac{\zeta_l}{\zeta_1} + \sum_{s \in \widetilde{D} \setminus D_0} \frac{q^s_l}{\zeta_1} \right) x_1 + \sum_{s \in D_0} x_s q^s_l - \sum_{s \in \widetilde{D} \setminus D_0} x_s \frac{q^s_l}{\zeta_s} \right| \leq \\
&\leq N_Q \cdot \sum_{l \in D} |x_l|
\end{aligned}$$

and $N_Q = \max\{N_{Q,s} : s \in D\}$, where

$$\begin{aligned}
N_{Q,1} &= \sum_{l=1}^{\ell} \left| \left(\frac{\zeta_l}{\zeta_1} + \sum_{s \in \widetilde{D} \setminus D_0} \frac{q^s_l}{\zeta_1} \right) \right|, \\
N_{Q,s} &= \sum_{l=1}^{\ell} |q^s_l| \text{ for } s \in D_0, \\
N_{Q,s} &= \sum_{l=1}^{\ell} \left| \frac{q^s_l}{\zeta_s} \right| \text{ for } s \in \widetilde{D} \setminus D_0.
\end{aligned}$$

If $N_Q = N_{Q,s}$ for some $s \in D$, then putting $x_s = 1$ and $x_l = 0$ for $l \in \{1, \ldots, \ell\} \setminus \{s\}$, we get that $\|(Id - Q)(x)\|_1 = N_{Q,s}$, which means that $\|(Id - Q)\|_1 = N_{Q,s}$. Let us notice that

$$\sum_{l\in\widetilde{D}\setminus D_0}\left|\left(\frac{\zeta_l}{\zeta_1}+\sum_{s\in\widetilde{D}\setminus D_0}\frac{q_l^s}{\zeta_1}\right)\right|=0$$

(by (22)) which means that

$$\sum_{l\in D_0}\left|\left(\frac{\zeta_l}{\zeta_1}+\sum_{s\in\widetilde{D}\setminus D_0}\frac{q_l^s}{\zeta_1}\right)\right|=\sum_{l\in D_0}\left|\left(\sum_{s\in\widetilde{D}\setminus D_0}\frac{q_l^s}{\zeta_1}\right)\right|=0.$$

Moreover, by (22),

$$\sum_{l=1}^{\ell}|q_l^s|=1+\sum_{l=d+1}^{\ell}|q_l^s| \text{ for } s\in D_0$$

and

$$\sum_{l=1}^{\ell}\left|\frac{q_l^s}{\zeta_s}\right|=1+\sum_{l=d+1}^{\ell}\left|\frac{q_l^s}{\zeta_s}\right| \text{ for } s\in\widetilde{D}\setminus D_0.$$

By the above, we get (35). □

Proof of Example 1. In that case $D=\{1,2\}$. Suppose that $\zeta_2=0$. Now $g^1=\left(\frac{1}{\zeta_1},0,0,0\right)$, $g^2=(0,1,0,0)$ (see (15)), $q^1=\zeta=(\zeta_1,0,\zeta_3,\zeta_4)$. Consequently, $q^2=(0,1,q_3^2,q_4^2)$ (see (22)) and $D_0=\widetilde{D}=\{2\}$ (see (14)). The above applied to the definition of number $M_{Q,l}$ (see (34)) lead to the following

$$M_{Q,3}=\left|\frac{\zeta_3}{\zeta_1}\right|+\left|q_3^2\right|,\quad M_{Q,4}=\left|\frac{\zeta_3}{\zeta_1}\right|+\left|q_4^2\right|$$

and

$$\|Id-Q\|_\infty=\max\left\{1,\left|\frac{\zeta_3}{\zeta_1}\right|+\left|q_3^2\right|,\left|\frac{\zeta_4}{\zeta_1}\right|+\left|q_4^2\right|\right\}. \tag{A1}$$

Hence

$$\|Id-Q\|_\infty\geq\max\left\{1,\left|\frac{\zeta_3}{\zeta_1}\right|,\left|\frac{\zeta_4}{\zeta_1}\right|\right\}.$$

If $\left|\frac{\zeta_4}{\zeta_1}\right|+\left|q_4^2\right|\leq\max\left\{1,\left|\frac{\zeta_3}{\zeta_1}\right|,\left|\frac{\zeta_4}{\zeta_1}\right|\right\}$ or $\left|\frac{\zeta_3}{\zeta_1}\right|+\left|q_3^2\right|\leq\max\left\{1,\left|\frac{\zeta_3}{\zeta_1}\right|,\left|\frac{\zeta_4}{\zeta_1}\right|\right\}$, then the projection

$$Q(x)=x-\widetilde{g}^1(x)\cdot\zeta-\widetilde{g}^2(x)\cdot q^2 \tag{A2}$$

satisfies (31). There are infinitely many such projections.

Now suppose that $\zeta_2\neq 0$. Then $g^1=\left(\frac{1}{\zeta_1},0,0,0\right)$, $g^2=\left(0,-\frac{1}{\zeta_2},0,0\right)$ (see (15)), $\zeta=(\zeta_1,\zeta_2,\zeta_3,\zeta_4)$ and $q^2=(0,-\zeta_2,q_3^2,q_4^2)$ (see (22)), $D_0=\emptyset$, $\widetilde{D}=\{2\}$ (see (14)).

By (34)

$$M_{Q,3} = \frac{|\zeta_3|}{|\zeta_1|} + |q_3^2| \left(\frac{1}{|\zeta_1|} + \frac{1}{|\zeta_2|} \right) \quad \text{and} \quad M_{Q,4} = \frac{|\zeta_4|}{|\zeta_1|} + |q_4^2| \left(\frac{1}{|\zeta_1|} + \frac{1}{|\zeta_2|} \right)$$

and

$$\|Id - Q\|_\infty = \max \left\{ 1, \frac{|\zeta_3|}{|\zeta_1|} + |q_3^2| \left(\frac{1}{|\zeta_1|} + \frac{1}{|\zeta_2|} \right), \frac{|\zeta_4|}{|\zeta_1|} + |q_4^2| \left(\frac{1}{|\zeta_1|} + \frac{1}{|\zeta_2|} \right) \right\}.$$

Hence

$$\|Id - Q\|_\infty \geq \max \left\{ 1, \left| \frac{\zeta_3}{\zeta_1} \right|, \left| \frac{\zeta_4}{\zeta_1} \right| \right\}.$$

If

$$\left| \frac{\zeta_4}{\zeta_1} \right| + |q_4^2| \left(\frac{1}{|\zeta_1|} + \frac{1}{|\zeta_2|} \right) \leq \max \left\{ 1, \left| \frac{\zeta_3}{\zeta_1} \right|, \left| \frac{\zeta_4}{\zeta_1} \right| \right\}$$

or

$$\left| \frac{\zeta_3}{\zeta_1} \right| + \left(\frac{1}{|\zeta_1|} + \frac{1}{|\zeta_2|} \right) \leq \max \left\{ 1, \left| \frac{\zeta_3}{\zeta_1} \right|, \left| \frac{\zeta_4}{\zeta_1} \right| \right\},$$

then a projection of form (A2) satisfies (31). There are infinitely many of such projections.
It is easy to see that, for any ζ_2 and projection Q_0 of form (A2),

$$\|Id - Q_0\|_\infty = 1 \text{ if and only if } |\zeta_3| \leq |\zeta_1| \text{ and } |\zeta_4| \leq |\zeta_1|$$

as well as

$$\text{if } |\zeta_3| = |\zeta_1| \text{ and } |\zeta_4| = |\zeta_1|, \text{ then } \|Id - Q_0\|_\infty = 1 \text{ if and only if } q_3^2 = q_4^2 = 0.$$

Moreover, if, $|\zeta_3| \leq |\zeta_1|$ and $|\zeta_4| < |\zeta_1|$ or $|\zeta_3| < |\zeta_1|$ and $|\zeta_4| \leq |\zeta_1|$, then there are infinitely many projections of the form (A2) for which $\|Id - Q_0\|_\infty = 1$. □

Proof of Example 2. As above $D = \{1, 2\}$.
Assume that $\zeta_2 = 0$. In that case $g^1 = \left(\frac{1}{\zeta_1}, 0, 0, 0 \right)$, $g^2 = (0, 1, 0, 0)$ (see (15)), $q^1 = \zeta = (\zeta_1, 0, \zeta_3, \zeta_4)$ and $q^2 = (0, 1, q_3^2, q_4^2)$ (see (22)), $D_0 = \widetilde{D} = \{2\}$ (see (14)).
By (36)

$$N_{Q,1} = \left| \frac{\zeta_3}{\zeta_1} \right| + \left| \frac{\zeta_4}{\zeta_1} \right| \quad \text{and} \quad N_{Q,2} = |q_3^2| + |q_4^2|.$$

By (35),

$$\|Id - Q\|_1 = 1 + \max \left\{ \frac{|\zeta_3| + |\zeta_4|}{|\zeta_1|}; |q_3^2| + |q_4^2| \right\}.$$

Hence, for every $Q \in \mathcal{P}\left(\mathbb{R}^\ell, g^1, g^2; \zeta\right)$,

$$\|Id - Q\|_1 \geq 1 + \frac{|\zeta_3| + |\zeta_4|}{|\zeta_1|}.$$

and the bound is attained for any projection with $|q_3^2| \geq \left|\frac{\zeta_3}{\zeta_1}\right|$ and $|q_4^2| \geq \left|\frac{\zeta_4}{\zeta_1}\right|$. There are infinitely many such projections save for the situation when $\zeta_3 = \zeta_4 = 0$ in which we have to take $q_3^2 = q_4^2 = 0$ and the unique minimal projection has the smallest possible norm 1.
Now suppose that $\zeta_2 \neq 0$. $g^1 = \left(\frac{1}{\zeta_1}, 0, 0, 0\right)$, $g^2 = \left(0, -\frac{1}{\zeta_2}, 0, 0\right)$ (see (15)), $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ with $\zeta_j \leq 0$ for all indices, and $q^2 = (0, -\zeta_2, q_3^2, q_4^2)$ (see (22)), $D_0 = \emptyset$, $\widetilde{D} = \{2\}$ (see (14))
By (36)

$$N_{Q,1} = \left|\frac{\zeta_3 + q_3^2}{\zeta_1}\right| + \left|\frac{\zeta_4 + q_4^2}{\zeta_1}\right|, \; N_{Q,2} = \frac{|q_3^2| + |q_4^2|}{|\zeta_2|}.$$

Now $|\zeta_1| \leq |\zeta_2|$, by (13). By (35),

$$\|Id - Q\|_1 = 1 + \max\left\{\left|\frac{\zeta_3 + q_3^2}{\zeta_1}\right| + \left|\frac{\zeta_4 + q_4^2}{\zeta_1}\right|; \frac{|q_3^2| + |q_4^2|}{|\zeta_2|}\right\}.$$

We consider four cases and calculate the norms for different projections.

1) $q_3^2 \leq 0 \wedge q_4^2 \leq 0$. Therefore, $\|Id - Q\|_1 = 1 + \frac{\zeta_3 + q_3^2}{\zeta_1} + \frac{\zeta_3 + q_3^2}{\zeta_1} \geq 1 + \frac{q_3^2 + q_4^2}{\zeta_2} \geq 0$ and if we take $q_3^2 = q_4^2 = 0$, then we achieve the minimal norm in this subfamily of projections $\|Id - Q\|_1 = 1 + \frac{|\zeta_3| + |\zeta_4|}{|\zeta_1|}$.

2) If $q_3^2 \geq 0 \wedge q_4^2 \leq 0$, $N_{Q,1} = \left|\frac{\zeta_3 + q_3^2}{\zeta_1}\right| + \frac{\zeta_4 + q_4^2}{\zeta_1} \geq \frac{q_3^2 + q_4^2}{\zeta_1} \geq \frac{q_3^2 + q_4^2}{\zeta_2} = N_{Q,2}$. Once again $\|Id - Q\|_1 = 1 + N_{Q,1}$ and the minimal norm in this subfamily is $\|Id - Q\|_1 = 1 + \frac{|\zeta_4|}{|\zeta_1|}$ which is obtained for $q_3^2 = -2\zeta_3 \geq 0$ and $q_4^2 = 0$.

3) If $q_3^2 \leq 0 \wedge q_4^2 \geq 0$, then we have to interchange the indices in 3) in order to obtain a similar result.

4) If $q_3^2 \geq 0 \wedge q_4^2 \geq 0$, we easily check that $N_{Q,1} \geq N_{Q,2}$ only if $\zeta_3 = \zeta_4 = 0$ and then the minimal norm is 1 which is attained for $q_3^2 = q_4^2 = 0$. On the other hand, if ζ_3, ζ_4 are not both equal to zero, then $\|Id - Q\|_1 = 1 + N_{Q,2}$ which means that the minimal possible norm is 1 and is attained again for $q_3^2 = q_4^2 = 0$.

□

Proof of Lemma 5. By Lemma 2, $\|Id - Q_0\|_\infty = \max\{1, M_{Q,l} : l \in \{d+1, \ldots, \ell\}\}$ (see (33)), where $M_{Q,l}$ is given by (34).

Suppose first that $\zeta_2 = 0$. Then $D_0 = \widetilde{D} = \{2\}$. Consequently $\widetilde{D} \backslash D_0 = \emptyset$ as well as, for $l \in \{3, \ldots, \ell\}$, $M_{Q,l} = \left|\frac{\zeta_l}{\zeta_1}\right| + \left|q_l^2\right|$ (see (34)). By the fact that $p \circ \zeta = 0$ (see (12)) we get that

$$-p_1 = \sum_{j=3}^{\ell} p_j \cdot \frac{\zeta_j}{\zeta_1}. \tag{A3}$$

Combining (18) and (22) we get that

$$-p_2 = \sum_{j=3}^{\ell} p_j q_j^2. \tag{A4}$$

By (A3) and (A4), the get the following:

$$-p_1 - p_2 \leq \sum_{j=3}^{\ell} p_j \cdot \left(\left|\frac{\zeta_j}{\zeta_1}\right| + \left|q_j^2\right|\right).$$

Hence, for $l \in \{3, \ldots, \ell\}$ and q^2 satisfying (18) and (22)

$$\max\left\{\left(\left|\frac{\zeta_l}{\zeta_1}\right| + \left|q_l^2\right|\right) : l \in \{3, \ldots, \ell\}\right\} \geq \frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j}$$

and consequently

$$\max\{M_{Q,l} : l \in \{3, \ldots, \ell\}\} \geq \frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j}. \tag{A5}$$

By the above, for every $Q \in \mathcal{P}\left(\mathbb{R}^\ell, V; p, \zeta\right)$

$$\|Id - Q\|_\infty \geq \frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j}.$$

For $l \in \{3, \ldots, \ell\}$ we put

$$q_l^2 = \frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j} - \frac{\zeta_l}{\zeta_1}. \tag{A6}$$

By (39), every $l \in \{3, \ldots, \ell\}$, $q_l^2 \geq 0$. Hence, for every $l \in \{3, \ldots, \ell\}$ and q_l^2 defined in (A6)

$$M_{Q,l} = \left|\frac{\zeta_l}{\zeta_1}\right| + \left|q_l^2\right| = \frac{\zeta_l}{\zeta_1} + \frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j} - \frac{\zeta_l}{\zeta_1} = \frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j},$$

which gives that, if projection Q_0 is determined by vectors ζ and q^2 of the form (A6), then it satisfies (40), i.e.

$$\|Id - Q_0\|_\infty = \max\left\{1, \frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j}\right\}$$

and Q_0 satisfies (29). Let us notice that

$$\max\{M_{Q,l} : l \in \{d+1, \ldots, \ell\}\} = \frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j}$$

if and only if, for every $l \in \{d+1, \ldots, \ell\}$,

$$\frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j} = M_{Q,l} \quad \text{and} \quad \left|\frac{\zeta_l}{\zeta_1}\right| + \left|q_l^2\right| = \frac{\zeta_l}{\zeta_1} + q_l^2.$$

Hence, if $\frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j} \geq 1$, then projection Q_0 satisfying (40) is the unique one.
If $\frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j} < 1$, then there are infinitely many projections Q_0 satisfying (40).
Now we assume that $\zeta_2 \neq 0$. Then $D_0 = \emptyset$ and $\widetilde{D} = \{2\}$. For $l \in \{3, \ldots, \ell\}$,

$$M_{Q,l} = \left|\frac{\zeta_l}{\zeta_1} + \frac{q_l^2}{\zeta_1}\right| + \left|\frac{q_l^2}{\zeta_2}\right|$$

(see (34)). By (18) and (22), we get that

$$-p_1 - p_2 \cdot \frac{\zeta_2}{\zeta_1} = \sum_{j=3}^{\ell} p_j \cdot \frac{\zeta_l}{\zeta_1} \quad \text{and} \quad p_2 \cdot \frac{\zeta_2}{\zeta_1} = \sum_{j=3}^{\ell} p_j \cdot \frac{q_l^2}{\zeta_1}. \tag{A7}$$

By (A7)

$$-p_1 = \sum_{j=3}^{\ell} p_j \cdot \left(\frac{\zeta_j}{\zeta_1} + \frac{q_j^2}{\zeta_1}\right). \tag{A8}$$

Additionally,

$$p_2 = \sum_{j=3}^{\ell} p_j \cdot \frac{q_j^2}{\zeta_2}. \tag{A9}$$

Combining (A8) and (A9), we get that for $l \in \{3, \ldots, \ell\}$,

$$\max\{M_{Q,l} : l \in \{d+1, \ldots, \ell\}\} = \max\left\{\left|\frac{\zeta_l}{\zeta_1} + \frac{q_l^2}{\zeta_1}\right| + \left|\frac{q_l^2}{\zeta_2}\right| : l \in \{d+1, \ldots, \ell\}\right\} \geq$$
$$\geq \frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j}.$$

As a result,

$$\|Id - Q\|_\infty \geq \frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j}$$

for $Q \in \mathcal{P}\left(\mathbb{R}^\ell, V; p, \zeta\right)$ determined by vectors ζ and q^2 of the form

$$q_l^2 = \left(\frac{-p_1 - p_2}{\sum_{j=3}^{\ell} p_j} - \frac{\zeta_l}{\zeta_1}\right) \cdot \left(\frac{1}{\zeta_1} + \frac{1}{\zeta_2}\right)^{-1}. \tag{A10}$$

It is easy to notice that for q_l^2 of the form (A10): $q_l^2 > 0$ and $\frac{\zeta_l}{\zeta_1} + \frac{q_l^2}{\zeta_1} > 0$. Reasoning as for $\zeta_2 = 0$ we get the result. □

Proof of Lemma 6. Let us suppose first that $\zeta_2 = 0$, i.e. $D_0 = \{2\}$ and so $\widetilde{D} \backslash D_0 = \emptyset$. By Lemma 2,

$$\|\mathrm{Id} - Q\|_1 = 1 + \max\{N_{Q,s} : s \in D\}$$

where

$$N_{Q,1} = \sum_{l=3}^{\ell} \left|\frac{\zeta_k}{\zeta_1}\right| \quad \text{and} \quad N_{Q,2} = \sum_{k=3}^{\ell} \left|q_k^2\right|$$

(see (35) and (36)). In addition $p \circ \zeta = 0$ (see (12)), thus

$$0 < -p_1 = \sum_{k=3}^{\ell} \left|\frac{\zeta_k}{\zeta_1}\right| p_k \leq \max\{p_j : j \in \{3, \ldots, \ell\}\} \cdot \sum_{k=3}^{\ell} \left|\frac{\zeta_k}{\zeta_1}\right|,$$

whence

$$N_{Q,1} \geq \frac{-p_1}{\max\{p_j : j \in \{3, \ldots, \ell\}\}} > 0.$$

Moreover $p \circ q^2 = 0$ (see (18)), thus

$$0 < -p_2 = \sum_{k=3}^{\ell} p_k q_k^2 \leq \max\{p_j : j \in \{3, \ldots, \ell\}\} \cdot \sum_{l=3}^{\ell} |q_k^2|,$$

whence

$$N_{Q,2} \geq \frac{-p_2}{\max\{p_j : j \in \{3, \ldots, \ell\}\}} > 0.$$

Assume in that $d = 2$, $\widetilde{D} \backslash D_0 = \{2\}$, i.e $\zeta_2 \neq 0$.
Additionally $p \circ \zeta = 0$ (see (12)), then $0 < -p_1 - \frac{\zeta_2}{\zeta_1} p_2 = \sum_{l=3}^{\ell} \left|\frac{\zeta_k}{\zeta_1}\right| p_k$ and $p \circ q^2 = 0$, so

$$0 < -p_2 = \sum_{k=3}^{\ell} p_k \frac{q_k^2}{-\zeta_2} \leq \max\{p_j : j \in \{3, \ldots, \ell\}\} \sum_{k=3}^{\ell} \frac{q_k^2}{-\zeta_2} \quad \text{and} \quad \zeta_2 < 0,$$

whence

$$N_{Q,2} \geq \frac{-p_2}{\max\{p_j : j \in \{3,\dots,\ell\}\}} > 0.$$

From this we calculate

$$p_2 \frac{\zeta_2}{\zeta_1} = \sum_{k=3}^{\ell} p_k \frac{q_k^2}{\zeta_1}.$$

Hence

$$-p_1 = \sum_{k=3}^{\ell} p_k \frac{\zeta_1 + q_k^2}{\zeta_1} \leq \max\{p_j : j \in \{3,\dots,\ell\}\} \cdot \sum_{k=3}^{\ell} \left| \frac{\zeta_1 + q_k^2}{\zeta_1} \right|.$$

Thus $N_{Q,1} \geq \frac{-p_1}{\max\{p_j : j \in \{3,\dots,\ell\}\}} > 0$.
Let now again $\zeta_2 = 0$ and assume that $p_2 \leq p_1$. Then we take any index $j_0 \geq 3$ for which $p_{j_0} = \max\{p_j : j \in \{3,\dots,\ell\}\}$ and define the vector q^2 by $q_2^2 = 1, q_{j_0}^2 = -\frac{p_2}{p_{j\,0}}$ and $q_j^2 = 0$ for other j. Clearly, $p \circ q^2 = 0$ as required and now

$$N_{Q,2} = -\frac{p_2}{p_{j_0}} = \frac{\max\{-p_1, -p_2\}}{\max\{p_j : j \in \{3,\dots,\ell\}\}}$$

which readily implies that

$$\|Id - Q_0\|_1 = 1 + N_{Q,2}$$

for projection Q_0 determined by ζ and q^2 according to (A6).
Similarly, if $\zeta_2 \geq 0$ but still $p_2 \leq p_1$, we define vector q^2 by $q_2^2 = -\zeta_2$, $q_{j_0}^2 = -\frac{\zeta_2 p_2}{p_{j\,0}}$ and $q_j^2 = 0$, for other j, so that

$$N_{Q,2} = \frac{\max\{-p_1, -p_2\}}{\max\{p_j : j \in \{3,\dots,\ell\}\}}.$$

This implies again that $\|Id - Q_0\|_1 = 1 + N_{Q,2}$ for the associated projection Q. □