

Exponential decay of transient values in discrete-time positive nonlinear systems

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*Dedicated to Professor Andrzej Świerniak
on the occasion of his 70th birthday*

The exponential decay of transient values in discrete-time nonlinear standard and fractional orders systems with linear positive linear part and positive feedbacks is investigated. Sufficient conditions for the exponential decay of transient values in this class of positive nonlinear systems are established. A procedure for computation of gains characterizing the class of nonlinear elements are given and illustrated on simple example.

Key words: exponential decay, transient value, discrete-time, fractional order, positive, nonlinear, feedback, system

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values for any nonnegative inputs and nonnegative initial conditions [1, 2, 10, 13, 17, 21]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollutions models and electrical circuits. A variety of models having positive behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive systems theory is given in the monographs [1, 2, 11, 13].

Mathematical fundamentals of the fractional calculus are given in the monographs [11, 18, 19]. The positive fractional linear systems have been investigated

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in [3, 4, 6–13, 21]. Positive linear systems with different fractional orders have been addressed in [1, 9, 21]. Linear positive electrical circuits have been investigated in [13]. The global stability of nonlinear systems with positive feedbacks and positive stable linear parts has been investigated in [6–8, 14] and the stability of discrete-time systems with delays in [20]. The exponential decay of transient values in continuous-time positive nonlinear systems has been analyzed in [5].

In this paper the exponential decay of transient values in discrete-time nonlinear standard and fractional positive systems with positive feedbacks will be addressed.

The paper is organized as follows. In Section 2 the basic definitions and theorems concerning the positivity and stability of fractional orders linear systems are recalled. The asymptotic stability of interval fractional linear systems with interval matrices are considered in Section 3. The main results of the paper are given in Sections 4 and 5. In Section 4 sufficient conditions for the global stability and the exponential decay of transient values in the positive nonlinear systems are established and procedures for computation of the gains characterizing the class of characteristics of nonlinear elements are given. In Section 5 the results of Section 4 are extended to the fractional nonlinear positive systems. The considerations are illustrated by numerical examples. Concluding remarks are given in Section 6.

The following notation will be used: \mathfrak{R} – the set of real numbers, $\mathfrak{R}^{n \times m}$ – the set of $n \times m$ real matrices, $\mathfrak{R}_+^{n \times m}$ – the set of $n \times m$ real matrices with nonnegative entries and $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$, M_n – the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n \times n$ identity matrix.

2. Fractional positive discrete-time linear systems

Consider the autonomous fractional discrete-time linear system

$$\Delta^\alpha x_{i+1} = Ax_i, \quad 0 < \alpha < 1, \quad i \in \mathbb{Z}_+, \quad (1)$$

where

$$\Delta^\alpha x_i = \sum_{j=1}^i c_j x_{i-j}, \quad (2a)$$

$$c_j = (-1)^j \binom{\alpha}{j}, \quad \binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0, \\ \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & \text{for } j = 1, 2, \dots \end{cases} \quad (2b)$$

is the fractional α -order difference of x_i and $x_i \in \mathfrak{R}^n$, $u_i \in \mathfrak{R}^m$ are the state and input vectors and $A \in \mathfrak{R}^{n \times n}$.

Substitution of (2) into (1) yields

$$x_{i+1} = A_\alpha x_i - \sum_{j=2}^{i+1} c_j x_{i-j+1}, \quad i \in Z_+, \quad (3a)$$

where

$$A_\alpha = A + I_n \alpha. \quad (3b)$$

Definition 1 [11, 20] *The fractional system (1) is called (internally) positive if $x_i \in \mathfrak{R}_+^n$, $i \in Z_+$ for any initial conditions $x_0 \in \mathfrak{R}_+^n$.*

Theorem 1 [11, 20] *The fractional system (1) is positive if and only if*

$$A_\alpha \in \mathfrak{R}_+^{n \times n}. \quad (4)$$

Proof is given in [11].

Definition 2 *The fractional positive system (1) is called asymptotically stable if*

$$\lim_{i \rightarrow \infty} x_i = 0 \quad \text{for all } x_0 \in \mathfrak{R}_+^n. \quad (5)$$

Theorem 2 [11] *The fractional positive system (1) is asymptotically stable if and only if one of the equivalent conditions is satisfied:*

1. *All coefficient of the characteristic polynomial*

$$p_A(z) = \det [I_n(z+1) - A] = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0 \quad (6)$$

are positive, i.e. $a_k > 0$ for $k = 0, 1, \dots, n-1$.

2. *There exists strictly positive vector $\lambda^T = [\lambda_1 \quad \dots \quad \lambda_n]^T$, $\lambda_k > 0$, $k = 1, \dots, n$ such that*

$$[A - I_n]\lambda < 0. \quad (7)$$

Proof is given in [11].

Theorem 3 *The positive system (1) is asymptotically stable if the sum of entries of each column (row) of the matrix A is less than one.*

Proof. The proof follows from condition (7) for $\lambda^T = [1 \quad \dots \quad 1]$ since $[1 \quad \dots \quad 1]^T A < [1 \quad \dots \quad 1]$ if the sum of entries of each column of the matrix A is less than 1. Proof for rows is similar. □

3. Fractional interval positive linear discrete-time systems

Consider the interval fractional positive discrete-time linear system (1) with the interval matrix $A \in \mathfrak{R}_+^{n \times n}$ defined by

$$A_1 \leq A \leq A_2 \quad \text{or equivalently} \quad A \in [A_1, A_2]. \quad (8)$$

Definition 3 *The interval fractional positive system with (8) is called asymptotically stable if the system is asymptotically stable for all matrices $A \in \mathfrak{R}_+^{n \times n}$ belonging to the interval $[A_1, A_2]$.*

Definition 4 *The matrix*

$$A = (1 - k)A_1 + kA_2, \quad 0 \leq k \leq 1, \quad A_1 \in \mathfrak{R}^{n \times n}, \quad A_2 \in \mathfrak{R}^{n \times n} \quad (9)$$

is called the convex linear combination of the matrices A_1 and A_2 .

Theorem 4 *The convex linear combination (9) is asymptotically stable if and only if the matrices $A_1 \in \mathfrak{R}^{n \times n}$ and $A_2 \in \mathfrak{R}^{n \times n}$ are asymptotically stable.*

For two fractional positive linear systems

$$x_{1,i+1} = A_1 x_{1,i}, \quad A_1 \in \mathfrak{R}_+^{n \times n} \quad (10a)$$

and

$$x_{2,i+1} = A_2 x_{2,i}, \quad A_2 \in \mathfrak{R}_+^{n \times n} \quad (10b)$$

there exists a strictly positive vector $\lambda \in \mathfrak{R}_+^n$ such that

$$A_1 \lambda < \lambda \quad \text{and} \quad A_2 \lambda < \lambda \quad (11)$$

if and only if the systems (10) are asymptotically stable.

Theorem 5 *If the matrices A_1 and A_2 of fractional positive systems (10) are asymptotically stable then their convex linear combination*

$$A = (1 - k)A_1 + kA_2 \quad \text{for} \quad 0 \leq k \leq 1 \quad (12)$$

is also asymptotically stable.

Proof. By condition (7) of Theorem 2 if the fractional positive linear systems (10) are asymptotically stable then there exists strictly positive vector $\lambda \in \mathfrak{R}_+^n$ such that

$$A_1 \lambda < \lambda \quad \text{and} \quad A_2 \lambda < \lambda. \quad (13)$$

Using (7) and (13) we obtain

$$A \lambda = [(1 - k)A_1 + kA_2] \lambda = (1 - k)A_1 \lambda + kA_2 \lambda < \lambda$$

for $0 \leq k \leq 1$. Therefore, if the positive linear systems (10) are asymptotically stable then their convex linear combination (12) is also asymptotically stable. \square

Theorem 6 *The interval positive systems (9) are asymptotically stable if and only if the positive linear systems (10) are asymptotically stable.*

Proof. By condition (7) of Theorem 2 if the matrices $A_1 \in \mathfrak{R}_+^{n \times n}$, $A_2 \in \mathfrak{R}_+^{n \times n}$ are asymptotically stable then there exists a strictly positive vector $\lambda \in \mathfrak{R}_+^n$ such that (7) holds. The convex linear combination (12) satisfies the condition $A\lambda < \lambda$ if and only if (13) holds. Therefore, the interval system (9) is asymptotically stable if and only if the positive linear system is asymptotically stable. \square

4. Exponential decay of transient values in feedback positive nonlinear systems with interval matrices

Consider the nonlinear multi-input multi-output feedback system shown in Fig. 1 which consists of the nonlinear element with matrix characteristic $u = f(e)$, positive linear part with interval matrices and feedback with gain matrix H .

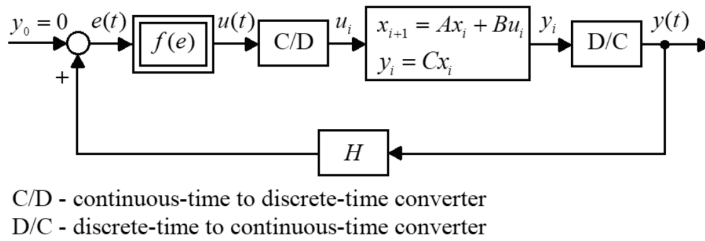


Figure 1: The positive nonlinear feedback system

The linear part is described by the equations

$$x_{i+1} = Ax_i + Bu_i, \quad i \in \mathbb{Z}_+ = \{0, 1, \dots\}, \quad (14a)$$

$$y_i = Cx_i, \quad (14b)$$

where $x_i \in \mathfrak{R}^n$, $u_i \in \mathfrak{R}^m$, $y_i \in \mathfrak{R}^p$ are the state, input and output vectors of the system $A \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times m}$, $C \in \mathfrak{R}^{p \times n}$ are interval

$$A_1 \leq A \leq A_2, \quad B_1 \leq B \leq B_2, \quad C_1 \leq C \leq C_2. \quad (14c)$$

The matrix characteristic of the nonlinear element satisfies the condition

$$u \leq Ke, \quad u_i = f(e_i) \leq k_{i1}e_1 + \dots + k_{ip}e_p, \quad i = 1, \dots, m, \quad (15a)$$

where

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad K = \begin{bmatrix} k_{11} & \cdots & k_{1p} \\ \vdots & \ddots & \vdots \\ k_{m1} & \cdots & k_{mp} \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ \vdots \\ e_p \end{bmatrix}, \quad (15b)$$

In general case the feedback matrix H is not square.

Theorem 7 *The multi-input multi-output nonlinear discrete-time system consisting of the positive linear part with interval matrices (14c), the matrix nonlinear element satisfying the condition (15) and the feedback with the matrix $H \in \mathfrak{R}_+^{m \times p}$ is globally stable if there exists a matrix K with positive entries such that the sum of entries of each column (row) of the matrix*

$$(1 - q)A_1 + qA_2 + BKHC = \begin{cases} A_1 + B_1K_1HC_1 \in \mathfrak{R}_+^{n \times n} & \text{for } q = 0, \\ A_2 + B_2K_2HC_2 \in \mathfrak{R}_+^{n \times n} & \text{for } q = 1 \end{cases} \quad (16)$$

is less than one.

Proof. As the Lyapunov [15, 16] function $V(x_i)$ we choose

$$V(x_i) = \lambda^T x_i \geq 0 \quad \text{for } x_i \in \mathfrak{R}_+^n, \quad i \in \mathbb{Z}_+, \quad (17)$$

where $\lambda \in \mathfrak{R}_+^n$ is strictly positive vector.

Using (17) and (14) we obtain

$$\begin{aligned} \Delta V(x_i) &= V(x_{i+1}) - V(x_i) = \lambda^T (Ax_i + Bu_i) - \lambda^T x_i \\ &= \lambda^T (Ax_i + Bf(e_i)) - \lambda^T x_i \leq \lambda^T [(A - I_n) + BKHC]x_i \end{aligned} \quad (18)$$

since $u \leq Ke = KHCx_i$.

By Theorem 3 from (18) it follows that $\Delta V(x_i) < 0$ if the sum of entries of each column (row) of the matrix (16) is less than one. \square

To find the maximal matrix K for which the nonlinear system shown in Fig. 1 is globally stable the following procedure can be used.

Procedure 1

Step 1. Find the matrix K_1 such that the sum of entries of each column of the matrix

$$A_1 + B_1K_1HC_1 \in \mathfrak{R}_+^{n \times n} \quad (19)$$

is less than one.

If $mp > n$ then we choose $mp - n$ nonnegative entries of the matrix K and the remaining its entries (components of vector k) we compute as the solution of the linear matrix equation

$$Gk = h, \quad (20)$$

where the matrix G and the column vector h are defined by the sum of entries of each column (row) of the matrix (19).

Step 2. In a similar way as in Step 1 find the matrix K_2 such that the sum of entries of each column of the matrix

$$A_2 + B_2K_2HC_2 \in \mathfrak{R}_+^{n \times n} \quad (21)$$

is less than one.

Step 3. Knowing K_1 and K_2 find the desired matrix K which satisfies the conditions (19) and (21).

Example 1 Consider the nonlinear system with the positive linear part with the interval matrices

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.4 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.5 \end{bmatrix}, & B_1 &= \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.3 \end{bmatrix}, & C_2 &= \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.4 \end{bmatrix}, \end{aligned} \tag{22}$$

the matrix nonlinear element satisfying the condition (15) and the matrix

$$H = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}. \tag{23}$$

Find the maximal matrix K for which the nonlinear system is globally stable.

Using Procedure 1 we obtain

Step 1. Using (19) and (22) we obtain

$$\begin{aligned} A_1 + B_1 K_1 H C_1 &= \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.4 \end{bmatrix} + \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \end{bmatrix} \begin{bmatrix} 0.4 & 0.3 \\ 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.3 \end{bmatrix} \\ &= \begin{bmatrix} 0.052k_{11} + 0.052k_{12} + 0.3 & 0.036k_{11} + 0.076k_{12} + 0.2 \\ 0.039k_{11} + 0.039k_{12} + 0.2 & 0.027k_{11} + 0.057k_{12} + 0.4 \end{bmatrix}. \end{aligned} \tag{24}$$

The sum of entries of the first column of the matrix (24) is $0.091k_{11} + 0.091k_{12} + 0.5$ and the sum of entries of the second column of the matrix (24) is $0.063k_{11} + 0.133k_{12} + 0.6$.

Solving the system of linear inequalities

$$\begin{cases} 0.091k_{11} + 0.091k_{12} + 0.5 < 1, \\ 0.063k_{11} + 0.133k_{12} + 0.6 < 1 \end{cases}. \tag{25}$$

we obtain $k_{11} < 4.725$ and $k_{12} < 0.769$.

Therefore, the maximal K_1 for which the matrix (24) is Schur is $K_1 = [4.725 \ 0.769]$.

Step 2. Using (21) and (22) we obtain

$$\begin{aligned} A_2 + B_2 K_2 H C_2 &= \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.5 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix} \begin{bmatrix} k_{12} & k_{21} \end{bmatrix} \begin{bmatrix} 0.4 & 0.3 \\ 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.4 \end{bmatrix} \\ &= \begin{bmatrix} 0.085k_{11} + 0.1k_{12} + 0.5 & 0.065k_{11} + 0.13k_{12} + 0.3 \\ 0.068k_{11} + 0.08k_{12} + 0.3 & 0.052k_{11} + 0.104k_{12} + 0.5 \end{bmatrix}. \end{aligned} \tag{26}$$

The sum of entries of the first column of the matrix (26) is $0.153k_{11} + 0.18k_{12} + 0.8$ and the sum of entries of the second column of the matrix (26) is $0.117k_{11} + 0.234k_{12} + 0.8$.

Solving the system of linear inequalities

$$\begin{cases} 0.153k_{11} + 0.18k_{12} + 0.8 < 1, \\ 0.117k_{11} + 0.234k_{12} + 0.8 < 1 \end{cases} \quad (27)$$

we obtain $k_{11} < 0.733$ and $k_{12} < 0.488$.

Therefore, the maximal K_2 for which the matrix (26) is Schur is $K_2 = [0.733 \ 0.488]$.

Step 3. Using the results obtained in Steps 1 and 2 we obtain that the maximal K for which the matrices (24) and (26) are Schur is $K = [0.733 \ 0.488]$.

Lemma 1 *If $z_k, k = 1, \dots, n$ are the eigenvalues of the matrix $A \in \mathfrak{X}_+^{n \times n}$, then the matrix $A - I_n \gamma$ has the eigenvalues $z_k + \gamma, k = 1, \dots, n, 0 \leq \gamma < 1$.*

Proof. Let $z_k, k = 1, \dots, n$ be the roots of the equation

$$\det [I_n z - A] = 0 \quad (28)$$

then

$$\det [I_n z - (A - I_n \gamma)] = \det [I_n(z + \gamma) - A] = 0. \quad (29)$$

Therefore, if $z_k, k = 1, \dots, n$ are the eigenvalues of the matrix A then $z_k + \gamma, k = 1, \dots, n$, are the eigenvalues of the matrix $A - I_n \gamma$. \square

From Lemma 1 it follows that if the eigenvalues of the matrix A are located in the unit circle then the eigenvalues of the matrix $A - I_n \gamma$ are located in the circle with the radius $1 - \gamma$. In this case transient values of the linear system are decreasing faster than $(1 - \gamma)^i$ for $i = 1, 2, \dots$.

Example 2 The asymptotically stable matrix with positive entries

$$A = \begin{bmatrix} 0.6 & 0.2 \\ 0.1 & 0.7 \end{bmatrix} \quad (30)$$

with the characteristic polynomial

$$\det [I_2 z - A] = \begin{vmatrix} z - 0.6 & -0.2 \\ -0.1 & z - 0.7 \end{vmatrix} = z^2 - 1.3z + 0.4 \quad (31)$$

has the eigenvalues: $z_1 = 0.5, z_2 = 0.8$.

The matrix

$$A_\gamma = A - I_2 \gamma = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.5 \end{bmatrix}, \quad \gamma = 0.2 \quad (32)$$

has the characteristic polynomial

$$\det [I_2z - A_\gamma] = \begin{vmatrix} z - 0.4 & -0.2 \\ -0.1 & z - 0.5 \end{vmatrix} = z^2 - 0.9z + 0.18 \quad (33)$$

and the eigenvalues $z_1 = 0.3, \quad z_2 = 0.6$.

This simple example confirms the Lemma 1.

From the above considerations and Theorem 7 we have the following theorem.

Theorem 8 *Transient values in the multi-input multi-output nonlinear system consisting of the positive linear part with interval matrices (14c), the matrix nonlinear element satisfying the condition (15) and the feedback with the matrix H are decreasing faster than $(1 - \gamma)^i$ for $i = 1, 2, \dots$ if the sum of entries of each column(row) of the matrix*

$$\begin{aligned} & (1 - q)(A_1 - I_n\gamma) + q(A_2 - I_n\gamma) + BKHC \\ &= \begin{cases} A_1 - I_n\gamma + B_1K_1HC_1 \in \mathfrak{R}_+^{n \times n} & \text{for } q = 0, \\ A_2 - I_n\gamma + B_2K_2HC_2 \in \mathfrak{R}_+^{n \times n} & \text{for } q = 1 \end{cases} \end{aligned} \quad (34)$$

is less than one.

5. Fractional positive feedback nonlinear systems

Consider the nonlinear system shown in Fig. 1 which consists of the positive linear part, nonlinear element with characteristic $u = f(e)$ and the scalar gain feedback h . The linear part is described by the equations (3) with interval matrix A satisfying the condition (8). To simplify the notion it is assumed that the scalar characteristic $u = f(e)$ satisfies the condition

$$0 < f(e) < ke, \quad 0 \leq k < \infty. \quad (35)$$

The following theorem gives sufficient conditions for the decreasing of the transient values in the nonlinear system faster than $(1 - \gamma)^i$ for $i = 1, 2, \dots$

Theorem 9 *Transient values in the fractional nonlinear discrete-time system shown in Fig. 1 with the positive linear part (9), the nonlinear element satisfying the condition (35) and the scalar positive feedback are decreasing faster than $(1 - \gamma)^i$ for $i = 1, 2, \dots$ if the sum of entries of each column (row) of the matrix*

$$\begin{aligned} & (1 - q)(A_1 - I_n\gamma) + q(A_2 - I_n\gamma) + khBC \\ &= \begin{cases} A_1 - I_n\gamma + khB_1C_1 \in \mathfrak{R}_+^{n \times n} & \text{for } q = 0, \\ A_2 - I_n\gamma + khB_2C_2 \in \mathfrak{R}_+^{n \times n} & \text{for } q = 1 \end{cases} \end{aligned} \quad (36)$$

is less than one.

Proof. As the Lyapunov function $V(x)$ for the system with not interval matrices we choose

$$V(x_i) = \lambda^T x_i \geq 0 \quad \text{for } x_i \in \mathfrak{X}_+^n, \quad i \in \mathbb{Z}_+, \quad (37)$$

where $\lambda \in \mathfrak{X}_+^n$ is strictly positive vector.

Using (37) and (35) we obtain

$$\begin{aligned} \Delta V(x_i) &= V(x_{i+1}) - V(x_i) = \lambda^T x_{i+1} - \lambda^T x_i \\ &= \lambda^T \left(A_\alpha x_i + \sum_{k=2}^{\infty} c_k x_{i-k+1} + B u_i \right) - \lambda^T x_i \\ &= \lambda^T \left[(A_\alpha - I_n) x_i + \sum_{k=2}^{\infty} c_k x_{i-k+1} + h B f(e_i) \right] \\ &\leq \lambda^T \left[(A_\alpha - I_n + k h B C) x_i + \sum_{k=2}^{\infty} c_k x_{i-k+1} \right] \end{aligned} \quad (38)$$

Taking into account that

$$\sum_{k=2}^{\infty} c_k x_{i-k+1} \leq \sum_{k=2}^{\infty} c_k x_i = (1 - \alpha) x_i \quad (39)$$

we obtain

$$\Delta V(x_i) = \lambda^T [A_\alpha + I_n \alpha - I_n + k h B C + (1 - \alpha) I_n] x_i = \lambda^T [A + k h B C] x_i < 0. \quad (40)$$

For the nonlinear system with interval matrices (9) we obtain (36). \square

The considerations can be easily extended to the systems with multi-input multi-output nonlinear systems satisfying the condition (15).

6. Concluding remarks

The exponential decay of transient values in the discrete-time nonlinear standard and fractional orders with positive linear parts and linear positive feedbacks has been investigated. Sufficient conditions for the exponential decay of transient values in this class of positive nonlinear systems have been established (Theorems 7 and 9). Procedure for computation of gains characterizing the class of nonlinear elements are given and illustrated on simple example. The considerations have been extended to fractional order nonlinear systems. An open problem is an extension of the consideration to different fractional orders nonlinear systems.

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