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Multi-attribute decision-making based on q-rung dual hesitant power dual Maclaurin symmetric mean operator and a new ranking method

Li LI, Jun WANG and Chunliang JI

The ability of q-rung dual hesitant fuzzy sets (q-RDHFSs) in dealing with decision makers' fuzzy evaluation information has received much attention. This main aim of this paper is to propose new aggregation operators of q-rung dual hesitant fuzzy elements and employ them in multi-attribute decision making (MADM). In order to do this, we first propose the power dual Maclaurin symmetric mean (PDMSM) operator by integrating the power geometric (PG) operator and the dual Maclaurin symmetric mean (DMSM). The PG operator can reduce or eliminate the negative influence of decision makers' extreme evaluation values, making the final decision results more reasonable. The DMSM captures the interrelationship among multiple attributes. The PDMSM takes the advantages of both PG and DMSM and hence it is suitable and powerful to fuse decision information. Further, we extend the PDMSM operator to q-RDHFSs and propose q-rung dual hesitant fuzzy PDMSM operator and its weighted form. Properties of these operators are investigated. Afterwards, a new MADM method under q-RDHFSs is proposed on the basis on the new operators. Finally, the effectiveness of the new method is testified through numerical examples.

Key words: q-rung dual hesitant fuzzy sets, power geometric, dual Maclaurin symmetric mean, power dual Maclaurin symmetric mean, multi-attribute decision-making

1. Introduction

Multiple attribute decision-making (MADM) theory aims to rank feasible alternatives based on decision makers' (DMs') evaluation values under multiple

L. Li is with the School of Economics and Management, Beihang University, Beijing 100191, China.

J. Wang (corresponding author, e-mail: junwang_jg@mail.buct.edu.cn) is with the School of Economics and Management, Beijing University of Chemical Technology, Beijing 100029, China.

Chunliang Ji is with the School of Economics and Management, Beijing Jiaotong University, Beijing 100044, China.

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attributes. MADM has a wide range of applications and it has gotten extensive interests [1–6]. One of the methods to get the rank of candidate alternatives is to arrange all the possible alternatives according to their corresponding overall evaluation values, which can be obtained by aggregating attribute values provided by DMs under multiple attributes. Hence, aggregation operator (AO) is an efficient and powerful tool in MADM. The power average (PA) operator proposed by Prof. Yager [7], which allows argument values to support each other in the information aggregation process, is a powerful information aggregation technology. The most prominent characteristic of the PA operator is that it is capable to reduce or eliminate the negative effect of unduly high or low arguments on the information aggregation results. In MADM process, the PA operator can reduce the bad influence of DMs extreme evaluation values, making the final decision results more reasonable. In addition, due to the complexity and uncertainty of MADM problems, the intuitionistic fuzzy sets [8], hesitant fuzzy sets [9], dual hesitant fuzzy sets [10], Pythagorean fuzzy sets [11] and q-rung orthopair fuzzy sets (q-ROFSs) [12], etc., have been extensively employed to express DMs' complicated and fuzzy evaluation values. Hence, the traditional PA operator has been extensively to the above-mentioned decision-making environment to propose new MADM methods [13–17]. Moreover, there usually exists interrelationship between multiple attributes when calculating the comprehensive evaluation values. The Maclaurin symmetric mean (MSM) [18] operator has the ability of reflecting the interrelationship among multiple correlated. Similarly, the MSM operator has been also widely generalized to different decision-making environments. Qin and Liu [19] originally extended MSM to IFSs and proposed weighted intuitionistic fuzzy MSM operator to deal with MADM problems. Thereafter, Wang and Liu [20] and Liu PD and Liu WQ [21] proposed new intuitionistic fuzzy MSM operator under Schweizer-Sklar operations and interaction operations, respectively. Sun and Xia [22] continued to investigate MSM under interval-valued IFSs. Wei and Lu [23] and Wei et al. [24] extended MSM to PFSs and interval-valued PFSs, respectively. Similarly, Wei and his colleagues studied the new forms of MSM under q-ROFSs [25] and interval-valued q-ROFSs [26], respectively. Besides, the MSM operator has been extended to probabilistic linguistic sets [27], two-dimensional uncertain linguistic sets [28], etc.

Recently, Teng et al. [29] combined PA with MSM and proposed the power Maclaurin symmetric mean (PMSM) operator. The PMSM operator takes the full advantages of PA and MSM. That is to say, the PMSM operator has the capability of reducing the negative influence of DMs' unreasonable evaluation values and reflecting the interrelationship among multiple input arguments. Due to these characteristics, the PMSM has been incorporated in Pythagorean fuzzy linguistic sets [29], interval-valued intuitionistic fuzzy sets [30], and q-ROFSs [31], and a series of novel AOs have been proposed. In these references, the advantages and superiorities of these new AOs were investigated in detail, and they were also successfully applied in solving practical MADM problems. In the theory of AO, each AO has its dual form. For example, the dual form of Bonferroni mean (BM) [32] is geometric Bonferroni mean (GBM) [33]. The dual form of Heronian mean (HM) [34] is geometric Heronian mean (GHM) [35]. The dual form of power BM operator [36] is power GBM (PGBM) operator [37]. The dual form of power HM operator [38] is power GHM (PGHM) operator [39]. Hence, the PMSM operator also has a dual form, i.e. the power dual Maclaurin symmetric mean (PDMSM). Actually, the PDMSM is a combination of the power geometric (PG) [40] with the dual Maclaurin symmetric mean (DMSM) [41]. The PDMSM has the similar characteristics as the PMSM operator. In addition, the PDMSM is more powerful than the PGBM and PGHM, as it proceeds the interrelationship among multiple input variables. Therefore, it is necessary to investigate the PDMSM as well as its applications in decision-making. Nevertheless, up to present, nothing has been done on the PDMSM and its applications. This is the first motivation of this paper. In our works, we give the definition of PDMSM operator, investigate its properties and further study its applications.

Additionally, the q-RDHFS proposed by Xu et al. [42] is an efficient tool to depict DMs' evaluation values. The advantages of the q-RDHFS are two-fold. First, due to its lax constraint that the sum of q-th power of membership degree and q-th power of non-membership degree is less than or equal to one, the q-RDHFS gives enough freedom for DMs to comprehensively express their evaluation values. Second, the q-RDHFSs effectively deal with DMs' hesitancy when providing their evaluation information. In [42], Xu et al. proposed a MADM method based on q-rung dual hesitant fuzzy (q-RDHF) HM operators. However, Xu et al.'s [42] method still have some shortcomings. First, it fails to effectively deal with DMs' unreasonable evaluation values. In other word, if DMs provide extreme evaluation values, the decision results produced by Xu et al.'s [42] method maybe become unreasonable or unreliable. Second, it does not consider the interrelationship that exists in multiple attributes.

Based on the above analysis, the main purpose of this paper is to propose novel decision-making method, which can overcome the drawbacks of Xu et al.'s [42] method. The main works and contributions of this paper are three-fold. First, a new AO, called PDMSM is proposed, which fulfills the theory vacancy. We combine the PG with DMSM, and present the definition of the PDMSM operator. Characteristics of the PDMSM operator are also discussed. Second, novel AOs for q-RDHF elements (q-RDHFEs) based on PDMSM are proposed. We generalize the PDMSM into q-RDHFSs and introduce the q-rung dual hesitant fuzzy PDMSM operator and its weighted form. The properties and special cases of the proposed AOs are also investigated. Third, a novel MADM method based on the proposed AOs is presented, and a series of numerical examples are further provided to show the validity and merits of our new method.

To clearly report our works and contributions, we organize the rest of this paper as follows. Section 2 reviews basic concepts. Section 3 develops a new score function and ranking method. Section 4 proposes a series of AOs and discusses their properties. Section 5 gives the main steps of a new MADM method. Section 6 demonstrates the effectiveness advantages of our method through numerical examples. Summarization and future research directions are given in Section 7.

2. Basic concepts

In this section, we review basic notions such as q-RDHFSs, PG, and DMSM operators. These concepts are important for the rest of this manuscript.

2.1. The q-rung dual hesitant fuzzy sets

Definition 1 *[42] Let X be a given fixed set, a q-RDHFS A defined on X is expressed as*

$$A = \{\langle x, h_A(x), g_A(x)\rangle \quad | x \in X\}, \tag{1}$$

where $h_A(x)$ and $g_A(x)$ are two sets of some values in the interval $[0, 1]$, denoting the possible MD and NMD of the element $x \in X$ to the set A, such that $0 \leqslant \gamma$, $\eta \leqslant 1$ and $\gamma^q + \eta^q \leqslant 1$, where $\gamma \in h_A(x)$ and $\eta \in g_A(x)$ for all $x \in X$. For convenience, the ordered pair $(h_A(x), g_A(x))$ is called a q-RDHFE, which can be denoted by $d = (h, g)$ for simplicity.

Xu et al. [42] gave some basic operations of q-RDHFEs.

Definition 2 *[42] Let $d_1 = (h_1, g_1)$, $d_2 = (h_2, g_2)$ and $d = (h, g)$ be any three q-RDHFEs and λ be a positive real number, then*

(1) $d_1 \oplus d_2 = \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2} \left\{\left\{\left(\gamma_1^q + \gamma_2^q - \gamma_1^q\gamma_2^q\right)^{1/q}\right\}, \{\eta_1\eta_2\}\right\}$;

(2) $d_1 \otimes d_2 = \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2} \left\{\{\gamma_1\gamma_2\}, \left\{\left(\eta_1^q + \eta_2^q - \eta_1^q\eta_2^q\right)^{1/q}\right\}\right\}$;

(3) $\lambda d = \bigcup_{\gamma\in h,\eta\in g} \left\{\left\{\left(1 - (1 - \gamma^q)^\lambda\right)^{1/q}\right\}, \{\eta^\lambda\}\right\}$;

(4) $d^\lambda = \bigcup_{\gamma\in h,\eta\in g} \left\{\{\gamma^\lambda\}, \left\{\left(1 - (1 - \eta^q)^\lambda\right)^{1/q}\right\}\right\}$.

Xu et al. [42] proposed a method to compare any two q-RDHFEs.

Definition 3 *[42] Let $d = (h, g)$ be a q-RDHFE, then the score function of d is defined as*

$$S(d) = \left(\frac{1}{\#h}\sum_{\gamma\in h}\gamma\right)^q - \left(\frac{1}{\#g}\sum_{\eta\in g}\eta\right)^q, \tag{2}$$

and the accuracy function of d is expressed as

$$H(d) = \left(\frac{1}{\#h}\sum_{\gamma\in h}\gamma\right)^q + \left(\frac{1}{\#g}\sum_{\eta\in g}\eta\right)^q, \tag{3}$$

For any two q-RDHFEs d_1 and d_2,

(1) If $S(d_1) > S(d_2)$, then $d_1 > d_2$;

(2) If $S(d_1) = S(d_2)$, then

if $H(d_1) > H(d_2)$, then $d_1 > d_2$;

if $H(d_1) = H(d_2)$, then $d_1 = d_2$.

In the following, we present the distance measure between any two q-RDHFEs.

Definition 4 *Let $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$ be any two q-RDHFEs, then the distance between d_1 and d_2 is expressed as*

$$\mathrm{dis}\,(d_1, d_2) = \frac{\sum_{i=1}^{\#h}\left|\left(\gamma^1_{\sigma(i)}\right)^q - \left(\gamma^2_{\sigma(i)}\right)^q\right| + \sum_{j=1}^{\#g}\left|\left(\eta^1_{\sigma(j)}\right)^q - \left(\eta^2_{\sigma(j)}\right)^q\right|}{\#h + \#g}, \tag{4}$$

where $\#h$ denotes the number of values of h_1 and h_2, and $\#g$ represents the number of values of g_1 and d_2. $\sigma(i)$ is a permutation of $(1, 2, \ldots, n)$, satisfying $\gamma^1_{\sigma(i)} \leqslant \gamma^1_{\sigma(i+1)}$, $\gamma^2_{\sigma(i)} \leqslant \gamma^2_{\sigma(i+1)}$, $\eta^1_{\sigma(j)} \leqslant \eta^1_{\sigma(j+1)}$, and $\eta^2_{\sigma(j)} \leqslant \eta^2_{\sigma(j+1)}$, where $\gamma^1_{\sigma(i)} \in h_1$, $\gamma^2_{\sigma(i)} \in h_2$, $\eta^1_{\sigma(j)} \in g_1$ and $\eta^2_{\sigma(j)} \in g_2$.

Remark 1 *From Definition 4, it is noted that when calculating the distance between any two q-RDHFEs, the two q-RDHFEs should have the numbers of MDs and NMDs. In other word, let $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$ be any two q-RDHFEs, then $\#h_1$ should be equal to $\#h_2$, and $\#g_1$ should be equal to $\#g_2$. However, in most situations, $\#h_1 \neq \#h_2$, and $\#g_1 \neq \#g_2$. To operate correctly, we usually extend the shorter one by adding some values until both of them have the same length. In addition, if DMs are optimistic about their evaluation values, they usually add the maximum value. If DMs are pessimistic about their decisions, they usually add the minimum value. In this paper, we assume DMs are optimistic* *about their decision values. Hence, we extend the shorter q-RDHFE by adding the maximum values. For example, let* $d_1 = \{\{0.1, 0.2\}, \{0.7, 0.8, 0.9\}\}$ *and* $d_2 = \{\{0.6, 0.7, 0.8\}, \{0.3, 0.4\}\}$ *be two q-RDHFEs (*$q = 2$*), then we can extend* d_1 *to* $\{\{0.1, 0.2, 0.2\}, \{0.7, 0.8, 0.9\}\}$ *and extend* d_2 *to* $\{\{0.6, 0.7, 0.8\}, \{0.3, 0.4, 0.4\}\}$. *Then, according to Eq. (4), we can obtain* $\mathrm{dis}(d_1, d_2) = 0.4883$.

2.2. Power geometric and dual Maclaurin symmetric mean

In the following, we review some basic AOs that will be used in the following sections.

Definition 5 *[40]. Let* a_i $(i = 1, 2, \ldots, n)$ *be a collection of non-negative real number, then PG operator is defined as*

$$PG\left(a_1, a_2, \ldots, a_n\right) = \prod_{i=1}^{n} a_i^{1+T(a_i) \Big/ \sum\limits_{i=1}^{n}(1+T(a_i))}, \tag{5}$$

where $T(a_i) = \sum\limits_{j=1; i \neq j}^{n} \mathrm{Sup}(a_i, a_j)$*, and* $\mathrm{Sup}(a_i, a_j)$ *is the support measure for* a_i *from* a_j*, satisfying the following conditions:*

(1) $\mathrm{Sup}\left(a_i, a_j\right) \in [0, 1]$*;*

(2) $\mathrm{Sup}\left(a_i, a_j\right) = \mathrm{Sup}\left(a_j, a_i\right)$*;*

(3) $\mathrm{Sup}\left(a_i, a_j\right) \geqslant \mathrm{Sup}\left(a_m, a_n\right)$*, if* $\left|a_i - a_j\right| \leqslant \left|a_m - a_n\right|$.

Qin and Liu [41] proposed the dual form of MSM, called DMSM operator.

Definition 6 *[41]. Let* a_i $(i = 1, 2, \ldots, n)$ *be a set of nonnegative real numbers and* $k = 1, 2, \ldots, n$*. If*

$$DMSM^{(k)}\left(a_1, a_2, \ldots, a_n\right) = \frac{1}{k}\left(\prod_{1 \leqslant i_1 < \ldots < i_k \leqslant n}\left(\sum_{j=1}^{k} a_{i_j}\right)^{1/C_n^k}\right), \tag{6}$$

then $DMSM^{(k)}$ *is called the dual Maclaurin symmetric mean (DMSM) operator, where* $(i_1, i_2, \ldots, i_k)$ *traverses all the k-tuple combinations of* $(i = 1, 2, \ldots, n)$ *and* C_n^k *is the binomial coefficient.*

3. A new score function and ranking method for q-RDHFEs

In this section, we aim to propose a new ranking method for q-RDHFEs by introducing a new score function, which enriches the theory of q-RDHFSs. Obviously, the positive ideal q-RDHFE is $d^+ = \{\{1\}, \{0\}\}$ and the negative ideal q-RDHFE is $d^- = \{\{0\}, \{1\}\}$. From Definition 4, we can obtain the distance between the q-RDHFE $d = (h, g)$ and the positive ideal q-RDHFE $d^+ = \{\{1\}, \{0\}\}$ as follows:

$$\operatorname{dis}\left(d, d^+\right) = \frac{\sum_{i=1}^{\#h} \left|\gamma_{\sigma(i)}^q - 1\right| + \sum_{j=1}^{\#g} \left|\eta_{\sigma(j)}^q\right|}{\#h + \#g} = \frac{\sum_{i=1}^{\#h} \left(1 - \gamma_{\sigma(i)}^q\right) + \sum_{j=1}^{\#g} \eta_{\sigma(j)}^q}{\#h + \#g}, \tag{7}$$

and the distance between the q-RDHFE $d = (h, g)$ and the negative ideal q-RDHFE $d^- = \{\{0\}, \{1\}\}$ as follows:

$$\operatorname{dis}(d, d^-) = \frac{\sum_{i=1}^{\#h} \left|\gamma_{\sigma(i)}^q\right| + \sum_{j=1}^{\#g} \left|\eta_{\sigma(j)}^q - 1\right|}{\#h + \#g} = \frac{\sum_{i=1}^{\#h} \gamma_{\sigma(i)}^q + \sum_{j=1}^{\#g} \left(1 - \eta_{\sigma(j)}^q\right)}{\#h + \#g}. \tag{8}$$

It is easy to find out that the smaller the distance $\operatorname{dis}(d, d^+)$ is, the bigger the q-RDHFE d is; the larger the distance $\operatorname{dis}(d, d^-)$ is, the bigger q-RDHFE d is. Hence, we propose a new score function for a q-RDHFE $d = (h, g)$. Based on this, we propose a novel ranking method for q-RDHFEs.

Definition 7 *Let $d = (h, g)$ be a q-RDHFE, $d^+ = \{\{1\}, \{0\}\}$ be the positive ideal q-RDHFE and $d^- = \{\{0\}, \{1\}\}$ be the negative ideal q-RDHFE, then the score function of d is defined as*

$$\begin{aligned} S(d) &= \frac{\operatorname{dis}\left(d, d^-\right)}{\operatorname{dis}\left(d, d^-\right) + \operatorname{dis}\left(d, d^+\right)} \\ &= \frac{\sum_{i=1}^{\#h} \gamma_{\sigma(i)}^q + \sum_{j=1}^{\#g} \left(1 - \eta_{\sigma(j)}^q\right)}{\sum_{i=1}^{\#h} \gamma_{\sigma(i)}^q + \sum_{j=1}^{\#g} \left(1 - \eta_{\sigma(j)}^q\right) + \sum_{i=1}^{\#h} \left(1 - \gamma_{\sigma(i)}^q\right) + \sum_{j=1}^{\#g} \eta_{\sigma(j)}^q}, \end{aligned} \tag{9}$$

For any two q-RDHFEs $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$, then if $S(d_1) \geqslant S(d_2)$, we have $d_1 \geqslant d_2$; if $S(d_1) < S(d_2)$, then we have $d_1 < d_2$.

Example 1 Let $d_1 = \{\{0.1, 0.2\}, \{0.7, 0.8, 0.9\}\}$ and $d_2 = \{\{0.6, 0.7, 0.8\}, \{0.3, 0.4\}\}$ be two q-RDHFEs ($q = 3$), then we can get that $S(d_1) = 0.285$ and $S(d_2) = 0.596$. Hence, we have $d_1 < d_2$ because $S(d_1) < S(d_2)$.

4. Power dual Maclaurin symmetric mean operator and its extension to q-RDHFSs

If we combine the PG with DMSM operators, the PDMSM operator can be obtained. Evidently, the PDMSM operator takes the advantages of both PG and DMSM operators. Hence, the PDMSM operator is useful and powerful in information aggregation process. In this section, we first give the definition of the PDMSM operator. Afterwards, we extend the PDMSM to q-RDHFSs and propose some novel AOs of q-RDHFEs.

4.1. The power dual Maclaurin symmetric

Definition 8 *Let a_i $(i = 1, 2, \dots, n)$ be a collection of crisp numbers and $k = 1, 2, \dots, n$. The PDMSM operator is expressed as*

$$PDMSM^{(k)}(a_1, a_2, \dots, a_n) = \frac{1}{k}\left(\prod_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(\sum_{j=1}^{k} a_{i_j}^{\frac{n\left(1+T\left(a_{i_j}\right)\right)}{\sum_{t=1}^{n}(1+T(a_t))}}\right)^{1/C_n^k}\right), \tag{10}$$

where $T(a_i) = \sum_{j=1;i\neq j}^{n} \mathrm{Sup}\left(a_i, a_j\right)$, and $\mathrm{Sup}\left(a_i, a_j\right)$ is the support measure for a_i from a_j, satisfying properties presented in Definition 5. If we assume

$$\varepsilon_i = \frac{1 + T(a_i)}{\sum_{t=1}^{n}(1 + T(a_t))}, \tag{11}$$

then Eq. (10) can be written as

$$PDMSM^{(k)}(a_1, a_2, \dots, a_n) = \frac{1}{k}\left(\prod_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(\sum_{j=1}^{k} a_{i_j}^{n\varepsilon_{i_j}}\right)^{1/C_n^k}\right), \tag{12}$$

where $0 \leqslant \varepsilon_i \leqslant 1$ and $\sum_{i=1}^{n}\varepsilon_i = 1$.

Remark 2 *From Definition 8, we can see the PDMSM operator is based on the combination of PG and DMSM. Hence, the PDMSM absorbs the advantages* *and superiorities of PG and DMSM. In other word, the PDMSM reduces the bad influence of the unduly high or low inputs variables on the final aggregation results and captures the interrelationship among multiple attributes. In addition, the PDMSM is the dual form of the PMSM operator. Moreover, some special cases of the PDMSM operator with different parameters can be obtained.*

Case 1: If $k = 1$, then the PDMSM operator reduces to PG operator, i.e.

$$PDMSM^{(k)}\left(a_1, a_2, \ldots, a_n\right) = \bigotimes_{i=1}^{n} a_i^{\varepsilon_{i_j}} = PG\left(a_1, a_2, \ldots, a_n\right). \tag{13}$$

Case 2: If $k = 2$, then the PDMSM operator reduces to the power geometric Bonferroni mean (PGBM) [43] operator, i.e.,

$$\begin{aligned} PDMSM^{(k)}\left(a_1, a_2, \ldots, a_n\right) &= \frac{1}{2}\left(\bigotimes_{i,j=1, i\neq j}^{n}\left(a_i^{n\varepsilon_i} \oplus a_j^{n\varepsilon_j}\right)^{\frac{2}{n(n-1)}}\right) \\ &= PGBM^{1,1}\left(a_1, a_2, \ldots, a_n\right). \end{aligned} \tag{14}$$

Definition 8 presents the PDMSM operator for crisp numbers. Given the good performance of PDMSM operator in aggregating information, it is necessary to extend PDMSM to q-RDHFSs. In the following, we present novel AOs of q-RDHFEs and discuss their properties.

4.2. The q-rung dual hesitant fuzzy power dual Maclaurin symmetric mean operator

By extending the powerful PDMSM operator to q-RDHFSs, we propose some novel AOs for q-RDHFEs and investigate their properties.

Definition 9 *Let d_i $(i = 1, 2, \ldots, n)$ be a collection of q-RDHFEs and $k = 1, 2, \ldots, n$. The q-rung dual hesitant fuzzy power dual Maclaurin symmetric mean (q-RDHFPDMSM) operator is expressed as*

$$q\text{-}RDHFPDMSM^{(k)}\left(d_1, d_2, \ldots, d_n\right) = \frac{1}{k}\left(\bigotimes_{\substack{1\leqslant i_1<\ldots \\ <i_k\leqslant n}}\left(\bigoplus_{j=1}^{k} d_{i_j}^{\frac{n\left(1+T\left(d_{i_j}\right)\right)}{\sum_{t=1}^{n}\left(1+T\left(d_t\right)\right)}}\right)^{1/C_n^k}\right), \tag{15}$$

where $T(d_i) = \sum_{j=1, i\neq j}^{n} \mathrm{Sup}\left(d_i, d_j\right)$, and $\mathrm{Sup}\left(d_i, d_j\right)$ is the support measure for d_i from d_j, satisfying the following condition:

(1) $0 \leqslant \mathrm{Sup}\left(d_i, d_j\right) \leqslant 1$*;*

(2) $\mathrm{Sup}\left(d_i, d_j\right) = \mathrm{Sup}\left(d_j, d_i\right)$*;*

(3) $\mathrm{Sup}\left(d_i, d_j\right) \geqslant \mathrm{Sup}\left(d_m, d_n\right)$*, if and only if* $\mathrm{dis}\left(d_i, d_j\right) \leqslant \mathrm{dis}\left(d_m, d_n\right)$.

For convenience, we assume

$$\sigma_i = \frac{1 + T\left(d_i\right)}{\sum_{t=1}^{n}\left(1 + T\left(d_t\right)\right)}, \tag{16}$$

then Eq. (15) can be written as

$$q\text{-}RDHFPDMSM^{(k)}\left(d_1, d_2, \ldots, d_n\right) = \frac{1}{k}\left(\bigotimes_{1 \leqslant i_1 < \ldots < i_k \leqslant n}\left(\bigoplus_{j=1}^{k} d_{i_j}^{n\sigma_{i_j}}\right)^{1/C_n^k}\right), \tag{17}$$

where $0 \leqslant \sigma_i \leqslant 1$ *and* $\sum_{i=1}^{n} \sigma_i = 1$.

According to the operations of q-RDHFEs presented in Definition 2, we can obtain the calculation result of the q-RDHFPDMSM operator.

Theorem 1 *Let* $d_i = \left(h_i, g_i\right)$ $(i = 1, 2, \ldots, n)$ *be a collection of q-RDHFEs and* $k = 1, 2, \ldots, n$*. The aggregated value by the q-RDHFPDMSM operator is still a q-RDHFE and*

$$\begin{aligned} q\text{-}RDHFPDMSM^{(k)}\left(d_1, d_2, \ldots, d_n\right) = \bigcup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \\ \left\{\left\{\left(1 - \left(1 - \prod_{1 \leqslant i_1 < \ldots < i_k \leqslant n}\left(1 - \prod_{j=1}^{k}\left(1 - \gamma_{i_j}^{nq\sigma_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right)^{1/q}\right\},\right. \\ \left.\left\{\left(1 - \prod_{1 \leqslant i_1 < \ldots < i_k \leqslant n}\left(1 - \prod_{j=1}^{k}\left(1 - \left(1 - \eta_{i_j}^{q}\right)^{n\sigma_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{qk}}\right\}\right\}. \end{aligned} \tag{18}$$

Proof. According to Definition 4, we can obtain

$$d_{i_j}^{n\sigma_{i_j}} = \bigcup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{\left\{\gamma_{i_j}^{n\sigma_{i_j}}\right\}, \left\{\left(1 - \left(1 - \eta_{i_j}^{q}\right)^{n\sigma_{i_j}}\right)^{1/q}\right\}\right\},$$

and

$$\bigoplus_{j=1}^{k} d_{i_j}^{n\sigma_{i_j}} = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{ \left\{ \left(1-\prod_{j=1}^{k}\left(1-\gamma_{i_j}^{nq\sigma_{i_j}}\right)\right)^{1/q} \right\}, \left\{ \prod_{j=1}^{k}\left(1-\left(1-\eta_{i_j}^{q}\right)^{n\sigma_{i_j}}\right)^{1/q} \right\} \right\}.$$

Thus,

$$\left(\bigoplus_{j=1}^{k} d_{i_j}^{n\sigma_{i_j}}\right)^{1/C_n^k} = \bigcup_{\gamma_{i_j}\in h_{i_j},\eta_{i_j}\in g_{i_j}} \left\{ \left\{ \left(1-\prod_{j=1}^{k}\left(1-\gamma_{i_j}^{nq\sigma_{i_j}}\right)\right)^{\frac{1}{qC_n^k}} \right\}, \left\{ \left(1-\left(1-\prod_{j=1}^{k}\left(1-\left(1-\eta_{i_j}^{q}\right)^{n\sigma_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{1/q} \right\} \right\}.$$

Then,

$$\bigotimes_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(\bigoplus_{j=1}^{k} d_{i_j}^{n\sigma_{i_j}}\right)^{1/C_n^k} = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{ \left\{ \prod_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\prod_{j=1}^{k}\left(1-\gamma_{i_j}^{nq\sigma_{i_j}}\right)\right)^{\frac{1}{qC_n^k}} \right\}, \left\{ \left(1-\prod_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\eta_{i_j}^{q}\right)^{n\sigma_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{1/q} \right\} \right\}.$$

Hence,

$$\frac{1}{k}\left(\bigotimes_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\bigoplus_{j=1}^{k} d_{i_j}^{n\sigma_{i_j}}\right)^{\frac{1}{C_n^k}}\right) = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{ \left\{ \left(1-\left(1-\prod_{\substack{1\leqslant i_1<\\ \ldots<i_k\leqslant n}}\left(1-\prod_{j=1}^{k}\left(1-\gamma_{i_j}^{nq\sigma_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right)^{\frac{1}{q}} \right\}, \left\{ \left(1-\prod_{\substack{1\leqslant i_1<\\ \ldots<i_k\leqslant n}}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\eta_{i_j}^{q}\right)^{n\sigma_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{qk}} \right\} \right\}.$$

Theorem 2 (Idempotency): *Let $d_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ be a collection of q-RDHFEs, if $d_i = d = (h, g)$ for all i, and d has only one MD and one NMD, then,*

$$q\text{-}RDHFPDMSM^{(k)}(d_1, d_2, \ldots, d_n) = d. \tag{19}$$

Proof. As $d_i = d = (h, g)$ $(i = 1, 2, \ldots, n)$ and d only has one MD and one NMD, then we have $\mathrm{Sup}\left(d_i, d_j\right) = 1$ for $i, j = 1, 2, \ldots, n$ $(i \neq j)$. Thus, $\sigma_{i_j} = 1/n$ $(i, j = 1, 2, \ldots, n; i \neq j)$ holds for all i. According to Theorem 1, we have

$$\begin{aligned}
&q\text{-}RDHFPDMSM^{(k)}\left(d_1, d_2, \ldots, d_n\right)\\
&= \bigcup_{\substack{\gamma_{i_j} \in h_{i_j},\\ \eta_{i_j} \in g_{i_j}}} \left\{ \left\{ \left(1 - \left(1 - \prod_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \prod_{j=1}^{k} \left(1 - \gamma_{i_j}^{nq\sigma_{i_j}} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{1/q} \right\}, \right.\\
&\qquad \left. \left\{ \left(1 - \prod_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \prod_{j=1}^{k} \left(1 - \left(1 - \eta_{i_j}^{q} \right)^{n\sigma_{i_j}} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{qk}} \right\} \right\}\\
&= \bigcup_{\substack{\gamma_{i_j} \in h_{i_j},\\ \eta_{i_j} \in g_{i_j}}} \left\{ \left\{ \left(1 - \left(1 - \prod_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \prod_{j=1}^{k} \left(1 - \gamma_{i_j}^{q} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{1/q} \right\}, \right.\\
&\qquad \left. \left\{ \left(1 - \prod_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \prod_{j=1}^{k} \left(\eta_{i_j}^{q} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{qk}} \right\} \right\}\\
&= \bigcup_{\substack{\gamma_{i_j} \in h_{i_j},\\ \eta_{i_j} \in g_{i_j}}} \left\{ \left\{ \left(1 - \left(\left(1 - \gamma_{i_j}^{q} \right)^{k} \right)^{\frac{1}{k}} \right)^{1/q} \right\}, \left\{ \left(\left(\eta_{i_j}^{q} \right)^{k} \right)^{\frac{1}{qk}} \right\} \right\} = (h, g) = d.
\end{aligned}$$

Theorem 3 (Boundedness) *Let $d_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ be a collection of q-RDHFEs, $d^- = \min\left(d_1, d_2, \ldots, d_n\right)$ and $d^+ = \max\left(d_1, d_2, \ldots, d_n\right)$, then*

$$x \leqslant q\text{-}RDHFPDMSM^{(k)}\left(d_1, d_2, \ldots, d_n\right) \leqslant y, \tag{20}$$

where

$$x = \frac{1}{k}\left(\bigotimes_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\bigoplus_{j=1}^{k}(d^-)^{n\sigma_{i_j}}\right)^{1/C_n^k}\right)$$

and

$$y = \frac{1}{k}\left(\bigotimes_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\bigoplus_{j=1}^{k}(d^+)^{n\sigma_{i_j}}\right)^{1/C_n^k}\right).$$

Proof. From Definition 3, we can obtain

$$(d^-)^{n\sigma_{i_j}} \leqslant d_{i_j}^{n\sigma_{i_j}},$$

and

$$\bigoplus_{j=1}^{k}(d^-)^{n\sigma_{i_j}} \leqslant \bigoplus_{j=1}^{k} d_{i_j}^{n\sigma_{i_j}}.$$

Therefore,

$$\left(\bigoplus_{j=1}^{k}(d^-)^{n\sigma_{i_j}}\right)^{1/C_n^k} \leqslant \left(\bigoplus_{j=1}^{k} d_{i_j}^{n\sigma_{i_j}}\right)^{1/C_n^k},$$

thus,

$$\bigotimes_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\bigoplus_{j=1}^{k}(d^-)^{n\sigma_{i_j}}\right)^{1/C_n^k} \leqslant \bigotimes_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\bigoplus_{j=1}^{k} d_{i_j}^{n\sigma_{i_j}}\right)^{1/C_n^k}.$$

Finally,

$$\frac{1}{k}\left(\bigotimes_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\bigoplus_{j=1}^{k}(d^-)^{n\sigma_{i_j}}\right)^{1/C_n^k}\right) \leqslant \frac{1}{k}\left(\bigotimes_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\bigoplus_{j=1}^{k} d_{i_j}^{n\sigma_{i_j}}\right)^{1/C_n^k}\right),$$

which means that $x \leqslant$ *q-RDHFPDMSM*$^{(k)}\,(d_1, d_2, \ldots, d_n)$.

Similarly, we can also prove *q-RDHFPDMSM*$^{(k)}(d_1, d_2, \ldots, d_n) \leqslant y$. Thus, the proof of Theorem 3 is completed. In addition, it is worth to point out that the q-RDHFPDMSM operator does not has the property of monotonicity. This is because the PG operator fails to provide the monotonicity property. Hence, the PDMSM and q-RDHFPDMSM operators do not exhibit the monotonicity property either.

In the following, we investigate the special cases of the q-RDHFPDMSM operator with respect to the parameters k and q.

Special case 1: If $k = 1$, then the q-RDHFPDMSM operator reduces to the q-rung dual hesitant fuzzy power geometric (q-RDHFPG) operator, i.e.

$$\begin{aligned} &q\text{-}RDHFPDMSM^{(1)}(d_1, d_2, \ldots, d_n) \\ &= \bigcup_{\substack{\gamma_i \in h_i, \\ \eta_i \in g_i}} \left\{ \left\{ \prod_{i=1}^{n} \gamma_i^{\sigma_i} \right\}, \left\{ \left(1 - \prod_{i=1}^{n} \left(1 - \eta_i^q\right)^{\sigma_i} \right)^{1/q} \right\} \right\} \\ &= \bigotimes_{i=1}^{n} d_i^{\sigma_{ij}} = q\text{-}RDHFPG(d_1, d_2, \ldots, d_n). \end{aligned} \tag{21}$$

Special case 2: If $k = 2$, then the q-RDHFPDMSM operator reduces to the q-rung dual hesitant fuzzy power geometric Bonferroni mean (q-RDHFPGBM) operator, i.e.

$$\begin{aligned} &q\text{-}RDHFPDMSM^{(2)}(d_1, d_2, \ldots, d_n) \\ &= \bigcup_{\substack{\gamma_i \in h_i, \\ \gamma_j \in h_j, \\ \eta_i \in g_i, \\ \eta_j \in g_j}} \left\{ \left\{ \left(1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \gamma_i^{nq\sigma_i}\right) \left(1 - \gamma_j^{nq\sigma_j}\right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \right)^{1/q} \right\}, \right. \\ &\quad \left. \left\{ \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \left(1 - \eta_i^q\right)^{n\sigma_i}\right) \left(1 - \left(1 - \eta_j^q\right)^{n\sigma_j}\right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2q}} \right\} \right\} \\ &= \frac{1}{2} \left(\bigotimes_{\substack{i,j=1, \\ i \neq j}}^{n} \left(d_i^{n\sigma_i} \oplus d_j^{n\sigma_j} \right)^{\frac{2}{n(n-1)}} \right) \\ &= q\text{-}RDHFPGBM^{1,1}(d_1, d_2, \ldots, d_n). \end{aligned} \tag{22}$$

Special case 3. If $q = 1$, then the q-RDHFPDMSM operator reduces to the dual hesitant fuzzy power dual Maclaurin symmetric mean (DHFPDMSM) operator, i.e.

$$
\begin{aligned}
&q\text{-}RDHFPDMSM_{q=1}^{(k)}\left(d_1, d_2, \ldots, d_n\right) \\
&= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{ \left\{ 1 - \left(1 - \prod_{1\leqslant i_1<\ldots<i_k\leqslant n} \left(1 - \prod_{j=1}^{k} \left(1 - \gamma_{i_j}^{n\sigma_{i_j}} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right\}, \right. \\
&\qquad \left. \left\{ \left(1 - \prod_{1\leqslant i_1<\ldots<i_k\leqslant n} \left(1 - \prod_{j=1}^{k} \left(1 - \left(1 - \eta_{i_j} \right)^{n\sigma_{i_j}} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right\} \right\} \\
&= DHFPDMSM\left(d_1, d_2, \ldots, d_n\right). \qquad (23)
\end{aligned}
$$

Special case 4. If $q = 2$, then the q-RDHFPDMSM operator reduces to the dual hesitant Pythagorean fuzzy power Maclaurin symmetric mean (DHPFPDMSM) operator, i.e.

$$
\begin{aligned}
&q\text{-}RDHFPDMSM_{q=2}^{(k)}\left(d_1, d_2, \ldots, d_n\right) \\
&= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{ \left\{ \left(1 - \left(1 - \prod_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}} \left(1 - \prod_{j=1}^{k} \left(1 - \gamma_{i_j}^{2n\sigma_{i_j}} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{1/2} \right\}, \right. \\
&\qquad \left. \left\{ \left(1 - \prod_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}} \left(1 - \prod_{j=1}^{k} \left(1 - \left(1 - \eta_{i_j}^2 \right)^{n\sigma_{i_j}} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{2k}} \right\} \right\}. \qquad (24)
\end{aligned}
$$

Special case 5. If $q = 3$, then the q-RDHFPDMSM operator reduces to dual hesitant Fermatean fuzzy PDMSM (DHFFPDMSM) operator, i.e.,

$$
\begin{aligned}
&q\text{-}RDHFPDMSM_{q=3}^{(k)}\left(d_1, d_2, \ldots, d_n\right) \\
&= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{ \left\{ \left(1 - \left(1 - \prod_{1\leqslant i_1<\ldots<i_k\leqslant n} \left(1 - \prod_{j=1}^{k} \left(1 - \gamma_{i_j}^{3n\sigma_{i_j}} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{1/3} \right\}, \right. \\
&\qquad \left. \left\{ \left(1 - \prod_{1\leqslant i_1<\ldots<i_k\leqslant n} \left(1 - \prod_{j=1}^{k} \left(1 - \left(1 - \eta_{i_j}^3 \right)^{n\sigma_{i_j}} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{3k}} \right\} \right\}. \qquad (25)
\end{aligned}
$$

Remark 3 *We can discover more special cases of the proposed q-RDHFPDMSM operator by adopting some special parameters. For instance, if $k = q = 1$ in the q-RDHFPDMSM operator, then the dual hesitant fuzzy power geometric operator is obtained. If $k = 2$ and $q = 1$, then the q-RDHFPDMSM operator reduces to the dual hesitant fuzzy power geometric Bonferroni mean operator. The other special AOs, such as dual hesitant Pythagorean fuzzy power geometric operator, dual hesitant Pythagorean fuzzy power geometric Bonferroni mean operator, the dual hesitant Fermatean fuzzy power geometric operator, and dual hesitant Fermatean fuzzy power geometric Bonferroni mean operator, etc.*

4.3. The q-rung dual hesitant fuzzy power weighted dual Maclaurin symmetric mean operator

As seen in above section, the q-RDHFPDMSM operator does not consider the importance of q-RDHFEs. However, the weights of input q-RDHFEs usually play an important role in the aggregation results. Therefore, it is necessary to take the weights of aggregated q-RDHFEs into account. By considering the weight vector, we propose the weighted form of q-RDHFPDMSM operator.

Definition 10 *Let d_i $(i = 1, 2, \ldots, n)$ be a collection of q-RDHFEs and $k = 1, 2, \ldots, n$. Let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector, such that $0 \leqslant w_i \leqslant 1$ and $\sum_{i=1}^{n} w_i = 1$. The q-RDHFPWDMSM operator is expressed as*

$$\begin{aligned}&q\text{-}RDHFPWDMSM^{(k)}(d_1, d_2, \ldots, d_n)\\&\quad= \frac{1}{k}\left(\bigotimes_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(\bigoplus_{j=1}^{k} d_{i_j}^{\frac{n w_{i_j}\left(1+T\left(d_{i_j}\right)\right)}{\sum_{t=1}^{n} w_t\left(1+T\left(d_t\right)\right)}}\right)^{1/C_n^k}\right),\end{aligned} \tag{26}$$

where $T(d_i) = \sum_{j=1; i\neq j}^{n} \operatorname{Sup}(d_i, d_j)$, and $\operatorname{Sup}(d_i, d_j)$ is the support measure for d_i from d_j, satisfying the properties in Definition 9. To simplify Eq. (26), we assume

$$\delta_i = \frac{w_i\left(1 + T\left(d_i\right)\right)}{\sum_{t=1}^{n} w_t\left(1 + T\left(d_t\right)\right)}, \tag{27}$$

then Eq. (26) can be written as

$$q\text{-}RDHFPWDMSM^{(k)}(d_1, d_2, \ldots, d_n) = \frac{1}{k}\left(\bigotimes_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\bigoplus_{j=1}^{k} d_{i_j}^{n\delta_{i_j}}\right)^{1/C_n^k}\right), \tag{28}$$

where $0 \leqslant \delta_i \leqslant 1$ *and* $\sum_{i=1}^{n}\delta_i = 1$.

Based on the operational rules of q-RDHFEs, the following decision result is obtained.

Theorem 4 *Let* $d_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ *be a collection of q-RDHFEs and* $k = 1, 2, \ldots, n$. *The aggregated value by the q-RDHFPWDMSM operator is also a q-RDHFE and*

$$\begin{aligned} &q\text{-}RDHFPWDMSM^{(k)}(d_1, d_2, \ldots, d_n) = \bigcup_{\gamma_{i_j}\in h_{i_j},\eta_{i_j}\in g_{i_j}} \\ &\left\{\left\{\left(1-\left(1-\prod_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\prod_{j=1}^{k}\left(1-\gamma_{i_j}^{nq\delta_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right)^{1/q}\right\},\right. \\ &\left.\left\{\left(1-\prod_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\eta_{i_j}^{q}\right)^{n\delta_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{qk}}\right\}\right\}. \end{aligned} \tag{29}$$

The proof of Theorem 4 is similar to that of Theorem 1. Moreover, it is easy to prove that the q-RDHFPWDMSM operator has the property of boundedness.

5. A novel MADM method under q-RDHFSs

This section proposes a new decision-making method wherein attribute values are in the form of q-RDHFEs. Let $X = \{X_1, X_2, \ldots, X_m\}$ be m feasible alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ be a set of n attributes. The weight vector of attributes is $w = (w_1, w_2, \ldots, w_n)^T$, such that $0 \leqslant w_j \leqslant 1$ and $\sum_{j=1}^{n} w_j = 1$. For attribute C_j $(j = 1, 2, \ldots, n)$ of alternative X_i $(i = 1, 2, \ldots, m)$, DMs employ a q-RDHFE

$d_{ij} = \left(h_{ij}, g_{ij}\right)$ to express their evaluation value. Finally, a q-rung dual hesitant fuzzy decision matrix $D = (d_{ij})_{m\times n}$ is obtained. In the following, based on the proposed AOs we introduce a decision-making to determine the optimal alternative.

Step 1. Normalize the original decision matrix according to the following formula

$$d_{ij} = \begin{cases} \left(h_{ij}, g_{ij}\right) & C_j \in I_1\,, \\ \left(g_{ij}, h_{ij}\right) & C_j \in I_2\,, \end{cases} \tag{30}$$

where I_1 and I_2 represent the benefit type and cost type of attributes, respectively.

Step 2. Calculate the support $\mathrm{Sup}\left(d_{if}, d_{it}\right)$ by

$$\mathrm{Sup}\left(d_{if}, d_{it}\right) = 1 - \mathrm{dis}\left(d_{if}, d_{it}\right), \tag{31}$$

where $f, t = 1, 2, \ldots, n$; $f \neq t$.

Step 3. Calculate the overall support $T(d_{ij})$ by

$$T\left(d_{ij}\right) = \sum_{t,f=1,t\neq f}^{n} \mathrm{Sup}\left(d_{if}, d_{it}\right). \tag{32}$$

Step 4. Compute the power weight δ_{ij} associated with q-RDHFEd_{ij} by

$$\delta_{ij} = \frac{w_j\left(1 + T\left(d_{ij}\right)\right)}{\sum_{j=1}^{n} w_j\left(1 + T\left(d_{ij}\right)\right)}. \tag{33}$$

Step 5. For alternative X_i ($i = 1, 2, \ldots, m$), use the q-RDHFPWDMSM operator to fuse the attribute values, i.e.

$$d_i = q\text{-}RDHFPWDMSM^{(k)}\left(d_{i1}, d_{i2}, \ldots, d_{in}\right). \tag{34}$$

Hence, a set of overall evaluation values of alternatives are derived.

Step 6. Calculate the score values of alternatives.

Step 7. Rank alternatives according to their evaluation values and select the best one.

6. Numerical examples

Example 2 Let's look at an online teaching platform selection problem. With the widespread of the COVID-19 virus, to protect the health of teachers and students, more and more schools and universities start to launch online teaching. Hence, before implementing online teaching, one of the most important objects is to select a suitable online teaching platform. Suppose that there are four possible teaching platforms that to be evaluated. The four platforms can be denoted as X_1, X_2, X_3 and X_4 for convenience. A set of decision-making experts are invited to evaluate the performance of the four alternatives. Suppose the DMs assess the four possible candidates under four attributes, i.e., stability (C_1), word of mouth (C_2), price competitiveness (C_3), and usability (C_4). The weight vector of the for attributes is $w = (0.23, 0.36, 0.15, 0.26)^T$. The DMs use q-RDHFEs to express their evaluation values and the original decision matrix is presented in Table 1.

Table 1: The q-rung dual hesitant decision matrix provided by DMs

	C_1	C_2	C_3	C_4
X_1	$\{\{0.2, 0.3, 0.4\}, \{0.6\}\}$	$\{\{0.4, 0.7\}, \{0.1\}\}$	$\{\{0.1\}, \{0.5, 0.6\}\}$	$\{\{0.6\}, \{0.2, 0.1\}\}$
X_2	$\{\{0.1, 0.2\}, \{0.7\}\}$	$\{\{0.6, 0.8\}, \{0.1\}\}$	$\{\{0.4, 0.5\}, \{0.2, 0.3\}\}$	$\{\{0.7\}, \{0.1, 0.2, 0.3\}\}$
X_3	$\{\{0.7, 0.8\}, \{0.1\}\}$	$\{\{0.3, 0.4\}, \{0.6\}\}$	$\{\{0.7\}, \{0.1, 0.3\}\}$	$\{\{0.5, 0.7\}, \{0.1, 0.2, 0.3\}\}$
X_4	$\{\{0.6\}, \{0.2\}\}$	$\{\{0.5, 0.6\}, \{0.2\}\}$	$\{\{0.3\}, \{0.4, 0.5\}\}$	$\{\{0.5, 0.7\}, \{0.1, 0.2, 0.3\}\}$

6.1. The decision-making process

Step 1. As all the attributes are benefit type, the original decision matrix does not need to be normalized.

Step 2. Calculate the support between d_{if} and d_{it}, that is, $\mathrm{Sup}(d_{if}, d_{it})$ according to Eq. (31). For convenience, we use the symbol S^{ft} to represent the value $\mathrm{Sup}(d_{if}, d_{it})$ $(f, t = 1, 2, 3, 4;\ i = 1, 2, 3, 4;\ f \neq t)$. Hence, we obtain the following results.

$$\begin{aligned}
S^{12} &= S^{21} = (0.7835, 0.6463, 0.6737, 0.9697);\\
S^{13} &= S^{31} = (0.9626, 0.7923, 0.9512, 0.8793);\\
S^{14} &= S^{41} = (0.8056, 0.6660, 0.9160, 0.9512);\\
S^{23} &= S^{32} = (0.8140, 0.8570, 0.7503, 0.8850);\\
S^{24} &= S^{42} = (0.9285, 0.9342, 0.8022, 0.9694);\\
S^{34} &= S^{43} = (0.8177, 0.8954, 0.9526, 0.8616).
\end{aligned}$$

Step 3. Calculate the support $T(d_{ij})$ according to Eq. (32). For convenience, we used the symbol T_{ij} to denote the value $T(d_{ij})$, and we can obtain the following matrix:

$$T = \begin{bmatrix} 2.5517 & 2.5260 & 2.5943 & 2.5518 \\ 2.1046 & 2.4375 & 2.5447 & 2.4956 \\ 2.5409 & 2.2261 & 2.6541 & 2.6708 \\ 2.8002 & 2.8241 & 2.6259 & 2.7822 \end{bmatrix}.$$

Step 4. Calculate the power weight δ_{ij} associated with the q-RDHFE d_{ij} according to Eq. (33), and we have:

$$\delta_{ij} = \begin{bmatrix} 0.2302 & 0.3577 & 0.1519 & 0.2602 \\ 0.2105 & 0.3648 & 0.1567 & 0.2679 \\ 0.2341 & 0.3339 & 0.1576 & 0.2744 \\ 0.2314 & 0.3644 & 0.1440 & 0.2603 \end{bmatrix}.$$

Step 5. For alternative X_i $(i = 1, 2, 3, 4)$, utilize the q-RDHFPWDMSM operator to calculate the overall evaluation d_i $(i = 1, 2, 3, 4)$ (suppose that $k = 2$ and $q = 3$). As the aggregation results are so complicated, we omit them here.

Step 6. Calculate the score values $S(d_i)$ $(i = 1, 2, 3, 4)$ of the overall evaluation values, and we can get

$$S(d_1) = 0.2720, \quad S(d_2) = 0.4169, \quad S(d_3) = 0.4817, \quad S(d_4) = 0.7938.$$

Step 7. According to the score values $S(d_i)$ $(i = 1, 2, 3, 4)$, the ranking orders of the alternatives can be determined, that is $X_4 \succ X_3 \succ X_2 \succ X_1$, and X_4 is the optimal alternative.

6.2. Validity test

In this subsection, we attempt to discuss the validity and effectiveness of our proposed MADM method. In order to do this, we use our method based on the q-RDHFPWDMSM operator and some exiting decision-making methods to solve some examples and compare their results. These methods involve that proposed by Wang et al. [44] based on the dual hesitant fuzzy weighted geometric (DHFWG) operator, that developed by Tang et al. [45] based on the dual hesitant Pythagorean fuzzy generalized geometric weighted Heronian mean (DHPFGGWHM) operator and that introduced by Xu et al. [42] based on the q-rung dual hesitant fuzzy weighted geometric Heronian mean (q-RDHFWGHM). To better illustrate the validity of our proposed method, we provide the following numerical examples. It is worth pointing out that different MADM methods use different score functions. Hence, to make the decision results comparative, we use the same score function in this subsection. More specifically, we adopt our proposed novel score function to calculate the final score values of comprehensive evaluation values.

6.2.1. Comparison with Wang et al.'s method

Example 3 (Adopted from [44]). Let's consider a potential evaluation of emerging technology commercialization problem and there are five possible emerging technology enterprises to be evaluated, which can be denoted as X_i $(i = 1, 2, 3, 4, 5)$. A group of experts evaluate the performance of the five alternatives under four attributes, i.e., C_1: the technical advancement; C_2: the technical advancement; C_3: the industrialization infrastructure, human resources and financial conditions; C_4: the employment creation and the development of science and technology. The weight vector of attributes is $w = (0.20, 0.15, 0.35, 0.30)^T$. DMs use hesitant fuzzy sets to express their evaluations and the decision matrix is listed in Table 2. We use Wang et al.'s [44] method and our proposed MADM approach to solve Example 3 and present the decision results in Table 3. As seen in Table 3, Wang et al.'s [44] method and our method produce the same ranking result, i.e., $X_2 \succ X_3 \succ X_5 \succ X_1 \succ X_4$. This indicates the validity of our proposed method.

Table 2: The decision matrix of Example 3

	C_1	C_2	C_3	C_4
X_1	$\{\{0.3, 0.4\}, \{0.6\}\}$	$\{\{0.4, 0.5\}, \{0.3, 0.4\}\}$	$\{\{0.2, 0.3\}, \{0.7\}\}$	$\{\{0.4, 0.5\}, \{0.5\}\}$
X_2	$\{\{0.6\}, \{0.4\}\}$	$\{\{0.2, 0.4, 0.5\}, \{0.4\}\}$	$\{\{0.2\}, \{0.6, 0.7, 0.8\}\}$	$\{\{0.5\}, \{0.4, 0.5\}\}$
X_3	$\{\{0.5, 0.7\}, \{0.2\}\}$	$\{\{0.2\}, \{0.7, 0.8\}\}$	$\{\{0.2, 0.3, 0.4\}, \{0.6\}\}$	$\{\{0.5, 0.6, 0.7\}, \{0.3\}\}$
X_4	$\{\{0.7\}, \{0.3\}\}$	$\{\{0.6, 0.7, 0.8\}, \{0.2\}\}$	$\{\{0.1, 0.2\}, \{0.3\}\}$	$\{\{0.1\}, \{0.6, 0.7, 0.8\}\}$
X_5	$\{\{0.6, 0.7\}, \{0.2\}\}$	$\{\{0.2, 0.3, 0.4\}, \{0.5\}\}$	$\{\{0.4, 0.5\}, \{0.2\}\}$	$\{\{0.2, 0.3, 0.4\}, \{0.5\}\}$

Table 3: Decision results of Example 3 using different decision-making methods

Decision-making methods	Score values $S(d_i)$ $(i = 1, 2, 3, 4, 5)$	Ranking results
Wang et al.'s [44] method based on the DHFWG operator	$S(d_1) = 0.3533$, $S(d_2) = 0.4215$, $S(d_3) = 0.4052$, $S(d_4) = 0.3328$ $S(d_5) = 0.4043$	$X_2 \succ X_3 \succ X_5 \succ X_1 \succ X_4$
Our method based on the q-RDHFPWDMSM operator $(q = k = 1)$	$S(d_1) = 0.3550$, $S(d_2) = 0.4280$, $S(d_3) = 0.4073$, $S(d_4) = 0.3403$, $S(d_5) = 0.4012$	$X_2 \succ X_3 \succ X_5 \succ X_1 \succ X_4$

6.2.2. Comparison with Tang et al.'s method

We continue to compare our result with that developed by Tang et al. [45]. Here, we adopt Example 2 as an illustrative example. The original decision matrix is presented in Table 1. We use Tang et al.'s [45] decision-making method and our new MADM method to solve Example 2 and present the decision results in Table 4. As seen from Table 4, the two decision-making methods produce the same ranking order $X_4 \succ X_3 \succ X_2 \succ X_1$. This also indicates the effectiveness of our proposed MADM method.

Table 4: Decision results of Example 2 using different methods

Decision-making methods	Score values $S(d_i)$ $(i = 1, 2, 3, 4)$	Ranking results
Tang et al.'s [45] method based on the DHPFGGWHM operator ($\zeta = 2$, $\xi = 2$)	$S(d_1) = 0.3102$, $S(d_2) = 0.3982$, $S(d_3) = 0.4277$, $S(d_4) = 0.8492$	$X_4 > X_3 > X_2 > X_1$
Our method based on the q-RDHFPWDMSM operator ($q = 2$, $k = 1$)	$S(d_1) = 0.4225$, $S(d_2) = 0.4937$ $S(d_3) = 0.5311$, $S(d_4) = 0.6708$	$X_4 > X_3 > X_2 > X_1$

6.2.3. Comparison with Xu et al.'s method

Example 4 (Adopted from [42]). A company is now selecting an appropriate supplier. In order to make a wise choice, the company invites a series of decision experts to evaluate four candidates, i.e. X_1, X_2, X_3, and X_4. The DMs evaluate the performance of the alternatives from four aspects, i.e., relationship closeness (C_1), product quality (C_2), price competitiveness (C_3), and delivery performance (C_4). The weight vector of attributes is $w = (0.17, 0.32, 0.38, 0.13)^T$. The original decision matrix is listed in Table 5. We use Xu et al.'s [42] method and our developed MADM approach to solve Example 4 and present the decision results in Table 6. As seen from Table 6, the two MADM methods produce the same ranking orders of alternatives, i.e., $X_3 > X_2 > X_4 > X_1$ and the best alternative is X_3, which also proves the effectiveness of our proposed method.

Table 5: The decision matrix of Example 4

	C_1	C_2	C_3	C_4
X_1	$\{\{0.3, 0.4\}, \{0.6\}\}$	$\{\{0.7, 0.9\}, \{0.1\}\}$	$\{\{0.4\}, \{0.2, 0.3\}\}$	$\{\{0.5, 0.6\}, \{0.2\}\}$
X_2	$\{\{0.2, 0.3\}, \{0.5\}\}$	$\{\{0.6, 0.7\}, \{0.2\}\}$	$\{\{0.7, 0.8\}, \{0.2\}\}$	$\{\{0.6\}, \{0.1, 0.2, 0.3\}\}$
X_3	$\{\{0.4\}, \{0.2, 0.3\}\}$	$\{\{0.2, 0.3, 0.4\}, \{0.6\}\}$	$\{\{0.7, 0.8\}, \{0.1\}\}$	$\{\{0.7\}, \{0.2, 0.3\}\}$
X_4	$\{\{0.6, 0.7\}, \{0.3\}\}$	$\{\{0.5\}, \{0.4\}\}$	$\{\{0.3, 0.4\}, \{0.5\}\}$	$\{\{0.4, 0.6\}, \{0.1, 0.2\}\}$

Table 6: Decision results of Example 4 using different methods

Decision-making methods	Score values $S(d_i)$ $(i = 1, 2, 3, 4)$	Ranking results
Xu et al.'s [42] method based on the q-RDHFWGHM operator	$S(d_1) = 0.1463$, $S(d_2) = 0.2548$, $S(d_3) = 0.2553$, $S(d_4) = 0.1056$	$X_3 > X_2 > X_4 > X_1$
Our method based on the q-RDHFPWDMSM operator ($q = 3$; $k = 2$)	$S(d_1) = 0.2325$, $S(d_2) = 0.2714$, $S(d_3) = 0.3931$, $S(d_4) = 0.1890$	$X_3 > X_2 > X_4 > X_1$

6.3. The influence of the parameters on the decision results

The above subsection has revealed the validity and correctness of our proposed method. In this subsection, we shall investigate influence of parameters in our MADM method on the final decision results. We first discuss the impact of the parameter k on the results. Afterwards, we continue to study the influence of the parameter q on the decision results.

6.3.1. The influence of the parameter k on the results

To investigate the influence of the parameter k on the final decision results, we assign different parameters in the q-RDHFPWDMSM operator and present the decision results of Example 2 in Table 7 (in these situations, the we assume the parameter q to be a fixed number). As it is seen from Table 7, no matter what the parameter k is, the final ranking orders of alternatives are always $X_4 \succ X_3 \succ X_3 \succ X_1$. However, it is worth pointing out that different values of parameter k have different meanings. For example, when $k = 1$ or $k = 4$, then our method does not consider the interrelationship among attributes. In other words, when $k = 1$ or $k = 4$, our method is suitable to handle MADM problems where attributes are independent. When $k = 2$, then our method takes the interrelationship between any two attributes into consideration. When $k = 3$, then our method considers the interrelationship among any three attributes. Hence, DMs can determine the value of k according to actual needs in practical MADM problems.

Table 7: Decision results of Example 2 with different parameter k in the q-RDHFPWDMSM operator ($q = 3$)

	Score values $S(d_i)$ $(i = 1, 2, 3, 4)$	Ranking orders
$k = 1$	$S(d_1) = 0.3991$, $S(d_2) = 0.4492$, $S(d_3) = 0.4831$, $S(d_4) = 0.6449$	$X_4 \succ X_3 \succ X_2 \succ X_1$
$k = 2$	$S(d_1) = 0.2720$, $S(d_2) = 0.4169$, $S(d_3) = 0.4817$, $S(d_4) = 0.7938$	$X_4 \succ X_3 \succ X_2 \succ X_1$
$k = 3$	$S(d_1) = 0.2898$, $S(d_2) = 0.4305$, $S(d_3) = 0.5089$, $S(d_4) = 0.7966$	$X_4 \succ X_3 \succ X_2 \succ X_1$
$k = 4$	$S(d_1) = 0.4521$, $S(d_2) = 0.5421$, $S(d_3) = 0.6101$, $S(d_4) = 0.6601$	$X_4 \succ X_3 \succ X_2 \succ X_1$

6.3.2. The influence of the parameter q on the decision results

In the followings, we continue to study the influence of the parameter q on the final decision results. In order to do this, we assign different values of q in the q-RDHFPWDMSM operator and present the decision results of Example 2 in Table 8 (in these situations, we assume the parameter k to be a fixed value). As it is seen from Table 8, where different values of q are assigned in the q-RDHFPWDMSM operator, the ranking orders of alternatives are the same, i.e., $X_4 \succ X_3 \succ X_2 \succ X_1$. However, we notice that the score values of comprehensive evaluations are different, when different parameters of q are employed. More specifically, the increase of the value of q leads to the decrease of the score values. Hence, how to select an appropriate value of q is a prominent question. In [42], Xu et al. proposed a method to determine the parameter q in q-rung dual hesitant fuzzy MADM problems. For more details, we suggest readers referring the publication [42].

Table 8: Decision results of Example 2 with different parameter q in the q-RDHFPWDMSM operator ($k = 2$)

	Score values $S(d_i)$ $(i = 1, 2, 3, 4)$	Ranking orders
$q = 1$	$S(d_1) = 0.4671$, $S(d_2) = 0.5902$, $S(d_3) = 0.6489$, $S(d_4) = 0.7109$	$X_4 > X_3 > X_2 > X_1$
$q = 2$	$S(d_1) = 0.3299$, $S(d_2) = 0.4837$, $S(d_3) = 0.5517$, $S(d_4) = 0.7893$	$X_4 > X_3 > X_2 > X_1$
$q = 3$	$S(d_1) = 0.2720$, $S(d_2) = 0.4169$, $S(d_3) = 0.4817$, $S(d_4) = 0.7938$	$X_4 > X_3 > X_2 > X_1$
$q = 4$	$S(d_1) = 0.2475$, $S(d_2) = 0.3752$, $S(d_3) = 0.4330$, $S(d_4) = 0.7874$	$X_4 > X_3 > X_2 > X_1$
$q = 5$	$S(d_1) = 0.2369$, $S(d_2) = 0.3483$, $S(d_3) = 0.3980$, $S(d_4) = 0.7815$	$X_4 > X_3 > X_2 > X_1$
$q = 10$	$S(d_1) = 0.2287$, $S(d_2) = 0.3033$, $S(d_3) = 0.3207$, $S(d_4) = 0.7722$	$X_4 > X_3 > X_2 > X_1$
$q = 20$	$S(d_1) = 0.2286$, $S(d_2) = 0.2968$, $S(d_3) = 0.2982$, $S(d_4) = 0.7714$	$X_4 > X_3 > X_2 > X_1$

6.4. Advantages of our proposed method

In this section, we investigate the advantages and superiorities of our proposed method through comparative analysis.

6.4.1. The ability of making the decision results more reasonable and reliable

Our decision-making method is based on the PDMSM operator, which is a combination of the PG and DMSM operators. As discussed in some existing literature [46–49], the PG operator can reduce the adverse impact of extreme evaluation values on the final decision results, and it has been successfully applied in decision-making to fuse attribute values. As our method is based on the q-RDHFPWDMSM operator, which inherits the advantage of PG. Hence, our proposed method has the ability of dealing with DMs' unduly high or low evaluation values and makes the final decision results more reliable. This characteristic makes our method more practical in modern MADM problems. This is because in present-day decision-making problems, DMs usually come from different fields and they have different background, education experience, occupational history, individual preferences, etc. Hence, some of them may provide some ultra-evaluation values, which obviously negatively affects the final decision results. As pointed above, our method can reduce or even eliminate the bad influence of unreasonable evaluation values on the final decision ranking orders. Therefore, our method is more adequate to deal with practical MADM problems than those introduced by Wang et al. [44], Tang et al. [45] and Xu et al. [42].

6.4.2. The ability of capturing the interrelationship among multiple attributes

As mentioned above, our method is based on the PDMSM operator, and hence it has the capability of reflecting the interrelationship among attributes. In most decision-making situations, attributes are usually correlated and such kind of interrelationship among attributes should be taken into consideration. MADM problems with relevant attributes have been widely studied [50–53], which indicates the necessity of considering the interrelationship among attributes in decision-making problems. Hence, our method is more powerful than those proposed by Wang et al. [44]. In addition, our method can consider the interrelationship among multiple input attributes values. Hence, our method is more powerful and flexible than those put forward by Tang et al. [45] and Xu et al. [42]. Based on the above section, we can easily find out that the most prominent advantage of our method is that it not only reduces the negative impact of DMs' extreme evaluation values, but also simultaneously takes the interrelationship among attributes into account. This characteristic makes our method more reliable and reasonable than those based on PBM or PHM operators.

6.4.3. The ability of effectively describing decision-making information

In the framework of our decision-making method, q-RDHFSs are used to express DMs' preference information over a set of alternatives. As discussed above, the constraint of q-RDHFSs is that the sum of q-th power of MD and q-th power of non-membership degree is equal to or less than one. Hence, our method can handle wider decision-making situations than Wang et al.'s [44] and Tang et al.'s [45] methods, which are based on DHFEs and DPHFSs, respectively. In other words, our method can full depict DMs' evaluation values and hence it is more suitable to deal with realistic MADM problems.

6.5. Summary of the characteristics of our developed method

To better demonstrate the characteristics and features of above-mentioned MADM methods, we provide Table 9.

Based on the above table, we give a summation of our method' advantages over some existing decision-making methods. First, compared with Wang et al.'s [44] method, our method has the following three advantages: 1) Our method can deal with decision-making situations in which the sum of MD and NMD is greater than one; 2) Our method can consider the interrelationship among attributes; 3) Our method can more effectively handle DMs' unreasonable evaluation values. Second, compared with Tang et al.'s [45] method, our approach has the following superiorities: 1) It has larger application range as it allows the sum of q-th power of MD and q-th power of NMD to equal to or smaller than one; 2) It has the ability of capturing the interrelationship among multiple attributes, making it more sufficient to cope with actual-life MADM problems; 3) It reduces the bad

Table 9: Characteristics of different MADM methods

	Whether it permits the sum of MD and NMD to be greater than one	Whether it permits the square sum of MD and NMD to be greater than one	Whether it considers the interrelationship between any two attributes	Whether it considers the interrelationship among multiple attributes	Whether it effectively deals with DMs' unreasonable evaluation values
Wang et al.'s [44] method	No	No	No	No	No
Tang et al.'s [45] method	Yes	No	Yes	No	No
Xu et al.'s [42] method	Yes	Yes	Yes	No	No
Our proposed method in the present study	Yes	Yes	Yes	Yes	Yes

effect that cased by DMs' extreme evaluation values, making the final decision results more reliable and reasonable. Third, compared with the decision-making presented by Xu et al.'s [42], our new decision-making method has the following advantages: 1) It manages the interrelationship among multiple attributes; 2) It has the capability of effectively dealing with DMs' unreasonable evaluation values.

7. Conclusions

The q-RDHFS is efficient to deal with DMs' complicated evaluation information in MADM process. This paper proposed novel MADM method under q-RDHFSs by introducing new AOs for q-RDHFEs. In order to do this, we first proposed the dual form of PMSM operator, i.e. the PDMSM operator, which is a combination of PG with DMSM operators. Then, we generalized PDMSM operator into q-RDHFSs and put forward the q-RDHFPDMSM and q-RDHFPWDMSM operators. These operators effectively deal with DMs' extreme or unreasonable evaluation values and reflect the interrelationship among multiple attributes. We further gave a MADM method and showed its calculation process through numerical example. We also tried to illustrate the advantages of our method through comparative analysis. In future works, we shall investigate more applications of our MADM method. We will apply the new decision-making method in more realistic problems, such as mutual fund evaluation [54], investment selection [55], quality assessment of Smart Watch appearance design [56], renewable energy source selection [57], etc. In addition, we shall investigate more MADM methods based on the q-RDHFSs.

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