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COMMERCIAL AIRPLANE TRAJECTORY OPTIMIZATION BY A CHEBYSHEV PSEUDOSPECTRAL METHOD

The paper presents application of direct pseudospectral Chebyshev method for solving a commercial airplane trajectory optimization problem. This method employs Nth-degree Lagrange polynomial approximations for the state and control variables with the values of these variables at the Chebyshev-Gauss-Lobatto (CGL) points as the expansion coefficients. This process is converted to a nonlinear programming problem (NLP) with the state and control values at the CGL points as unknown NLP parameters. The kinetic model of flight is formulated, where it is assumed that an airplane is a particle and the motion takes place in the vertical plane. The method is implemented in Matlab using sequential quadratic programming algorithm (SQP) as an efficient solver. Sensitivity analyses are performed concerning the influence of the degree of discretization and the initial approximation on the solution. Three examples of optimized trajectories in presence of wind are shown.

NOMENCLATURE

a – speed of sound,
c_{fl} – cost of fuel per kg,
c_i – specific fuel consumption,
c_t – cost of airplane operation per hour,
c_x – drag coefficient,
c_z – lift coefficient,
g – gravity acceleration,

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h – altitude,
J – cost function,
m – airplane mass,
M – Mach number,
P_s – thrust,
P_x – drag,
P_z – lift,
S – reference area of the airplane,
u – horizontal wind velocity,
v – vertical wind velocity,
V – airspeed,
V_g – ground speed,
w_{fl} – fuel mass flow per hour,
x – horizontal coordinate,
α – angle of attack,
γ – flight path angle,
ε – angle between the airspeed and the ground speed,
η – throttle coefficient,
ρ – air density,
τ – time,
$(.)_0$ – initial value,
$(.)_f$ – final value,
$(.)^{max}$ – maximal value.

1. Introduction

The methods of classical calculus of variations were successfully applied to many airplane performance problems at the beginning of the fifties. The minimum-time-to-climb problem was considered employing simplified Lagrange-Euler equation (Rutowski, 1954). In this problem, the supersonic fighter should increase its speed and altitude operating with full thrust. The applied method indicates that acceleration in a dive at Mach number $M = 1$ is a part of the optimum climbing path. At the beginning of seventies one can observe attempts of extension of this approach to the minimum fuel consumption problem. It was connected with the fuel crisis. A difficulty arose, however. For relatively simple airplane kinetic models, the flight with partial throttle and constant velocity referring to the cruise appeared to be non-optimal. Such regime of flight refers to a singular arc in calculus of variations. The so-called "chattering" solution, where the throttle chatters between maximal and minimal value with the frequency going to infinity provides the best performance. The detailed discussion of such a case one can find in (Maroński, 1988). For civil transportation, the unstable regime of flight is rather unacceptable, therefore some authors assume "a priori" the cruise with constant velocity computing the parameters of such flight. Barman and Erzberger (1976) assume that each optimal profile consists of segments: climb, constant cruise and descent. Their method is relatively simple and does not contain singularities. It cannot be extended for the airplane kinetic model involving the variations of the mass of the airplane, therefore it may be applied for a short-haul airplane only (Krawczyk, 1983).

For more advanced models, application of sophisticated numerical methods is necessary. They may be divided into two groups: indirect methods basing on necessary conditions of optimality (for example Pontryagin's maximum principle) and direct methods. In these methods the optimal control problem is converted to a parameter optimization problem by assuming that the control functions are known functions of time involving a number of unknown parameters. The values of these parameters are optimized using parameter optimization methods. For example, Reader and Hull (1975) used fifth-order series of Chebyshev polynomials to minimize the time to climb of a high performance airplane. The experience of one of the authors of this paper with direct methods has been unsatisfactory for years (Maroński, Łucjanek, 1979). However, the great progress in development of these methods has happened. Direct methods have several advantages over the indirect methods. They are easy to program. They may be used for advanced airplane kinetic models including inequality constraints imposed on the state variables. That is why the authors decided to pay their attention to these methods.

2. Problem formulation

In the case of a specific airplane, which has to complete its given mission, the most important factor which is highly connected with the performance of the airplane is the operational procedure. For airlines, the main goal is to minimize the direct operating costs (DOC). In the year 1997, DOC made up on average 51.4% of all operational costs (Burrows et al., 2001). The most essential in these costs is fuel participation and categories depending on time i.e.: inspection, maintenance, overhaul, airplane depreciation, pilots and cabin crew. These are about 35% of the DOC. That is why in formulations of the optimization problem the following indices are used: the cost of the fuel per kg c_{fl} and the cost of the airplane operation per hour c_t. The ratio c_t/c_{fl} is called the cost index (*CI*) and it is often used in financial analysis and in the flight management systems (FMSs), which are installed on the airplane flight decks to control performance. The cost index is not used directly in this analysis, but it is taken into consideration by using c_t and c_{fl} in the following way:

$$J = c_{fl}(m_0 - m_f) + c_t(\tau_f - \tau_0) = \int_{\tau_0}^{\tau_f} (c_{fl} w_{fl} + c_t)\, d\tau, \tag{1}$$

where $(m_0 - m_f)$ is mass of all fuel used.

The equations of airplane motion are derived assuming that:

- the airplane is a particle,
- the motion takes place in the vertical plane,
- Earth is an inertial system, flat with constant gravitational acceleration,
- the thrust vector P_s is parallel to the mean aerodynamic chord,
- the air density and speed of sound are changing due to altitude,
- engine characteristics depend on altitude and airplane velocity,
- the variation of airplane mass depends only on the fuel consumption,
- critical angles of attack are not exceeded,
- the flight is subsonic ($M < 1$).

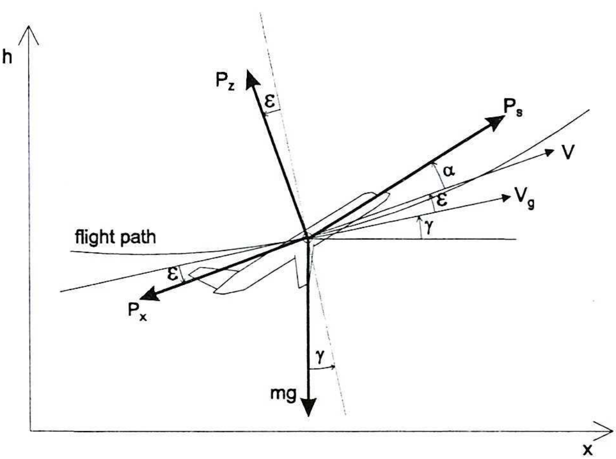

Fig. 1. Forces acting on the airplane

Equations of an airplane motion are as follows:

$$m\dot{V}_g = \eta P_S^{\max}\cos(\alpha + \varepsilon) - P_X\cos\varepsilon - P_Z\sin\varepsilon - mg\sin\gamma, \tag{2}$$

$$mV_g\dot{\gamma} = P_Z\cos\varepsilon + \eta P_S^{\max}\sin(\alpha + \varepsilon) - P_X\sin\varepsilon - mg\cos\gamma, \tag{3}$$

$$\dot{x} = V_g \cos\gamma, \tag{4}$$

$$\dot{h} = V_g \sin\gamma, \tag{5}$$

$$\dot{m} = -w_{fl}, \tag{6}$$

where: $\varepsilon = \arcsin\left(\frac{1}{V}(u\sin\gamma - v\cos\gamma)\right)$,
$V = \sqrt{V_g^2 - 2V_g(u\cos\gamma + v\sin\gamma) + u^2 + v^2}$
are the geometrical relations resulting from Fig. 2. In equations (2), (3) the lift P_z and the drag P_x are respectively:

$$P_z = \frac{1}{2}\rho V^2 S c_z, \quad P_x = \frac{1}{2}\rho V^2 S c_x.$$

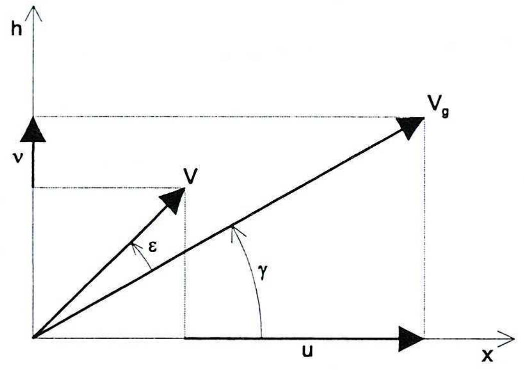

Fig. 2. Dependences between velocities: v, u, V, V_g

In addition, there are the following dependences used for describing: drag coefficient, lift coefficient, thrust and fuel mass flow per hour respectively:

$$c_X = c_X(M, \alpha), \quad c_Z = c_Z(M, \alpha), \quad P_S = \eta P_S^{\max}(M, h), \quad w_{fl} = \eta P_S^{\max} c_i(M, h)$$

According to the earlier assumption, the trust vector P_S is parallel to the mean aerodynamic chord, and in general it is not collinear to the airspeed vector V. On the basis of this fact, the angle of attack α is the angle between P_S and V. Relations (2) and (3) are the equations of motion in natural coordinate system and they follow on the Newton's second law. The first one comes from projection of forces on the ground speed direction, and the second one – on the perpendicular direction. Equations (4) and (5) are the additional kinematical relations. Equation (6) characterizes variation of the airplane mass due to fuel consumption. The state variables in this model are: ground speed V_g, flight path angle γ, horizontal coordinate x, altitude h, airplane mass m, and the control variables are: throttle coefficient η and angle of attack α. The time is the independent variable.

The problem is formulated as follows. We should minimize the performance index (1). The state equations describing the airplane motion (2–6) should be satisfied in every point of the path. The state variables V_g, γ, x, h are given in initial and final points of the path. The state variable m representing the mass of the airplane is given in the initial point of the path. Its final value is not known "a priori", and it results from the computed optimal path including inequality constraints.

3. Numerical analysis

In this part, there are examples of trajectory optimization for the airliner of Boeing 767 class. First analysis concerns the method itself: convergence, accuracy and iteration process. Next, wind influences on airplane trajectory are investigated by using three different wind conditions.

To solve NLP problem referred by Fahroo and Ross (2000), the function *fmincon* is used. It is one of the available functions from *Optimization Toolbox* of MATLAB. *Fmincon* is an implementation of the sequential quadratic programming method (SQP), which represents state-of-the-art in nonlinear programming methods (Matlab, 2000). It allows us to closely mimic Newton's method for constrained optimization just as it is done for an unconstrained optimization. At each major iteration, an approximation of the Hessian of the Lagrangian function is made using a quasi-Newton updating method. This is then used to generate a quadratic programming subproblem. The obtained solution is used to form a search direction for a line search procedure. The constrained quasi-Newton methods guarantee super linear convergence by accumulating second order information regarding the Kuhn-Tucker equations using a quasi-Newton updating procedure. A non-linear constraint problem can often be solved in fewer iterations than an unconstrained problem using SQP. One of the reasons for this are limits on the feasible area, thus the optimizer can make well-informed decisions regarding the search of direction and step length.

In this paper, the airplane is regarded as a particle. It allows us use very simple description of aerodynamic properties. Sufficient to this, is the knowledge on the drag-lift polar or relations $c_z(M, \alpha)$, $c_x(M, \alpha)$ and the reference area S. In this paper, the data are taken for Boeing 767 class airliner.

Long range commercial airplanes (more then a few thousand kilometer range) flying with the Mach number $M < 1$ are mostly jet propelled without after-burning. In Boeing 767 airplane, there are jet engines of GE CFM6 class. The data for this type of engine are taken from (Cichosz et al., 1980). In altitude-velocity characteristics the value of normalized maximal thrust (180000N at $h = 0$ for one engine) and the corresponding specific fuel consumption c_i are introduced. These data are taken for Mach number from the interval <0;0.9> and for the altitude up to 14000m. Visualizations are given in Fig. 3 and Fig. 4. The surface spreaded on well-known points is generated linearly so it is not smooth. In the numerical code sought values of the maximal thrust P_S^{max} and the specific fuel consumption c_i are obtained interpolating two variables function employing splines.

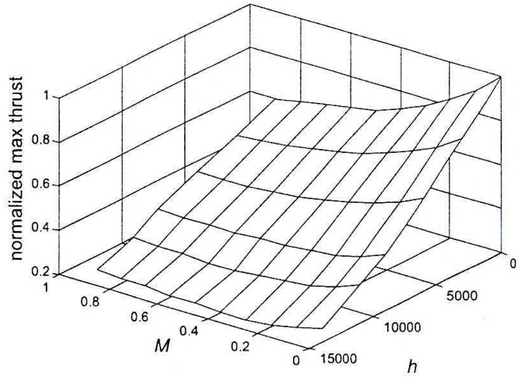

Fig. 3. Normalized maximal thrust versus M and h

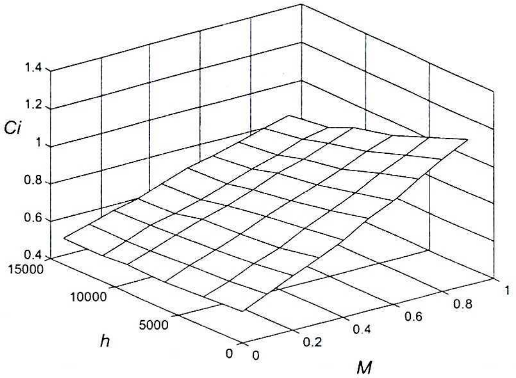

Fig. 4. Specific fuel consumption versus M and h

To estimate performance of the airplane, the unified description of atmosphere is applied. It was introduced by *ICAO* in 1962 and it is called

Standard Atmosphere. In numerical analysis, such features described by *Standard Atmosphere* as speed of sound and air density changing according to altitude are used.

Given are: range of flight $x_f = 4000$ km and take off mass 150000 kg. Assumed values of c_{fl} and c_t are: $c_{fl} = 0.5$ (1kg of fuel costs 0.5$), $c_t = 0$ (such a case refers to minimization of fuel consumption). Equality constraints are equations (2–6). In addition, there are the following inequality constraints:

$$-5 \text{ m/s} \leq \dot{h} \leq 5 \text{ m/s},$$

$$\frac{P_Z}{mg} \leq 1.2,$$

$$V \leq 265 \text{ m/s}.$$

The first condition refers to climb and descent velocity up to 5 m/s, the second one to the load factor. It is because of passenger and crew comfort. Third relation regards not exceeding $M = 0.9$ which, after taking into account the speed of sound at the altitude 14000 m, constrains the airspeed ($M = V/a \leq 0.9$). Additionally, the following constraints are imposed on variables in any node:

$$100 \text{ m/s} \leq V_g \leq 500 \text{ m/s},$$

$$500 \text{ m} \leq h \leq 14000 \text{ m},$$

$$-5° \leq \alpha \leq 15°,$$

$$0 \leq \eta \leq 1,$$

$$-5° \leq \gamma \leq 30°,$$

$$50000 \text{ kg} \leq m \leq 150000 \text{ kg},$$

$$2000 \text{ s} \leq \tau \leq 100000 \text{ s}.$$

In order to establish how the number of nodes influences the quality of the solution, five computations are performed for different number of nodes: N = 5, 10, 15, 20 and 30. Some solutions are shown in Fig. 5–8. Duration of the flight is shown in Tab. 1.

Table 1.

Influence of number of nodes on the duration of the flight

Case	Duration of the flight [s]
N = 5	17262
N = 10	16385
N = 15	16364
N = 20	16407
N = 30	16320

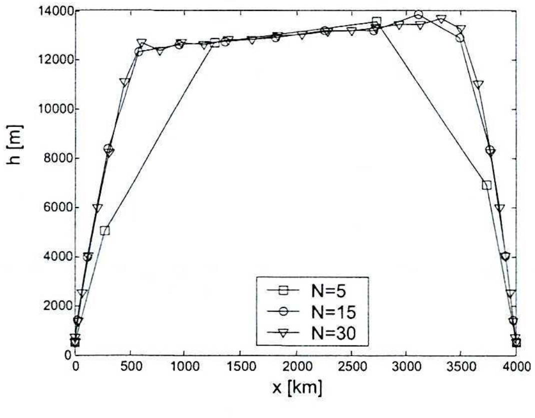

Fig. 5. Altitude for different number of nodes

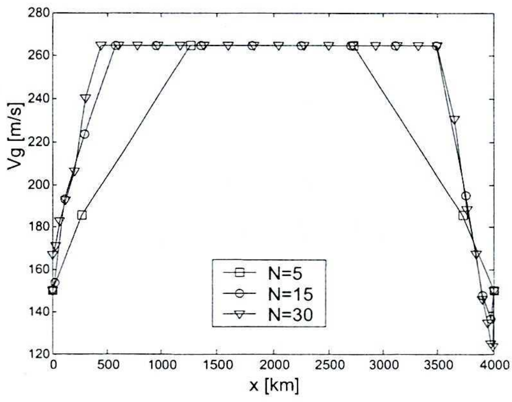

Fig. 6. Ground speed for different number of nodes

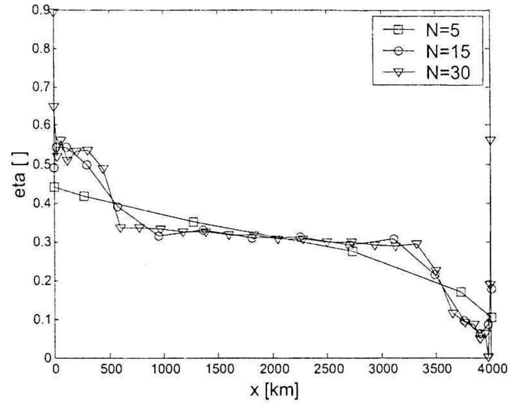

Fig. 7. Throttle coefficient for different number of nodes

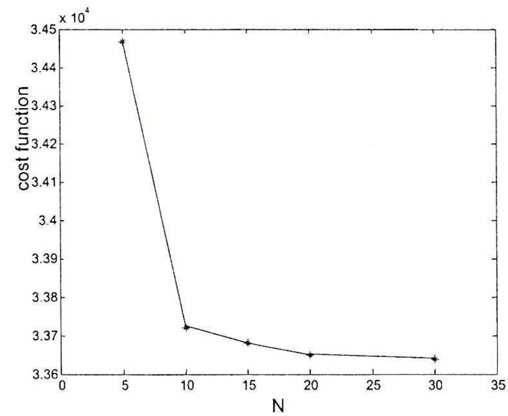

Fig. 8. Cost function for different number of nodes

In Fig. 8 the influence of the number of nodes on the cost function is shown. It should be emphasised, that even for few numbers of nodes (about 10) the solution is acceptable. Concerning Fig. 5–8, it is assumed that number of 15 nodes is sufficient for further analysis. It represents the compromise between the accuracy of the computations and the computing time. The computations for N = 5 for PC Athlon 1,4 GHz class takes about fifteen minutes, for N = 15 it takes a few hours, and for N =30 a few dozen hours.

Next, the influence of the initial approximation on the solution obtained by the pseudospectral Chebyshev method is considered. To start optimization process, one has to approximate the initial point of iterations. Very often it is difficult to provide initial approximation which gives fast convergence. For example, it may be linear whereas the solution is nonlinear. Therefore, the important feature of the method is that one can obtain solution even having "rough" initial approximation. To check sensitivity of the method on various starting points, two optimizations are performed. Results are given in Fig. 9, 10 and Tab. 2. For both cases, the final solutions are the same. The computed trajectories possess identical character and the durations of flight are comparable. Other important thing is that the time of computations is nearly the same in both cases.

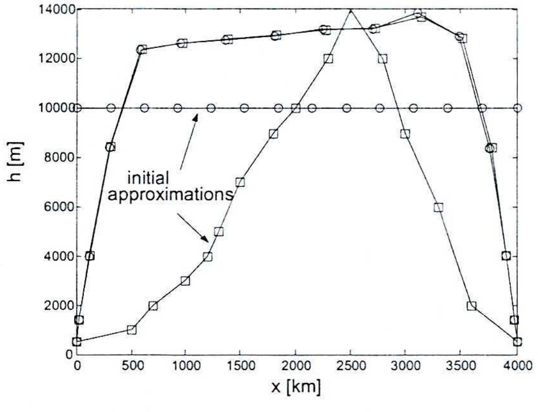

Fig. 9. Influence of the initial approximation on the altitude

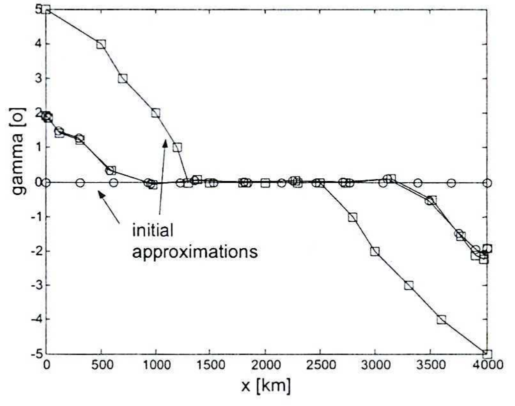

Fig. 10. Influence of initial approximation on the flight path angle

Table 2.

Influence of the initial approximation on the cost function and the duration of flight

Case	Cost function values	Duration of flight [s]
Linear approximation	33681	16364
Nonlinear approximation	33675	16431

To analyze the influence of wind on the trajectory, three wind conditions are proposed (see Fig. 11). Four optimized trajectories are shown in Fig. 12–14. The values of the cost function and the duration of flight are shown in Tab. 3.

Table 3.

Influence of wind on the cost function and the duration of the flight

Case	Cost function values	Duration of flight [s]
without a wind	33681	16364
wind 1	29503	14360
wind 2	39209	19265
wind 3	27697	13331

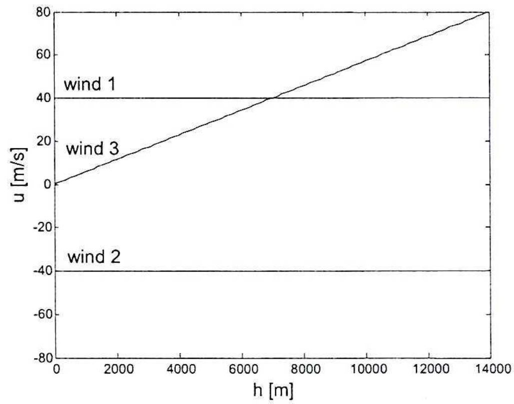

Fig. 11. Different wind conditions

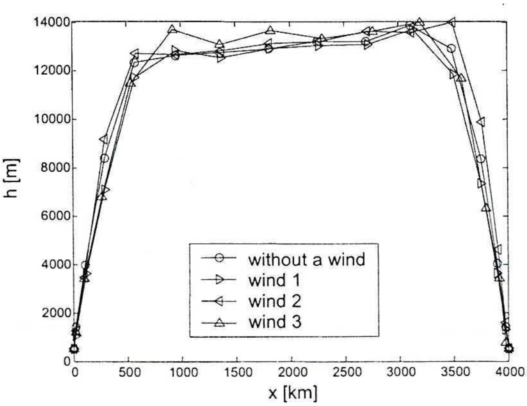

Fig. 12. Influence of wind on the altitude of flight

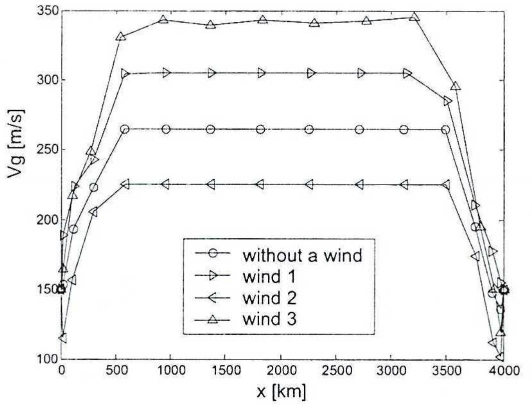

Fig. 13. Influence of wind on the ground speed

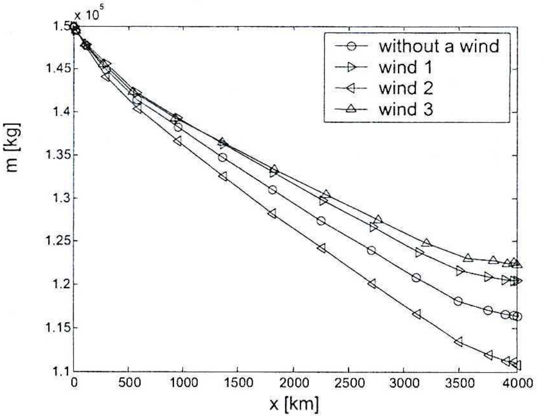

Fig. 14. Influence of wind on the mass of the airplane

4. Conclusions

In this paper, the pseudospectral Chebyshev method is applied for B767 class commercial airplane trajectory optimization. The direct operating costs are minimized. State and control variables are approximated by Lagrange Nth-degree polynomials. Values of these variables are computed at Chebyshev-Gauss-Lobatto points as the expansion coefficients, which are treated as parameters in NLP problem.

The airplane is modeled as a particle and the motion takes place in the vertical plane. Equality constraints are: kinetic equations of motion including presence of wind, kinematical equations, and the equation of weight variation as a result of fuel consumption. Inequality constraints are imposed on: velocity relative to air, climb and descent velocities and the load factor. The method is implemented in the Matlab. As a NLP *solver*, the function *fmincon* is used from the *Optimization Toolbox*. It is an implementation of SQP algorithm. The performed numerical analyses indicate that low degree of discretization generates satisfying results. Also, low sensitivity to the initial approximation is a good feature of the method. The results of computations concerning the presence of winds are reliable and show that the algorithm is effective.

Versatility and effectiveness of the pseudospectral Chebyshev method allows for a variety of applications. Considering performed trajectory optimizations, one may draw a conclusion that it is a useful tool for minimizing airlines' DOC as well as for supporting performance analyses made by airplane designers. The method is easy to program. It may be used for advanced airplane kinetic models that include inequality constraints imposed on the state variables.

The presented examples show that during the cruise the optimal altitudes of the flight increase with the distance as the mass of the airplane decreases (Fig. 5 and Fig. 12) and the optimal cruising velocities are almost constant (Fig. 6 and Fig. 13). For different wind conditions, the cruising altitudes are nearly the same, whereas the cruising velocities are considerably different (Fig. 12 and Fig. 13).

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Optymalizacja trajektorii lotu samolotu komunikacyjnego za pomocą pseudospektralnej metody Czebyszewa

Streszczenie

W pracy została zastosowana bezpośrednia, pseudospektralna metoda Czebyszewa do rozwiązania zadania optymalizacji trajektorii lotu samolotu komunikacyjnego klasy Boeing 767. W metodzie tej zmienne stanu i sterujące obliczane są w punktach Czebyszewa-Gaussa-Lobatto jako współczynniki rozwinięcia w funkcje Lagrange'a. Są one traktowane jako parametry w zagadnieniu programowania nieliniowego z ograniczeniami. Przedstawiono model lotu, w którym samolot traktowany jest jak punkt materialny poruszający się w płaszczyźnie pionowej. Metoda została zaimplementowana w programie Matlab wykorzystując algorytm sekwencyjnego programowania kwadratowego. Wykonano analizy wrażliwości ze względu na stopień dyskretyzacji i przybliżenie