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CONSTITUTIVE RELATION FOR AN ORTHOTROPIC BODY WITH DAMAGE

In the present paper, the damage of fiber-reinforced composite laminates is considered with the aim to examine the change of their mechanical properties.

The crucial issue for theoretical analysis is to construct constitutive relations, which take into account the development of damage, along with stress and strain. The paper is mainly focused on this problem.

In order to derive an adequate description, the author employs an approach based on polynomial invariants functions and invariants integrity basis by Adkins.

1. Introduction

One of the most important factors determining the load-bearing capacity of structural materials is the presence of internal defects, which can exists as initial defects and/or can initiate and develop in response to the applied external load.

In isotropic and homogenous materials one can formulate the deterioration problem, in most cases, in terms of fracture toughness i.e. the resistance to growth of single crack. Fracture mechanics provides very efficient methods to solve the problems of this class.

Deterioration of composite materials is more complex, due to the existence of at least two constituents, which differ in properties. The main feature of composites deterioration is multiplicity of cracking in various shapes and forms, which depend not only on the constituents' properties, but also on their geometrical arrangement.

Therefore, considering that issue, we must take into account the multitude of cracks. Damage of composite materials, expressed in terms of fracture

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mechanics is now in relatively early stage and has limited usage, mainly due to extremely sophisticated solutions of crack problem in anisotropic body. However, significant progress has been noticed in last years and one can expect further intensive development in this area, mainly due to the application of specialized, commercial numerical codes [1].

The consecutive, basic mechanisms of composite laminates damage are [2]: intralaminar matrix cracking, cracks coupling and interfacial debonding, delamination, large scale fiber breaking and at last – formation of a failure path leading to the total material deterioration.

It should be pointed out that not all of the specified mechanisms must necessarily occur. Besides, some of them can cover more or less wide range of laminate "life period" – it depends mainly on laminate ply stacking sequence and on applied load.

For most laminates under monotonically increasing tensile load, the predominant mechanism of laminate deterioration is transverse matrix cracking – see e.g. [3], [4]. That phenomenon is observed in wide range of applied load. The delamination and fibers breaking occurre nearly simultaneously with specimen failure. This was also observed in experiments, carried out by the author on symmetrical cross-ply and angle-ply specimens [5].

Transverse matrix cracking is characterized by cracking of the matrix along fibers in off-axis layers. This so-called primary matrix cracking results in an array of almost parallel cracks – named intralaminar cracks – lying within off-axis lamina.

Macroscopically observed (with use of e.g. optical methods) and measured (in e.g. standard tensile test) effect of gradual material deterioration results in a change of laminate's stiffness.

Despite the unquestionable importance and "attractiveness", the number of papers in this field is relatively small in comparison with other topics of the mechanics of composite materials – author's advanced searching in the World Wide Web justified such a conclusion [6].

It shows that the problem of damage influence on composite laminates characteristics is still open and far from final solution. It can be seen as an explanation to different approaches used in the formulation and description of the considered problem.

Let us mention only the basic approaches used in the analysis of laminates damage. They are: shear lag approach [7], [8], [9], self consistent approach [10], [11], micro-macro approach based on Mori-Tanaka average model [12], [13], minimum complementary potential energy method [14], variational mechanics approach [15], [16], approach based on the average crack opening displacement [7], [8], [17], [18], probabilistic approach [13], homogenization methods [19], [21].

In the present paper continuum damage mechanics (CDM) approach is applied (see e.g. [21], [22], [23], 24]). The criterion for classifying any work as falling in frame of CDM is the description of cracks field in terms of damage measures associated with CDM concepts. Taking into account anisotropic properties of composite materials, the second order damage tensor is often used (see e.g. [25], [26]) in damage analysis.

In order to evaluate the influence of damage state on the change of composite stiffness, the four general steps have to be taken. The first one is to model damage in form of intralaminar cracks; the second one is to establish an adequate constitutive equation, describing relationships between stress, strain and damage. The third step is to apply a suitable procedure for stiffness cal culation, taking into account both laminates stacking sequence and damage in individual layers. And the last one is to compare theoretical predictions with experimental data. Each of the listed steps needs a specific approach and relevant tools to be applied.

The present paper is focused on the second step, as it is crucial for overall analysis. Besides, the remaining issues have been previously reported in a number of papers by the author, e.g. [5], [27], [28], [29], [30].

2. Tensorial representation of the damage

The continuum damage approach to the cracked composite materials is based on an element of the volume of homogeneous continuum, containing a representative sample of damage entities.

By the damage entity we understand a single structural change, which in composite materials made of brittle constituents, under mechanical loads, can take a form of a matrix crack.

In the composite laminates we never observe a single dominant defect (for example crack), but a collection of damage entities. The collection of damage entities of the same, or at least, similar geometrical features, is called damage mode. In laminated fiber composites, under tension load, one can observe mainly the set of almost parallel cracks called intralaminar cracks. Depending on the laminate stacking sequence, one can distinguish several damage modes in a given laminate.

The set of damage modes is called simply a damage.

Let us consider the two states of a body, namely the initial "virgin" state and the actual one – the state in which internal damage is developing in the body in response to the applied external load. The transition from the first to the second state can be characterized by two vectors, namely displacement jump vector $\boldsymbol{b}$ (called also discontinuity vector) across a crack surface S and the vector $\boldsymbol{n}$ – a unit outward normal to the surface S.

Following classical papers by Vakulenko and Kachanov [31], [32], one can construct at any point on surface S of a damage entity second order tensor $\boldsymbol{d'}$ in the form of dyadic product of vectors $\boldsymbol{b}$ and $\boldsymbol{n}$, which fully defines the local geometry of a single defect, and takes the form:

$$\boldsymbol{d'} = \boldsymbol{b} \otimes \boldsymbol{n}\, d\,S \tag{1}$$

For "k" isolated defects, the total measure of the damage field is derived by summation of local damage over k entities. We obtain the following relation:

$$\boldsymbol{d'} = \sum_k \boldsymbol{b}_k \otimes \boldsymbol{n}_k\, d\,S_k \tag{2}$$

For the transition from "discrete" to the "continuous" model, the standard averaging procedure must be employed, by means of averaging the damage field described by (2) over a representative volume V, containing representative sample of cracks.

Denoting by symbol "<>" an average over volume V, we can express the damage tensor in the following form:

$$\langle \boldsymbol{d'} \rangle = \frac{1}{V} \sum_k \int_{S_k} \boldsymbol{b}_k \otimes \boldsymbol{n}_k\, d\,S_k \tag{3}$$

Discontinuity vector $\boldsymbol{b}$ can always be decomposed into the sum of two vectors – parallel $\beta\, \boldsymbol{n}$ and normal $\boldsymbol{\tau}$ to the vector $\boldsymbol{n}$. Thus:

$$\boldsymbol{b} = \beta \boldsymbol{n} + \boldsymbol{\tau} \tag{4}$$

where β denotes a multiplier characterizing crack opening displacement.

Using (3) and (4), one can decompose the averaged damage tensor into two others, namely: normal damage tensor $\langle \boldsymbol{d'} \rangle_n$ and shear damage tensor $\langle \boldsymbol{d'} \rangle_t$, which have the following forms:

$$\langle \boldsymbol{d'} \rangle_n = \boldsymbol{d} = \frac{1}{V} \sum_k \int_{S_k} \beta_k\, \boldsymbol{n}_k \otimes \boldsymbol{n}_k\, d\,S_k \tag{5}$$

$$\langle \boldsymbol{d'} \rangle_t = \frac{1}{V} \sum_k \int_{S_k} \boldsymbol{\tau}_k \otimes \boldsymbol{n}_k\, d\,S_k \tag{6}$$

The damage tensor in the form of eq. (5) characterizes the normal discontinuities while the tensor given by eq. (6) – tangential discontinuities. In terms of the 2-D crack behaviour being under consideration, the first tensor is suitable for the description of the damage field with cracks in mode I ("normal mode"), the second one – in mode II ("shear mode").

In further analyses, only symmetrical, second order, normal damage tensor (5) is retained, since intralaminar cracks are assumed to be active in mode I. The shear damage tensor will not be longer taken into consideration. It means that mode II (as well as mode III) is neglected.

For the set of "k" identical (more precisely speaking – nearly identical) damage entities creating individual damage mode with restriction to their normal behaviour, the damage tensor simplifies to the following form:

$$d_{ij} = \frac{1}{V} k \, b_i \, n_j \, S = \frac{1}{V} k \, \beta \, n_i \, n_j \, S \tag{7}$$

Surface S is understood as the projection of crack surface on a crack "midplane", a multiplier β is assumed as an averaged crack opening factor, depending on material stiffness, applied load and crack geometry.

Factor β has been calculated in frame of LEFM (possibility and admissibility of such an approach is discussed by Varna *et al.* [7]), with some additional assumptions and with the use of the concept of "imaginary strip" proposed by the author – see e.g. [27].

The above procedure takes into account – in approximate way – the constraint effect of neighbouring plies on cracking process developing in a given ply, contrary to e.g. Talreja concept [38] of a crack surface activity vector.

Finally, the m-th intralaminar damage mode within the layer, in its *on-axis* configuration is described by the following damage tensor:

$$d_{ij}^{m} = \frac{\pi t_m}{4 E_2} \rho_m \, v_m \, f(t_m / c_m) \, \sigma^{\infty} \, n_i \, n_j \tag{8}$$

where: E_2 – the transverse Young modulus of a single ply, c_m – width of "imaginary strip", $f(t_m / c_m)$ – the finite width correction factor for the stress intensity factor K_I, v_m – the ply volume fraction, σ^{∞} – the applied load. The symbol ρ_m denotes average cracks density within m-th damaged ply.

Cracks density ρ_m was determined experimentally – for details, see e.g. [27], [30].

3. Constituive relation for an orthotropic body with damage

To describe the mechanical behaviour of continuous body with damage it is necessary to define tensors specifying the stress σ, the deformation ε and the damage $\boldsymbol{d}$. The two last tensors are called the "kinematic" tensors.

The crucial point for the analysis of a body with damage is to construct appropriate constitutive relations, taking into account the damage state.

The above goal can be achieved with employing the theory of irreducible integrity basis and polynomial functions, connecting one of the mechanical tensors with the remaining tensors.

Rivlin and Ericksen [33] proposed an approach to the construction of constitutive relation in which stress tensor components are expressed as polynomials in the elements of kinematic matrices, with taking into account specific material symmetry (e.g. orthotropy). This direction was continued by Rivlin, Green, Noll, Pipkin, Adkins, Spencer and in Poland by e.g. Litewka [26], [34].

It is an important issue to construct a suitable polynomial in such a manner that it takes into consideration any symmetry properties of an anisotropic material.

The material symmetry can be effectively described by applying finite group of orthogonal transformations [35], which allows for transformation of the material by the use of any of the member of the group into a configuration being indistinguishable from its reference configuration.

In the present paper, we confine the analysis to considering the orthotropic materials, since composite laminates generally belong to this class of materials. For orthotropy (rhombic-dipyramidal symmetry), the group of orthogonal transformations consists of the following set of transformations:

$$\boldsymbol{I},\ \boldsymbol{C},\ \boldsymbol{R}_1,\ \boldsymbol{R}_2,\ \boldsymbol{R}_3,\ \boldsymbol{D}_1,\ \boldsymbol{D}_2,\ \boldsymbol{D}_3 \tag{9}$$

where $\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_1\, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3$ are, respectively, relations of identity, central inversion, three reflections of coordinate system in the plane normal to the axis x_1, x_2, x_3 and three rotations of coordinate system through 180° about axis x_1, x_1, x_3.

The theory of orthogonal groups, supported by the classical theory of invariant functions and theory of integrity basis, together with Hilbert theorem, leads to the conclusion that any polynomial function of any number of vectors and tensors, which is invariant under given orthogonal group can be expressed as a polynomial in members of integrity basis.

The above conclusion is the basis of Adkins's approach [36], [37] applied in the present paper.

Following Rivlin and Ericksen approach – one can express the stress tensor components for orthotropic material in a rectangular Cartesian system, as follows:

$$\sigma_{ij} = f_{ij}(e_{rs}, a_{pq}) \quad ; \quad (f_{ij} = f_{ji}) \tag{10}$$

where: e_{ij}, a_{ij} denote symmetrical, II order kinematic matrices and the functions f_{ij} are polynomials in the arguments indicated.

The coefficients, which appear in these polynomials, are material parameters, which do not depend on the position through the body as well as upon any deformation.

Adkins [36] derived the specific form of general relation (10), which was expressed in his original formulation in a system of curvilinear coordinates. Adkins's relation transformed to the rectangular system, applied in the present paper, has the following form:

$$\sigma_{ij} = A_{ijtt}\,\Theta_{tt} + \varepsilon_{rst}\,\varepsilon_{rst}\,A_{ijrs}\,(P^{(1)}_{rs;t} + P^{(2)}_{rs;t} + Q^{(1)}_{rs;t} + Q^{(2)}_{rs;t} + R_{rs;t}) \tag{11}$$

where:

$$A_{ijrs} = \begin{cases} 1 & \text{for } i = r,\ j = s \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

$$P^{(1)}_{rs;t} = e_{rs}\,\Theta^{(1)}_{tt} \qquad P^{(2)}_{rs;t} = a_{rs}\,\Theta^{(2)}_{tt} \qquad Q^{(1)}_{rs;t} = e_{rs}\,e_{ts}\,\Theta^{(3)}_{tt} \tag{13}$$

$$Q^{(2)}_{rs;t} = a_{rs}\,a_{ts}\,\Theta^{(4)}_{tt} \qquad R_{rs;t} = e_{rt}\,a_{ts}\,\Theta^{(5)}_{tt} + e_{st}\,a_{tr}\,\Theta^{(6)}_{tt} \tag{14}$$

and ε_{rst} denotes the Ricci's symbol.

The relation (11) is a form, invariant under the group of transformations (9), which describes the orthotropic symmetry of a material.

The functions Θ_{tt} and $\Theta^{(k)}_{tt}$ ($k = 1, 2, 3, 4, 5, 6$) are invariant polynomial functions of invariants system appropriate to the case where stress depends on two symmetric, kinematic matrices, namely e_{ij} and a_{ij}.

Taking into account the symmetry of matrices e_{ij} and a_{ij} the set of invariants consists of 23 invariants, which with the use of contracted notation by Voigt, has the following explicit form:

$$K_1 = e_{11} = e_1 \qquad K_2 = e_{22} = e_2 \qquad K_{12} = e_{33} = e_3$$

$$K_4 = a_{11} = a_1 \qquad K_5 = a_{22} = a_2 \qquad K_6 = a_{33} = a_3$$

$$K_3 = e_{12}^2 = e_6^2 \qquad K_{13} = e_{13}^2 = e_5^2 \qquad K_{14} = e_{23}^2 = e_4^2$$

$$K_7 = a_{12}^2 = a_6^2 \qquad K_8 = a_{13}^2 = a_5^2 \qquad K_9 = a_{23}^2 = a_4^2$$

$$K_{10} = e_{12}\, a_{12} = e_6\, a_6 \quad K_{15} = e_{13}\, a_{13} = e_5\, a_5 \quad K_{16} = e_{23}\, a_{23} = e_4\, a_4$$

$$K_{17} = e_{12}\, e_{23}\, e_{31} = e_4\, e_5\, e_6 \tag{15}$$

$$K_{18} = e_{12}\, e_{23}\, a_{13} = e_4\, e_6\, a_5 \quad K_{11} = a_{12}\, a_{23}\, a_{31} = a_4\, a_5\, a_6$$

$$K_{21} = e_{13}\, a_{12}\, a_{23} = e_5\, a_4\, a_6 \quad K_{20} = e_{12}\, e_{13}\, a_{23} = e_5\, e_6\, a_4$$

$$K_{23} = e_{23}\, a_{12}\, a_{13} = e_4\, a_5\, a_6 \quad K_{22} = e_{12}\, a_{13}\, a_{23} = e_6\, a_4\, a_5$$

In successive considerations we assume stress being linearly dependent upon both matrices e_{ij}, a_{ij}, therefore, the functions $Q^{(1)}_{rs;t}$ and $Q^{(2)}_{rs;t}$ can be neglected, and functions $\Theta^{(5)}_{tt}$, $\Theta^{(6)}_{tt}$ are reduced to constant multipliers, as otherwise the function $R_{rs,t}$ – eq. (14 b) – would be at least a quadratic function of kinematic matrices.

Therefore, eq. (14 b) takes the following form:

$$R_{rs:t} = g_1\, e_{rt}\, a_{ts} + g_2\, e_{st}\, a_{tr} \tag{16}$$

where g_1 and g_2 are constants.

Under the assumption previously made, and due to equation (12), the stresses are as follows:

$$\sigma_{ij} = A_{ijtt}\, \Theta_{tt} + \varepsilon_{rst}\, \varepsilon_{rst}\, A_{ijrs}\, (P^{(1)}_{rs;t} + P^{(2)}_{rs;t} + R_{rs;t}) \tag{17}$$

Let us further identify the symmetrical kinematic matrices e_{ij} and a_{ij} as respectively, a strain tensor ε_{ij} and damage tensor d_{ij}.

An assumption of linear dependence of stress on kinematic matrices is in fact equivalent to the assumption of strains and damage being small quantities.

We confine subsequent analysis to the in-plane behaviour of a laminate ply. Therefore, the stress, strain and damage tensors in Voigt's notation are as follows:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \sigma_{1} \\ \sigma_{2} \\ \sigma_{6} \end{bmatrix} \qquad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \varepsilon_{6} \end{bmatrix} \qquad \boldsymbol{d} = \begin{bmatrix} d_{11} \\ d_{22} \\ d_{12} \end{bmatrix} = \begin{bmatrix} d_{1} \\ d_{2} \\ d_{6} \end{bmatrix} \tag{18}$$

Using equations (12), (13 a, b) and (16), we derive from eq. (17) stresses in the following form:

$$\sigma_1 = \Theta_{11} \qquad \sigma_2 = \Theta_{22} \qquad \sigma_6 = \varepsilon_6\, \Theta_{33}^{(1)} + d_6\, \Theta_{33}^{(2)} \tag{19}$$

Now, we must define invariant polynomial functions Θ_{11}, Θ_{22}, $\Theta_{33}^{(1)}$ and $\Theta_{33}^{(2)}$, which in general are the functions of invariants set (15), but for in-plane case, being considered here, only the chosen invariants are of interest, namely K_1, K_2, K_4, K_5 and K_{10}, i.e. ε_1, ε_2, d_1, d_2 and $\varepsilon_6\, d_6$.

As we take into account linearity of the relations "stress vs. strain", as well as, "stress vs. damage", the relevant polynomials follow:

$$\Theta_{11} = \mathrm{a}_1\, \varepsilon_1 + a_2\, \varepsilon_2 + (a_3\, d_1 + a_4\, d_2)\, \varepsilon_1 + (a_5\, d_1 + a_6\, d_2)\, \varepsilon_2 + a_7\, d_6\, \varepsilon_6 + a_8 \tag{20}$$

$$\Theta_{22} = b_1\, \varepsilon_1 + b_2\, \varepsilon_2 + (b_3\, d_1 + b_4\, d_2)\, \varepsilon_1 + (b_5\, d_1 + b_6\, d_2)\, \varepsilon_2 + b_7\, d_6\, \varepsilon_6 + b_8 \tag{21}$$

$$\Theta_{33}^{(1)} = c_1\, d_1 + c_2\, d_2 + c_3 \qquad \Theta_{33}^{(2)} = f_1\, \varepsilon_1 + f_2\, \varepsilon_2 + f_3 \tag{22}$$

The coefficients a_1-a_8, b_1-b_8, c_1-c_3 and f_1-f_3 are material parameters.

Polynomial functions Θ_{11}, Θ_{22}, $\Theta_{33}^{(1)}$ and $\Theta_{33}^{(2)}$ can not be independent, as the stress derived with the use of these functions must satisfy the constitutive equation of general form:

$$\sigma_i = C_{ij}\, \varepsilon_j = C_{ij}^{o}\, \varepsilon_j + C_{ij}^{d}\, \varepsilon_j \qquad i,j = 1,2,6 \tag{23}$$

where C_{ij} denotes the stiffness matrix, which can be decomposed into two constituent matrices – the first one $\boldsymbol{C}^o$ referring to undamaged, "virgin" state and the other one $\boldsymbol{C}^d$ characterizing the change of stiffness due to damage. The stiffness matrix for in-plane case has the following components (in Voigt's notation):

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \tag{24}$$

We can now calculate stresses from eq. (23) and then compare with those given by eqs. (19) with the use of polynomial functions (20)-(22). Thus, we obtain the following relations:

$$\begin{gathered} C_{11}^{o} = a_1 \qquad C_{12}^{o} = C_{21}^{o} = a_2 = b_1 \qquad C_{16}^{o} = C_{61}^{o} = 0 \\ C_{22}^{o} = b_2 \qquad C_{26}^{o} = C_{62}^{o} = 0 \qquad C_{66}^{o} = c_3 \\ a_8 = b_8 = f_3 = 0 \qquad a_5 = b_3 \quad a_6 = b_4 \quad a_7 = f_1 \qquad b_7 = f_2 \\ C_{11}^{d} = a_3\, d_1 + a_4\, d_2 \qquad C_{12}^{d} = C_{21}^{d} = a_5\, d_1 + a_6\, d_2 \\ C_{22}^{d} = b_5\, d_1 + b_6\, d_2 \qquad C_{66}^{d} = c_1\, d_1 + c_2\, d_2 \\ C_{16}^{d} = C_{61}^{d} = a_7\, d_6 \qquad C_{26}^{d} = C_{62}^{d} = b_7\, d_6 \end{gathered} \tag{25}$$

From eqs. (25) we get the following set of independent material parameters:

$$\begin{matrix} A_1 = a_5 & A_2 = a_6 & A_3 = a_7 & A_4 = b_7 & A_5 = a_3 \\ A_6 = a_4 & A_7 = c_1 & A_8 = c_2 & A_9 = b_5 & A_{10} = b_6 \end{matrix} \tag{26}$$

Note that we have derived stiffness matrix for undamaged material with elements $C_{16}^{o} \neq 0$, $C_{26}^{o} \neq 0$. This result was expected, since it is typical for orthotropic laminate in principal material axis (1, 2) (so-called *special orthotropy*).

For a general case of any damage state, we obtained $C_{16}^{d} \neq 0$, $C_{26}^{d} \neq 0$, thus the initial special orthotropy is no longer retained, and is replaced by the so-called *general orthotropy.*

For the intralaminar damage being of interest, it has been shown in [27], [28], [29] that the only non-zero damage tensor component is d_2.

Using the last six relations in eq. (25) and eq. (26) we finally get the stiffness matrix (so-called *reduced* stiffness matrix) associated with the damage state in m-th damaged ply, expressed in principal material axis in the form:

$$C^{dm} = \begin{bmatrix} A_6\, d_2^m & A_2\, d_2^m & 0 \\ A_2\, d_2^m & A_{10}\, d_2^m & 0 \\ 0 & 0 & A_8\, d_2^m \end{bmatrix} \tag{27}$$

It follows from (27) that only four unknown material parameters, namely A_2, A_6, A_8 and A_{10} will be involved in further analysis.

The result obtained here – according to Adkins approach – is similar to that one derived by Talreja [22], [38] from considerations based on thermodynamics with internal state variables introduced by Coleman and Gurtin.

It can be proven that under certain assumptions, both approaches are formally equivalent. Therefore, the formal, mathematical approach applied in the present paper has also a physical background.

4. The stiffness matrix for a damaged laminate

The stiffness matrices C^o and C^d derived for a single "virgin" and damaged ply in on-axis configurations are a basis for evaluation of *transformed* stiffness matrices for a ply (i.e. in any reference coordinate system (x, y)).

In the present paper it is achieved by utilization of the transformation procedure accomplished by Tsai and Pagano, described in e.g. [39].

The next step is to derive the stiffness matrix for a laminate being a collection of plies, some of which can be damaged, while the remaining can be still intact.

In order to simplify the analysis, and have a possibility to compare theoretical predictions with experimental data, the class of laminates being of interest was restricted to the symmetrical laminates. Besides, it was assumed that laminate is loaded only by unidirectional tensile force.

Under the above limitations, the problem is confined to the analysis of *extensional* stiffness matrix $\boldsymbol{A}$, which has been calculated using the classical theory of lamination.

5. Engineering moduli for a damaged laminate

In order to determine engineering moduli, the compliance matrix $\boldsymbol{S}$ must be determined. Under restrictions being assumed in previous chapter, it can be shown that stiffness and compliance matrices satisfy the relation:

$$\boldsymbol{S} = \boldsymbol{A}^{-1}\, t \tag{28}$$

where t denotes the thickness of a laminate.

The engineering moduli of interest, namely longitudinal Young modulus, transverse Young modulus and major Poisson ratio, E_x^L, E_y^L and ν_{xy}^L, can be directly derived from compliance matrix by the use of the standard relations, given in e.g. [39].

After numbers of transformations and calculations, which are shown in [28], we finally get the desired moduli in the following form:

$$E_x^L = E_x^{oL} + V_0^d\,[A_{10} + A_6\,(\nu_{xy}^{oL})^2 - 2\,A_2\,\nu_{xy}^{oL}] \tag{29}$$

$$E_y^L = E_y^{oL} + V_0^d\,[A_6 + A_{10}\,(\nu_{yx}^{oL})^2 - 2\,A_2\,\nu_{yx}^{oL}] \tag{30}$$

$$\nu_{xy}^L = \nu_{xy}^{oL} + \frac{1 - \nu_{xy}^{oL}\,\nu_{yx}^{oL}}{E_y^{oL}}\,V_0^d\,[A_2 - A_6\,\nu_{xy}^{oL}] \tag{31}$$

The factor V_0^d takes into account the volume fraction of m-th damaged ply and the damage within this ply.

The three unknown parameters A_2, A_6, A_{10} have been determined in unidirectional tensile test carried out on specimen of code $[0/90_3]_s$, manufactured from carbon/epoxy prepreg tape Vicotex NCHR 174B. More details of specimens' manufacturing and their testing one can find in e.g. [27], [30].

6. Theoretical model versus experimental data

Detailed comparison of theoretical predictions and experimental data is given in e.g. [28], [29]. Here, just for illustration of the applied approach, only one example is shown in Figure 1. It refers to the cross-ply specimen of stacking sequence $[0, 90_4]_s$.

Very good agreement between theoretical predictions and experimental data regarding changes of longitudinal Young modulus and Poisson ratio and their correlation with cracks development is easily visible, and can be seen as confirmation of the correctness of theoretical description.

The present paper also shows that the theory of invariants, orthogonal groups and integrity basis, primary thought as a tool in the projection geometry, is still – after about 60 years from its origin – a useful tool in solving problems of modern continuum mechanics, including mechanics of composite laminates.

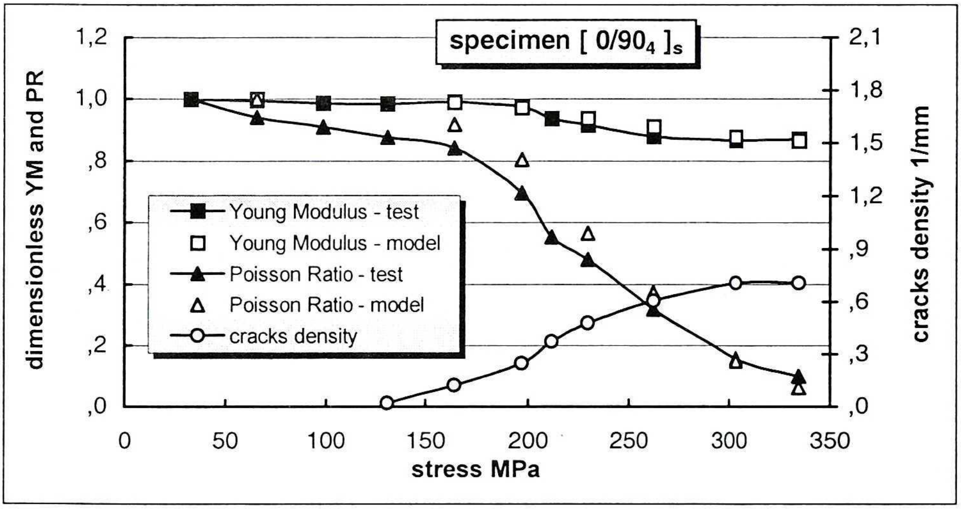

Fig. 1. Dimensionless longitudinal Young modulus (YM), Poisson ratio (PR) and crack density as functions of applied stress

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Równanie konstytutywne dla ciała ortotropowego z uszkodzeniami

Streszczenie

Praca dotyczy pękania wewnątrzwarstwowego laminatów kompozytowych o warstwach jednokierunkowo zbrojonych włóknami i jego wpływu na zmianę charakterystyk sztywnościowych.

Kluczowe znaczenie dla opisu teoretycznego ma taka konstrukcja równania konstytutywnego, która umożliwia właściwe uwzględnienie uszkodzeń na równi z naprężeniami i odkształceniami.

W celu osiągnięcia tego celu, w pracy wykorzystano podejście zaproponowane przez Adkinsa, oparte na teorii funkcji wielomianowych i nieredukowalnych baz niezmienniczych.