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SIMPLIFICATION OF THE MATRIX KINEMATICS METHOD FOR DETERMINATION OF ANGULAR VELOCITIES

The properties of matrix operations and the properties of Hartenberg-Denavit's co-ordinate system's transformation matrices were used for deriving a dependence facilitating an easier determination of the links' angular velocity vectors in the link-related co-ordination systems. The use of derived dependence does not require determining products of transformation matrices nor inverse matrices. The numbers of necessary algebraic operations for previous and simplified dependences was set up. The use of a simplified dependence was illustrated by a numerical example.

1. Introduction

At present, the most popular and quoted in numerous manuals [1], [2], [5], [6], [7], [8] method of kinematic analyzing the mechanisms is the matrix method, based on transformations of link-related co-ordination systems, published in 1955 by R. S. Hartenberg and J. Denavit as well as in their later works [3], [4]. The matrix method may be successfully used for analyzing the kinematics of two- and three-dimensional mechanisms. It plays a particular role in analyzing the kinematics of three-dimensional mechanisms (the robotics) [1], [2], [5], [6], [8]. In case when the dynamic model is constructed on basis of the Lagrange's 2nd order equations, the use of the said method means mainly to determine the location vectors of mobile joints, barycenters, the barycenter velocity vectors and the links' angular velocity vectors. The obtained dependences are in turn used for determining the potential energy equation - the vertical components of the barycenter location vectors - and for determining the kinetic energy - the squares of barycenter velocities and vectors of angular velocities.

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The dependences for determining the mechanism links' barycenter velocities are the most sophisticated ones. Complication of formulas depends directly on the number of degrees of freedom, the joints' geometry and the mass distribution. The dependences for determining the links' angular velocity vectors are significantly simpler, but their determination implies the necessity of determining matrices invert to the transformation matrices and their products or matrices invert to those being products of transformation matrices.

The kind of any numeric calculation algorithm to be used for some simple cases is not very important when modern PC-class computers are applied. However, if the growth of calculation capacity is associated with growing needs to solve more and more sophisticated tasks, the problem of minimizing the number of actions remains essential. The number of algebraic operation carried out is directly related to the accuracy of numeric calculations and to the labor consumption of analytic derivations.

Having used the properties of matrix operation and the properties of the Hartenberg-Denavit's co-ordinate system's transformation matrices, two dependences were derived for an easier determination of the links' angular velocity vectors in the link-related co-ordination systems. The first dependence allows us to determine the angular velocity vectors for any link without the need of calculating angular velocity components for the antecedent link. The second dependence is simpler, but it causes that getting the results depends on values obtained for the antecedent link.

2. Determination of angular velocity vector components

Many works present two dependences to be used for determining angular velocity vectors. The first one describes angular velocity vectors in an immobile co-ordinate system,

$$\omega_i = \omega_{i-1} + T_{1,i-1}\omega_{i,i-1}, \tag{1}$$

the second one is based on the first calculation, is made as an inverse transformation of previously calculated angular velocities, and determines values of the angular velocity vectors on the links-related co-ordination systems,

$$\omega_{ii} = T_{1i}^{-1}\omega_i , \tag{2}$$

whereas

$$\omega_{i,i-1} = \begin{bmatrix} 0 & 0 & \dot{\Theta}_i & 0 \end{bmatrix}^T , \tag{3}$$

$$T_{1i} = A_1 A_2 A_3 \ldots A_i = \prod_1^i A_i , \tag{4}$$

$$A_i = \begin{bmatrix} c\Theta_i & -s\Theta_i c\alpha_i & s\Theta_i s\alpha_i & l_i c\Theta_i \\ s\Theta_i & c\Theta_i c\alpha_i & -c\Theta_i s\alpha_i & l_i s\Theta_i \\ 0 & s\alpha_i & c\alpha_i & \lambda_i \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

where:

$s\Theta_i = \sin\Theta_i$, $c\Theta_i = \cos\Theta_i$, $s\alpha_i = \sin\alpha_i$, $c\alpha_i = \cos\alpha_i$,

l_i, α_i - distance and angle between the axes of rotational pair of the i-link,

λ_i, Θ_i - distance and rotation angle between links: i-1 and i,

T_{1i} - product of matrices A_1 through A_i whereas $T_{11} = A_1$.

The dependence (1) can be shown in another form, not depending on the antecedent calculation

$$\omega_i = \sum_{k=1}^{i} T_{1,k-1}\omega_{k,k-1}, \qquad \text{whereas } T_{1,0} = 1. \tag{6}$$

The values of angular velocity vectors, determined in an immobile co-ordinate system, are useless for any further dynamic calculations, and may be used only for calculating values of angular velocity vectors in the links-related co-ordinate systems needed for determining the rotational kinetic energy function. It seems therefore advisable to derive one dependence that directly determines the ω_{ii} values. After inserting the dependence (1) into (2) we obtain

$$\omega_{i,i} = T_{1,i}^{-1}\sum_{k=1}^{i} T_{1,k-1}\omega_{k,k-1}. \tag{7}$$

Because of

$$T_{1,i}^{-1} = A_i^{-1}A_{i-1}^{-1}A_{i-2}^{-1}\ldots A_1^{-1} = \prod_{i}^{1} A_i^{-1}, \qquad \text{whereas} \qquad T_{11}^{-1} = A_1^{-1} \tag{8}$$

and considering that

$$T_{1,i}^{-1}T_{1,i} = 1, \tag{9}$$

the dependence (7) can be written in a much simpler way

$$\omega_{i,i} = \sum_{k=1}^{k=i} T_{k,i}^{-1}\omega_{k,k-1}. \tag{10}$$

The values of angular velocity vectors do not depend on the mechanism dimensions. It means that the last column of the matrix $T_{k,i}^{-1}$, because of the form of the vectors $\omega_{k,k-1}$, is always multiplicated by zero and never affects the result. In this case, it can be always assumed that $\lambda_i = 0$ and $l_i = 0$. For the transformation matrices, where $\lambda_i = 0$ and $l_i = 0$, the transposed and the invert matrices are equivalent to each other

$$A_i^T = A_i^{-1} \quad \text{and} \quad A_i^{-1} = A_i^T, \quad \text{when} \quad \lambda_i = 0 \quad \text{and} \quad l_i = 0. \tag{11}$$

When in the dependence (10) the invert matrix $T_{k,i}^{-1}$ is replaced by the transposed matrix, and after considering the fact that the transposed product of matrices is equal to the product of transposed matrices calculated in the reverse order

$$\left(A_1 A_2\right)^T = A_2^T A_1^T , \tag{12}$$

as a result of transposing both sides of the dependence (10), we can obtain finally

$$\omega_{i,i}^T = \sum_{k=1}^{k=i} \omega_{k,k-1}^T T_{k,i} . \tag{13}$$

The so obtained dependence allows us to determine vectors of the links' angular velocities in the link's co-ordination system without the need of calculating inverse matrices. A certain inconvenience in the above dependence is the necessity of calculating the product of transformation matrices, and this makes the process of determining angular velocities when more sophisticated the number of links increases. This inconvenience can be avoided by using (similarly to the input dependences) the velocity vectors calculated previously for the antecedent link.

After substituting the dependence (1) into the dependence (2) we will get

$$\omega_{i,i} = T_{1,i}^{-1}\left(\omega_{i-1} + T_{1,i-1}\omega_{i,i-1}\right). \tag{14}$$

After the multiplication

$$\omega_{i,i} = T_{1,i}^{-1}\omega_{i-1} + T_{1,i}^{-1}T_{1,i-1}\omega_{i,i-1} , \tag{15}$$

and having considered the dependence (8)

$$T_{1,i}^{-1} = A_i^{-1}T_{1,i-1}^{-1} , \tag{16}$$

after inserting to (15) we get

$$\omega_{i,i} = A_i^{-1}T_{1,i-1}^{-1}\omega_{i-1} + A_i^{-1}T_{1,i-1}^{-1}T_{1,i-1}\omega_{i,i-1} . \tag{17}$$

Having replaced $T_{1,i-1}^{-1}T_{1,i-1} = 0$ and used the input dependence (2) it can be obtained

$$\omega_{i,i} = A_i^{-1}\left(\omega_{i-1} + \omega_{i,i-1}\right). \tag{18}$$

Having considered the dependences (11) and (12), as a result of double-sided transposition of the dependence (18) we get finally

$$\omega_{i,i}^T = \left(\omega_{i-1,i-1} + \omega_{i,i-1}\right)^T \cdot A_i^0 , \quad \text{wheras} \quad \omega_{0,0} = \left[0\ 0\ 0\ 0\right]^T , \tag{19}$$

A_i^0 - the matrix describing the transformation of the system i into the system $i-1$ where there is no need to determine the terms of the last column ($\lambda_i = 0$ and $l_i = 0$).

The above dependence can be also edited in another form

$$\omega_{i,i}^T = \left(\omega_{i-1,i-1} + \omega_{i,i-1}\right)^T \cdot B_i , \quad \text{wheras} \quad \omega_{0,0} = \left[0\ 0\ 0\right]^T \tag{20}$$

and

$$\omega_{i,i-1} = \left[0 \; 0 \; \dot{\Theta}_i \right]^T , \tag{21}$$

$$B_i = \begin{bmatrix} c\Theta_i & -s\Theta_i c\alpha_i & s\Theta_i s\alpha_i \\ s\Theta_i & c\Theta_i c\alpha_i & -c\Theta_i s\alpha_i \\ 0 & s\alpha_i & c\alpha_i \end{bmatrix}, \tag{22}$$

B_i - the 3rd order minor of the matrix A_i describing the transformation of the system i into the system $i-1$, created by means of deleting the last column and the last line.

3. Numbers of arithmetic operations

In course of calculation of the angular velocity vectors, the matrix 4×4 and the matrix 4×1 can be neglected and replaced by the matrices 3×3 and 3×1 accordingly. This applies, of course, to all dependences used in calculating the angular velocity vectors.

Table 1.
The specification of numbers of multiplications and additions during calculation of the angular velocity vectors for the link i

	according to dependences (1) and (2)	according to dependence (13)	according to dependence (20)
number of additions	$9+6\sum_{k=2}^{i}(9k-10)$ *	$\frac{9}{2}i(3i-1)$	$3(4i-1)$
number of multiplications	$9+9\sum_{k=2}^{i}(6k-7)$ *	$\frac{9}{2}i(3i-1)$	$9i$
* formula for the link $i>1$, for $i=1$ the numbers of multiplications and additions according to the comparable dependences is the same and equal to 9			

To show how much the calculations of the angular velocity vectors were simplified, the numbers of necessary operations according to formulas (1) and (2) should be compared to those of formulas (13) and (20), whereas in all comparable dependences the transformation matrices A_i were replaced by the matrices B_i.

Table 2.
The specification of numbers of multiplications and additions during calculation of the angular velocity vectors for $i=5$ and $i=10$

	number of links i	according to dependences (1) and (2)	according to dependence (13)	according to dependence (20)
number of additions	5	525	315	57
	10	2385	1305	117
number of multiplications	5	513	315	45
	10	2358	1305	90

When the dependence (20) is used, the number of arithmetic operations forms an arithmetic progression in the function of the number of links. The result for any next link is obtained after performing always the same number of actions.

4. Numerical example

For the mechanism shown on Fig. 1, the dependences determining values of the mechanism's links' angular velocity vectors in the link's co-ordination system were determined. The parameters of the transformation matrix of the link-related co-ordination systems are shown in Table 3. The calculations were made on basis of derived dependences (20).

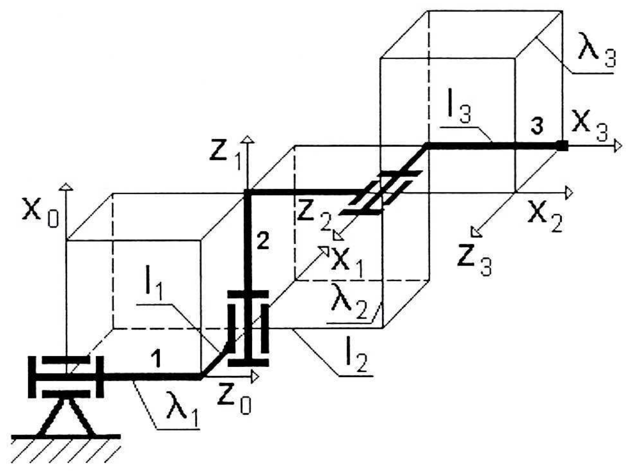

Fig. 1. An example of a mechanism dimensioned according to the Hartenberg's-Denavit's notation

Table 3.

Parameters of the transformation matrix

Transformation number			
parameter	1	2	3
λ_i	1	1	1
l_i	1	1	1
α_i	-90	90	90
Attention: „One" means, that the value exists			

To determine the matrix B_i only the values of the angles α_i must be known. The matrices B_i, determined on basis of the dependence (22) and values of the angles α_i given in the last line of the Table 3 are:

$$B_1 = \begin{bmatrix} c_1 & 0 & -s_1 \\ s_1 & 0 & c_1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} c_2 & 0 & s_2 \\ s_2 & 0 & -c_2 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From the dependence (20) for $i = 1$

$$\omega_{1,1}^T = \left(\omega_{0,0} + \omega_{1,0}\right)^T B_1,$$

therefore

$$\omega_{1,1}^T = \begin{bmatrix} 0 & 0 & \dot{\Theta}_1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & -s_1 \\ s_1 & 0 & c_1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\Theta}_1 & 0 \end{bmatrix},$$

for $i = 2$

$$\omega_{2,2}^T = \left(\omega_{1,1} + \omega_{2,1}\right)^T B_2,$$

$$\omega_{2,2}^T = \begin{bmatrix} 0 & -\dot{\Theta}_1 & \dot{\Theta}_2 \end{bmatrix} \begin{bmatrix} c_2 & 0 & s_2 \\ s_2 & 0 & -c_2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -\dot{\Theta}_2 s_2 & \dot{\Theta}_2 & \dot{\Theta}_1 c_2 \end{bmatrix},$$

for $i = 3$

$$\omega_{3,3}^T = \left(\omega_{2,2} + \omega_{3,2}\right)^T B_3,$$

$$\omega_{3,3}^T = \begin{bmatrix} -\dot{\Theta}_1 s_2 & \dot{\Theta}_2 & \dot{\Theta}_1 c_2 + \dot{\Theta}_3 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then $\omega_{3,3}^T = \begin{bmatrix} -\dot{\Theta}_1 s_2 c_3 + \dot{\Theta}_2 s_3 & \dot{\Theta}_1 s_2 s_3 + \dot{\Theta}_2 c_3 & \dot{\Theta}_1 c_2 + \dot{\Theta}_3 \end{bmatrix}.$

The so derived dependence for determining values of the angular velocity vectors gives it is due to a much smaller number of necessary algebraic operations that must be performed a substantial reduction of labor consumption to be used for deriving as well as the decrease of error possibility in case of using analytic transformations. The use of that dependence for numeric calculations reduces also the numeric error value.

5. Conclusions

As a result of the described above transformations a simple dependence was found that generally facilitates the method of determining angular velocities. The most important benefits are:

- the result needed for further calculations is obtained from one simple dependence,
- there is no need to determine products of transformation matrices,
- there is no need to determine invert matrices and their products,
- the result for a subsequent link is always obtained after performing the same number of actions,
- when the number of links grows, the performed actions are added,
- the labor consumption of analytic transformation is reduced,
- the accuracy of numeric calculations is increased.

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Uproszczenie metody macierzowej kinematyki w zakresie wyznaczania prędkości kątowych

Streszczenie

Wykorzystując właściwości działań na macierzach oraz właściwości macierzy przekształceń układów współrzędnych Hartenberga Denavita, wyprowadzono zależność pozwalającą na łatwiejsze wyznaczenie wektorów prędkości kątowych ogniw w układach współrzędnych związanych z ogniwami. Korzystanie z wyprowadzonej zależności nie wymaga wyznaczania iloczynów macierzy przekształceń oraz wyznaczania macierzy odwrotnych. Wykonano zestawienie liczb koniecznych działań algebraicznych, dla zależności dotychczas stosowanych i zależności uproszczonej. Stosowanie zależności uproszczonej zilustrowano przykładem liczbowym.